Complex Analysis II - MT433P

References

[1] Lars Ahlfors: *Complex Analysis: An Introduction to the Theory of Analytic Functions of One Complex Variable*

[2] Joseph Bak, Donald J. Newman: *Complex Analysis*ls

[3] Serge Lang: *Complex Analysis*

[4] John B Conway: *Functions of one complex variable*

[5] Steven G. Krantz: *Complex analysis: The geometric viewpoint*

1 Problems

1. Let $\mu, \gamma\colon [0,1] \to \mathbb{C} \setminus \{0\}$ be closed continuous curves with same base point, i.e. $\gamma(0) = \mu(0)$. Show that

$$w(\gamma * \mu; 0) = w(\gamma; 0) + w(\mu; 0)\ .$$

Hint: As usual we are sloppy as to the interval bounds. In this problem, γ and μ can be concatenated because they are both loops at the same point $\mu(0) = \gamma(0) = \gamma(1)$. By definition of the concatenation, $\gamma * \mu$ is the curve defined on $[0,2]$ given by

$$(\gamma * \mu)(t) = \begin{cases} \gamma(t) & \text{if } 0 \le t \le 1 \\ \mu(t-1) & \text{if } 1 \le t \le 2 \end{cases}\ .$$

Closedness now means that $(\gamma * \mu)(0) = (\gamma * \mu)(2)$.

2. Let $\gamma\colon [0, 2\pi] \to \mathbb{C}$ be the curve with

$$\gamma(t) = \cos(14t) + i\sin(42t)\ .$$

Check that $\gamma(t) \neq 0$ for all t and find the winding number of γ around 0.

What is the winding number of the curve $\mu\colon [0, 2\pi] \to \mathbb{C}$ be the curve with

$$\mu(t) = \cos(14t) + i\sin(35t)$$

around 0?

Hint: The curves are concatenations of simpler curves. You can use the expression of the winding number in terms of ray crossings.

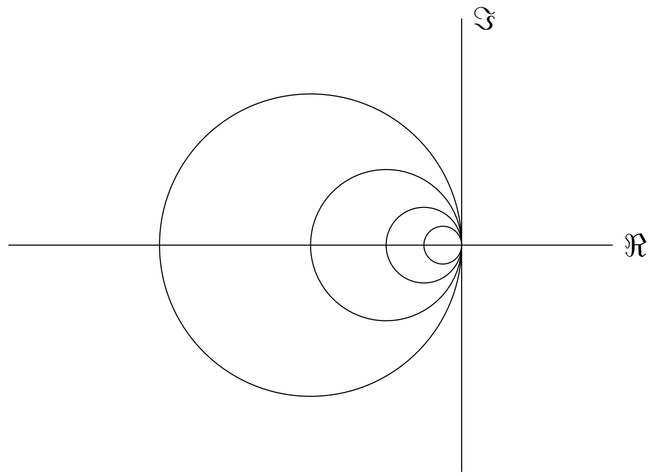
3. Is there a continuous closed curve $\gamma\colon [0,1] \to \mathbb{C}$ so that for every $n \in \mathbb{Z}$, there is a point $p_n \in \mathbb{C} \setminus \gamma([0,1])$ with $w(\gamma; p) = n$?

Solution: It suffices to find such a curve for all positive integers, $n \in \mathbb{N}$. To construct such a curve we concatenate circles of length $\frac{1}{2^n}$ (or any other summable sequence), for instance, we define

$$c(t) = -1 + e^{2\pi i t} \quad , \quad t \in [0,1]$$

and

$$\gamma(0) := 0 \quad , \quad \gamma(t) = \frac{1}{2^{n-1}} c(2^n(t - 2^{-n})) \quad \text{if} \quad 2^{-n} \leq t \leq 2^{-n+1} \; .$$



The winding number of this curve around the point $-3/2^n$ for $n \in \mathbb{N}$ is n.

4. Let $\gamma\colon [-1,1] \to \mathbb{C}$,

$$\gamma(t) = 2t + i\sin(\pi t) \; .$$

Compute

$$\int_\gamma \frac{1}{z+1} + \frac{1}{z-1} \; dz \; .$$

Solution: The integrand $f(z)$ is odd, i.e. $f(-z) = -f(z)$. The same holds for the curve, $\gamma(-t) = -\gamma(t)$. It follows that

$$\int_\gamma f(z) \; dz = \int_{-1}^{1} f(\gamma(t))\gamma'(t) \; dt = 0 \; .$$

5. Let $\gamma\colon [0,1] \to \mathbb{C}$ be a loop. Show that the set

$$\{p \in \mathbb{C} \,|\, p \in \mathbb{C} \setminus \gamma([0,1]), w(\gamma,p) \neq 0\}$$

is bounded.

Solution: The maps $\gamma\colon [0,1] \to \mathbb{C}$ and the modulus $|\cdot| : \mathbb{C} \to \mathbb{R}_0^+$ are continuous and $[0,1]$ is compact. Hence the image $|\gamma([0,1])|$ is also compact. Therefore there is $r \in \mathbb{R}_0^+$ so that $\gamma([0,1]) \subset \mathrm{B}_r(0)$. If $p \in \mathbb{C} \setminus \mathrm{B}_r(0)$, then

$$\Re\left(\frac{\gamma(t) - p}{-p}\right) > 0 \quad \text{for all} \quad t \in [0,1] \; .$$

Hence, with the standard branch ln of the logarithm

$$\ln\colon \mathbb{C} \setminus \mathbb{R}_0^- \to \mathbb{R}^+ + i(-\pi,\pi) \; ,$$

we can write this as

$$\frac{\gamma(t)-p}{-p} = \left|\frac{\gamma(t)-p}{-p}\right| e^{2\pi i\theta(t)}$$

with the angle function

$$\theta(t) = \frac{1}{2\pi}\Im\left(\ln\left(\frac{\gamma(t)-p}{-p}\right)\right) \in \left(-\frac{1}{2},\frac{1}{2}\right) .$$

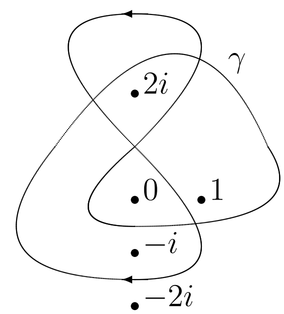
Since $\theta(1)-\theta(0)\in\mathbb{Z}$ we must have

$$w(\gamma,p) = w\left(\frac{\gamma-p}{-p},0\right) = \theta(1)-\theta(0) = 0 .$$

6. Compute the integral

$$\oint_\gamma \frac{z}{(e^z-1)(4+z^2)(z-1)^2}dz$$

over the simple closed curve γ shown in the picture. What are the winding numbers of the curve γ around the singularities of the integrand?



Solution: The integrand has a removable singularity at 0, simple poles at $\pm 2i$ and a pole of order 2 at 1. The winding numbers of γ around $0, -2i, +2i, 1$ are $-2, 0, 0, -1$ respectively. The residue at 0 is 0, thus we only need the residue at 1:

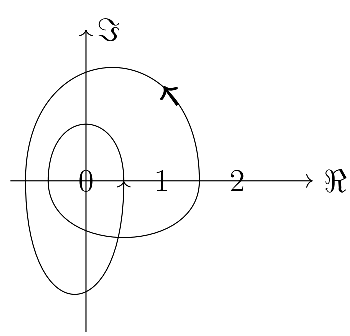
$$\textbf{Res}\left(\frac{z}{(e^z-1)(4+z^2)(z-1)^2}; z=1\right) = \left.\frac{d}{dz}\right|_{z=1}\left((z-1)^2\frac{z}{(e^z-1)(4+z^2)(z-1)^2}\right)$$

$$\left.\frac{d}{dz}\right|_{z=1}\left(\frac{z}{(e^z-1)(4+z^2)}\right) = \frac{5(e-1)-5e-2(e-1)}{25(e-1)^2} = \frac{-2e-3}{25(e-1)^2} .$$

Hence

$$\oint_\gamma \frac{z}{(e^z-1)(4+z^2)(z-1)^2}dz = -2\pi i\frac{-2e-3}{25(e-1)^2} .$$

7. Let γ be the curve shown in the picture. Compute $\displaystyle\oint_\gamma \frac{z^2e^{\frac{1}{z}}}{z-1}\,dz$.



Solution: The singularities of the integrand $f(z) = \frac{z^2 e^{\frac{1}{z}}}{z-1}$ are at 0, essential, and 1, a smple pole. The winding numbers are

$$w(\gamma, 0) = 2 \quad , \quad w(\gamma, 1) = 1 \ .$$

Since $z^2 e^{\frac{1}{z}}$ is holomorphic near $z = 1$, the residue of f and 1 is

$$\mathbf{Res}\left(\frac{z^2 e^{\frac{1}{z}}}{z-1}; z=1\right) = z^2 e^{\frac{1}{z}}\Big|_{z=1} \mathbf{Res}\left(\frac{1}{z-1}; z=1\right) = e \ .$$

The Laurent series of $f(z)$ at $z = 0$ is

$$f(z) = z^2 e^{\frac{1}{z}} \frac{1}{z-1} = \sum_{k=0}^{\infty} \frac{1}{k!} z^{2-k} \times (-1) \sum_{l=0}^{\infty} z^l = -\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{z^{l-k+2}}{k!} \ .$$

The residue is the coefficient with z^{-1}, hence

$$\mathbf{Res}\left(\frac{z^2 e^{\frac{1}{z}}}{z-1}; z=0\right) = -\sum_{k=3}^{\infty} \frac{1}{k!} = 1 + 1 + \frac{1}{2} - e = \frac{5}{2} - e \ .$$

By the Residue Theorem, the integral is

$$\oint_\gamma \frac{z^2 e^{\frac{1}{z}}}{z-1} \, dz = 2\pi i \left(2\left(\frac{5}{2} - e\right) + e\right) = 2\pi i (5 - e)$$

8. Let z_0 be an isolated, not removable singularity of $f(z)$. Is it possible that z_0 is a removable singularity of $e^{f(z)}$?

   Solution: If z_0 were a removable singularity of $e^{f(z)}$, then this has a holomorphic extension g. Then

$$\frac{g'(z)}{g(z)} = \frac{f'(z) e^{f(z)}}{e^{f(z)}} = f'(z) \ .$$

   Hence z_0 is a simple pole of f', impossible.

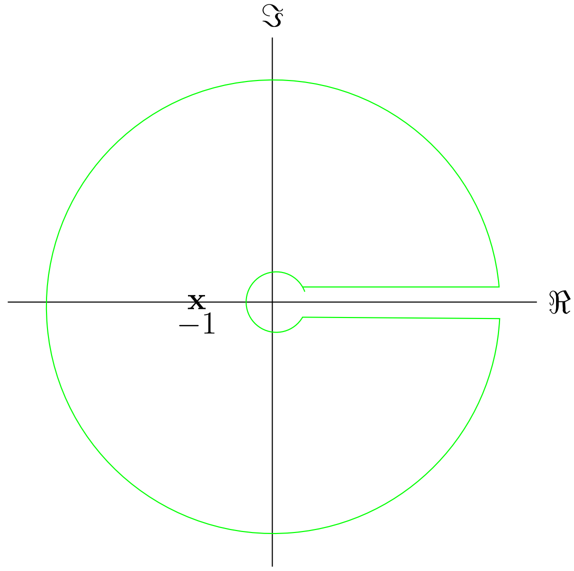
9. Compute the integral

$$\int_0^\infty \frac{1}{\sqrt{t}(1+t)} \, dt \ .$$

   Hint: Look at the curve below. You need to choose the branch of $\sqrt{t}$ defined on and "inside" this curve, i.e. so that, for instance,

$$\sqrt{e^{it}} = e^{it/2}$$

   for $0 \leq t < 2\pi$.



10. Show that the polynomial $p(z) = z^7 + z^2 + 1$ has all its zeros in the annulus $A_{3/4,3/2}(0) = \left\{z \in \mathbb{Z} \,\middle|\, \frac{3}{4} < |z| < \frac{3}{2}\right\}$.

    Solution: For $|z| = 3/2$ we have $|z^7| = 3^7/2^7 > 1 + 3^2/2^2 \geq |1 + z^2|$. By Rouché's Theorem, the polynomial has as many zeros in $\mathrm{B}_{3/2}(0)$ as z^7, i.e. 7 and thus all of the zeros of $p(z)$.

    For $|z| = 3/4$ we have $|z^7| = 3^7/4^7 < 1 - 3^2/4^2 \leq |1 + z^2|$. By Rouché's Theorem, the polynomial has as many zeros in $\mathrm{B}_{3/4}(0)$ as $1 + z^2$, i.e. none. There are also no zeros on $\partial\mathrm{B}_{3/4}(0)$. Thus all zeros of $p(z)$ lie in $A_{3/4,3/2}(0) = \mathrm{B}_{3/2}(0) \setminus \overline{\mathrm{B}}_{3/4}(0)$.

11. Show that the function $\cos(z) + z$ has $2n$ zeros (counting multiplicities) in the domain $\{z \in \mathbb{C} \,|\, 0 < \Re(z) < 2\pi n\}$.

    It follows from this that the zeros are all simple, how?

    Solution: If $z = 2n\pi + \lambda i$, $n \in \mathbb{N}_0$, $\lambda \in \mathbb{R}$, $t \in [0, 1]$ then

$$f_t(z) = \cos(z) + tz = \frac{e^{iz} + e^{-iz}}{2} + 2nt\pi + it\lambda = \frac{e^{\lambda} + e^{-\lambda}}{2} + 2nt\pi + it\lambda \neq 0 \ .$$

We estimate

$$\begin{aligned} |f_t(x+iy)| &= |\cos(x+iy) + t(x+iy)| \\ &= \left|\frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2} + t(x+iy)\right| \\ &\geq \frac{e^{|y|} - e^{-|y|}}{2} - t\,|x+iy| \\ &\geq \frac{e^{|y|}}{2} - \frac{1}{2} - |x| - |y| \end{aligned}$$

For $0 \leq x \leq 2n$ this is positive if

$$\frac{e^{|y|}}{2} - |y| > 2n + \frac{1}{2} \tag{1.1}$$

Let $y \in \mathbb{R}^+$ satisfy this. Such y exists because $\frac{e^t}{2} - t \xrightarrow{y\to\infty} = \infty$. Thus for any h, $h > y$, and all z on the boundary of

$$\Omega = \{z \,|\, 0 < \Re(z) < 2n \ , \ |\Im(z)| < h\}$$

we have that $f_t(z) \neq 0$ for all $t \in [0,1]$. By the argument principle, f_1 has the same number of zeros as $f_0(z) = \cos(z)$, in Ω, i.e. $2n$.

To see that all these are simple, first note that f_1 has no real zero because for $t \in \mathbb{R}_0^+$,

$$f(t) = \cos(t) + t = 1 + \int_0^t \underbrace{-\sin(u) + 1}_{\geq 0} \; du > 0 \; .$$

But f is real, i.e. $f(\overline{z}) = \overline{f(z)}$ and therefore zeros of f come in conjugate pairs $z, \overline{z}$. By the above we have two zeros in each strip $(2k\pi, 2(k+1)\pi) + i\mathbb{R}$ and these must be conjugates of each other. Since they are not real, they must be different.

Alternatively: A (at least) double zero is also a zero of the derivative

$$f'(z) = -\sin(z) + 1 \; .$$

But the zeros of this are all of the form $z = \frac{\pi}{2} + 2k\pi$, $k \in \mathbb{Z}$. However none of these is a zero of f,

$$f(\pi/2 + 2k\pi) = \cos(\pi/2) + \frac{\pi}{2} + 2k\pi \neq 0 \quad \text{for all} \quad k \in \mathbb{Z} \; .$$

12. Is there a non constant entire function that takes only real values on the unit circle?

Solution: Let $f \in \mathcal{O}(\mathbb{C})$ be so that $f(\mathrm{S}^1) \subset \mathbb{R}$, hence $\Im(f(z)) = 0$ for all $z \in \mathcal{S}^1 = \partial\mathbb{E}$. By the maximum principle, the function $\Im(f(z))$ can not have an extremum in $\mathbb{E}$, hence $f(z) = 0$ for all $z \in \mathbb{E}$. By the Identity Theorem 11.10, $f = 0$.

Alternatively, we apply the Schwarz reflection principle, Theorem 11.22, to the composition $f \circ c$ of such a function f with the Cayley transform $c \colon \mathcal{H} \to \mathbb{E}$. This composition is holomorphic on $\mathbb{C} \setminus \{-i\} \supset \overline{\mathcal{H}}$. The restriction of $f \circ c$ to the closed upper half plane $\overline{\mathcal{H}}$ satisfies the conditions of the Schwarz reflection principle and therefore extends holomorphically to all of $\mathbb{C}$. By the identity theorem this extension must coincide with $f \circ c$, hence the singularity of $f \circ c$ at $z = -i$ must be removable. But this is only possible if f is holomorphic at ∞ and therefore bounded. By Liouville's Theorem 11.8, f must be constant.

13. Show that for every function $g \in \mathcal{O}(\mathbb{C})$ with $0 \notin g(\mathbb{C})$, there is a function $f \in \mathcal{O}(\mathbb{C})$ so that $e^{f(z)} = g(z)$ for all $z \in \mathbb{C}$.

Hint: You can not simply define $f(z) := \ln(g(z))$ because ln can not be extended holomorphically to $\mathbb{C} \setminus \{0\}$. Review the definition of ln on the universal cover of $\mathbb{C} \setminus \{0\}$.

Solution: We must have $f'e^f = g'$, hence $f' = \frac{g'}{g}$. Thus we define

$$f(z) = \int_{l_z} \frac{g'(z)}{g(z)} \, dz + \ln(g(0))$$

where $l_z(t) = tz$ and $\ln(0)$ is any complex number with $e^{\ln(0)} = g(0)$.

14. Consider the domains

$$A_{0,1} = \mathbb{E} \setminus \{0\} = \{z \in \mathbb{C} \,|\, 0 < |z| < 1\} \quad \text{and} \quad A_{1,5} = \{z \in \mathbb{C} \,|\, 1 < |z| < 5\} \; .$$

(a) Is there a surjective holomorphic function

$$g \colon A_{0,1} \to A_{1,5} \; ?$$

Solution: Let $q\colon \mathbb{E} \to A_{1,\sqrt[20]{5}} \setminus \mathbb{R}^-$ be biholomorphic. Such a function exists by the Riemann Mapping Theorem. Then

$$f\colon \mathbb{E} \to A_{1,5} \quad , \quad f(z) = q(z)^{20}$$

is surjective and each $y \in A_{1,5}$ has at least 2 preimages. It follows that the restriction $f\colon \mathbb{E} \setminus \{0\} \to A_{1,5}$ is still surjective.

(b) Is there a biholomorphic function

$$h\colon A_{0,1} \to A_{1,5} \ ?$$

Solution: Assume h were such a function. Since h is bounded, the singularity of h at 0 can be removed, extending h to a holomorphic function

$$f\colon \mathbb{E} \to Y = A_{1,5} \cup \{f(0)\} \ .$$

Because of the Open Mapping Theorem, $q = f(0)$ can not lie in the boundary of Y, hence we must have $f(0) \in A_{1,5}$ and $Y = A_{1,5}$. Let $x \in A_{0,1}$ be the preimage of q under h, hence

$$f^{-1}(q) = \{0, x\} \ ,$$

and let $U, V \subset \mathbb{E}$ be disjoint open neighbourhoods of 0 and x respectively. By the Open Mapping Theorem, $f(U)$ and $f(V)$ are open neighbourhoods of q, hence $f(U) \cap f(V)$ is open and therefore there is a point $z \in f(U) \cap f(V) \setminus \{q\}$. Thus $z = f(u) = f(v)$ for some $u \in U$, $v \in V$, hence $u, v \in A_{0,1}$ and $u \neq v$. But this contradicts the injectivity of f. Thus there is no biholomorphic map $h\colon A_{0,1} \to A_{1,5}$.

15. Let $f\colon \mathbb{C} \to \mathbb{C}$ be continuous and assume that f is holomorphic on

$$\Omega = \mathbb{C} \setminus [-1,1] = \{z \in \mathbb{C} \,|\, \Im(z) \neq 0 \text{ or } |\Re(z)| > 1\} \ .$$

Show that $f \in \mathcal{O}(\mathbb{C})$. Is there a biholomorphic map $\Omega \to \mathbb{E} \setminus \{0\}$?

Hint: For the first question, use Morera's Theorem. For the second, try to map $[-1,1]$ to a ray.

Solution: Let γ be the boundary curve of a triangle. If this triangle lies in Ω, then the line integral of f over γ is 0 because of the Cauchy Integral Theorem. If the triangle intersects $[-1,1]$, then proceed as in the proof of the Schwarz Reflection Principle, Theorem 11.22.

The function $g(z) = \frac{1}{z+1}$ maps

$$\mathbb{C} \setminus [-1,1] \longrightarrow \mathbb{C} \setminus \left(\{0\} \cup [\frac{1}{2}, \infty)\right)$$

Since $\mathbb{C} \setminus [\frac{1}{2}, \infty)$ is simply connected, by the Riemann Mapping Theorem, there is biholomorphic map $h\colon \mathbb{C} \setminus [\frac{1}{2}, \infty) \to \mathbb{E}$. The composition $h \circ g\colon \mathbb{C} \setminus [-1,1] \to \mathbb{E} \setminus \{h(0)\}$ is also biholomorphic. If m is a Möbius transformation of $\mathbb{E}$ with $m(h(0)) = 0$, then $m \circ h \circ g$ maps $\mathbb{C} \setminus [-1,1]$ biholomorphically to $\mathbb{E} \setminus \{0\}$.

16. Let $A = \{z \in \mathbb{C} \,|\, 1 < |z| < 2\}$ and consider the function $f \in \mathcal{O}(\mathbb{C} \setminus \{0\})$, $f(z) = \frac{1}{z}$. Show that there is no sequence $(p_n)_{n \in \mathbb{N}} \in \mathbb{C}[z]^{\mathbb{N}}$ of polynomials converging uniformly on A to f.

Hint: Recall **uniform convergence**: If M is a set and (X, d) a metric space, then a sequence $(f_n)_{n \in \mathbb{N}} \in (X^M)^{\mathbb{N}}$ converges uniformly to $g \in X^M$ if

$$\lim_{n \to \infty} \sup \{d(f_n(m), g(m)) \,|\, m \in M\} = 0 \ .$$

Solution: The complex line integral is continuous with respect to uniform convergence on compact sets. If f were a uniform limit of polynomials p_n on A (uniform convergence on $\frac{3}{2}S^1$ would suffice), then there would be a polynomial p so that $|p(z) - f(z)| < 1/47$ for all z with $|z| = 3/2$. Since polynomials are entire, closed line integrals of polynomials vanish. Thus

$$2\pi = \left| \oint_{|z|=\frac{3}{2}} f(z)\ dz \right| = \left| \oint_{|z|=\frac{3}{2}} f(z) - p(z)\ dz \right| \leq \oint_{|z|=\frac{3}{2}} |f(z) - p(z)|\ dz \leq 2\pi \frac{3}{2} \frac{1}{47} \ ,$$

impossible.

17. For $n \in \mathbb{N}$ let f_n be the entire function

$$f_n(z) = \sum_{k=1}^{n} e^{kz^2} \ .$$

Show that $(f_n)_{n\in\mathbb{N}}$ converges compactly on the sector

$$S = \{x + iy \mid x, y \in \mathbb{R},\ y > |x|\}$$

Solution: Squaring maps S biholomorphically to the left half-plane ($\Re < 0$). In particular compact subsets are mapped to compact subsets in the left half-plane. Thus it suffices to show that

$$\sum_{k=1}^{\infty} e^{kz}$$

converges compactly on the left half-plane $\{z \in \mathbb{C} \mid \Re(z) < 0\}$. If $\Re(z) < \delta < 0$, then

$$\left| \sum_{k=M}^{\infty} e^{kz} \right| \leq \sum_{k=M}^{\infty} \left| e^{kz} \right| \leq \sum_{k=M}^{\infty} \left(e^{\delta}\right)^k = \frac{e^{M\delta}}{1 - e^{\delta}} \xrightarrow{M\to\infty} 0 \ .$$

Since every compact subset of the left half-plane is contained in a half-plane

$$\{z \in \mathbb{C} \mid \Re(z) < \delta\}$$

for some $\delta < 0$, we have shown uniform convergence on all compact sets.

18. For an open subset $U \subset \mathbb{C}$ consider the almost metric space $\mathbb{C}^U$ of functions $f\colon U \to \mathbb{C}$ with the ∞-distance,

$$\mathrm{d}_\infty(f, g) = \sup\{|f(u) - g(u)| \mid u \in U\} \in \mathbb{R}_0^+ \cup \{\infty\} \ .$$

Recall that this almost metric space is complete, any uniform Cauchy sequence converges in $\mathbb{C}^U$. Show that $\mathcal{O}(U) \subset \mathbb{C}^U$ is closed.

Hint: You can not use Morera right away, because the integral of an arbitrary function $f \in \mathbb{C}^U$ over a triangle boundary need not be defined. You need an intermediary X between $\mathcal{O}(U)$ and $\mathbb{C}^U$.

Solution: The set of continuous functions $C(U, \mathbb{C}) \subset \mathbb{C}^U$ is closed and the integral over a triangle boundary defines a continuous map $C(U, \mathbb{C}) \to \mathbb{C}$. If $(f_n)_{n\in\mathbb{N}} \in \mathcal{O}(U)^{\mathbb{N}} \subset C(U, \mathbb{C})^{\mathbb{N}}$ is a Cauchy sequence, then by completeness of $C(U, \mathbb{C}) \subset \mathbb{C}^U$, $f_n \xrightarrow{n\to\infty} f \in C(U, \mathbb{C})$. But since for every triangle boundary δ the map $g \mapsto \int_\delta g(z)\ dz$ is continuous, we have

$$\int_\delta f(z)\ dz = \int_\delta \lim_{n\to\infty} f_n(z)\ dz = \lim_{n\to\infty} \int_\delta f_n(z)\ dz = 0$$

by the Cauchy integral theorem. By Morera's Theorem, f is holomorphic.

2 The Residue Theorem

2.1 The Complex Line Integral over Continuous Curves

The complex line integral of a holomorphic function extends to continuous curves. Thus let $G \subset \mathbb{C}$ be a domain and $f \in \mathcal{O}(G)$. For every $z \in G$ let $r(z) > 0$ be so that $\mathrm{B}_{r(z)}(z) \subset G$. The Taylor series of f at z converges on $\mathrm{B}_{r(z)}(z)$. For each z we choose an antiderivative F_z of $f|_{\mathrm{B}_{r(z)}(z)}$.

If $\gamma\colon [a,b] \to \mathrm{B}_{r(z)}(z)$ is piecewise C^1 then

$$\int_\gamma f(z)\ dz = F_z(\gamma(b)) - F_z(\gamma(a))\ . \tag{2.1}$$

This gives the same value for all antiderivatives F_z of $f|_{\mathrm{B}_{r(z)}(z)}$. The complex line integral depends continuously on γ, and is additive under concatenation.

For continuous, not necessarily piecewise C^1-curves we use (2.1) as definition. This can also be extended to continuous curves in G not contained in a disk $\mathrm{B}_\epsilon(z) \subset G$. To this end, let $\gamma\colon [0,1] \to G$ be continuous. Let δ be a **Lebesgue number** of the open covering

$$\left\{\gamma^{-1}(\mathrm{B}_{r(z)}(z)) \,\middle|\, z \in G\right\}$$

of $[0,1]$. Thus if $J \subset [0,1]$ is a subinterval with $|J| < \delta$, then there is $z(J) \in G$ so that $\gamma(J) \subset \mathrm{B}_{r(z(J))}(z(J))$. We define the complex line integral of $f \in \mathcal{O}(G)$ over γ by summing up the differences (2.1) of a subdivision of the interval $[0,1]$,

$$0 = \tau_0 < \tau_1 < \tau_2 < \cdots \tau_{n-1} < \tau_n = 1 \quad , \quad \forall i : \tau_i - \tau_{i-1} < \delta\ ,$$

$$\int_\gamma f(z)\ dz := \sum_{i=1}^n \int_{\gamma|_{[\tau_{i-1},\tau_i]}} f(z)\ dz = \sum_{i=1}^n F_{z([\tau_{i-1},\tau_i])}(\gamma(\tau_i)) - F_{z([\tau_{i-1},\tau_i])}(\gamma(\tau_{i-1})) \tag{2.2}$$

Problem 2.3 *Check that this does not depend on the choices involved (list these first).*

Theorem 2.4 (Homotopy invariance of the integral) *Let $G \subset \mathbb{C}$ be a domain, $f \in \mathcal{O}(G)$ and $\gamma_0, \gamma_1\colon [0,1] \to G$ be continuous and homotopic relative end point. Then*

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)\ dz\ .$$

Proof: Let $\gamma_0 \simeq_H \gamma_1 \operatorname{rel} \partial$, i.e. $H\colon [0,1] \times [0,1] \to G$ is continuous, $H(s,t) = \gamma_s(t)$ for all $s \in \{0,1\}$, $t \in [0,1]$, and $H(s,t) = \gamma_0(t) = \gamma_1(t)$ for all $s \in [0,1]$, $t \in \{0,1\}$.

For $s \in [0,1]$ let $\gamma_s\colon [0,1] \to G$ be the curve with $\gamma_s(t) = H(s,t)$. The open covering

$$\left\{H^{-1}(\mathrm{B}_{r(z)}(z)) \,\middle|\, z \in G\right\}$$

of the compact set $[0,1]\times[0,1]$ has a Lebesgue number and therefore we can choose subdivisions

$$0=\sigma_0<\sigma_1<\sigma_2<\cdots\sigma_{m-1}<\sigma_m=1 \quad , \quad 0=\tau_0<\tau_1<\tau_2<\cdots\tau_{n-1}<\tau_n=1 \ ,$$

and $z(i,j)\in G$, $i=1\dots m$, $j=1\dots n$ so that

$$H([\sigma_{i-1},\sigma_i]\times[\tau_{j-1},\tau_j])\subset \mathrm{B}_{r(z(i,j))}(z(i,j)) \ .$$

We may assume that $m=1=\sigma_1$. Then

$$\int_{\gamma_0} f(z)\ dz = \sum_{j=1}^n F_{z(0,j)}(\gamma_0(\tau_j)) - F_{z(0,j)}(\gamma_0(\tau_{j-1}))$$

$$= \sum_{j=1}^n F_{z(1,j)}(\gamma_0(\tau_j)) - F_{z(1,j)}(\gamma_0(\tau_{j-1}))$$

$$= \sum_{j=1}^n F_{z(1,j)}(\gamma_0(\tau_j)) - F_{z(1,j)}(\gamma_1(\tau_j)) + F_{z(1,j)}(\gamma_1(\tau_j)) - F_{z(1,j)}(\gamma_1(\tau_{j-1})) + F_{z(1,j)}(\gamma_1(\tau_{j-1})) - F_{z(1,j)}(\gamma_0(\tau_{j-1}))$$

$$= \sum_{j=1}^n \underline{F_{z(1,j)}(\gamma_0(\tau_j)) - F_{z(1,j)}(\gamma_1(\tau_j))} + F_{z(1,j)}(\gamma_1(\tau_j)) - F_{z(1,j)}(\gamma_1(\tau_{j-1})) + \underline{F_{z(1,j-1)}(\gamma_1(\tau_{j-1})) - F_{z(1,j-1)}(\gamma_0(\tau_{j-1}))}$$

$$= \sum_{j=1}^n F_{z(1,j)}(\gamma_1(\tau_j)) - F_{z(1,j)}(\gamma_1(\tau_{j-1}))$$

$$= \int_{\gamma_1} f(z)\ dz \ .$$

The underlined terms cancel in the sum because they vanish at the ends. •

2.2 The Winding Number

Crucial for this definition is the following **Lifting Theorem**.

Theorem 2.5 *For every continuous curve $\gamma\colon[0,1]\to S^1$ and $\vartheta_0\in\mathbb{R}$ so that $\gamma(0)=e^{i\vartheta_0}$, there is a unique continuous function*

$$\vartheta\colon [0,1]\to\mathbb{R} \quad \textit{so that} \quad \forall t\in[0,1]:\gamma(t)=e^{i\vartheta(t)} \quad \textit{and} \quad \vartheta(0)=\vartheta_0 \ .$$

Corollary 2.6 *Let $a\in\mathbb{C}$ and $\gamma\colon[0,1]\to\mathbb{C}\setminus\{a\}$ be continuous. Let $\vartheta_0\in\mathbb{R}$ be so that*

$$\gamma(0)=a+|\gamma(0)-a|\,e^{i\vartheta_0} \ .$$

Then there is a unique continuous function $\vartheta_{\gamma,\vartheta_0}\colon[0,1]\to\mathbb{R}$ so that

$$\vartheta_{\gamma,\vartheta_0}(0)=\vartheta_0 \quad \textit{and} \quad \gamma(t)=a+|\gamma(t)-a|\,e^{i\vartheta_{\gamma,\vartheta_0}(t)} \ .$$

Angle functions of the same curve differ by a constant,

$$\vartheta_{\nu,\gamma}(t)-\vartheta_{\mu,\gamma}(t)=\nu-\mu\in 2\pi\mathbb{Z} \quad \textit{for all} \quad t\in\mathbb{R} \ .$$

Definition 2.7 *A curve* $\gamma\colon [a,b] \to X$ *is* **closed** *if* $\gamma(a) = \gamma(b)$.

The **winding number** *of a closed continuous curve* $\gamma\colon [0,1] \to \mathbb{C} \setminus \{a\}$ *is*

$$w(\gamma, a) = \frac{\vartheta(1) - \vartheta(0)}{2\pi} \in \mathbb{Z}$$

where $\vartheta\colon [0,1] \to \mathbb{R}$ *is any function so that* $\gamma(t) = a + |\gamma(t) - a|\, e^{i\vartheta(t)}$ *for all* $t \in [0,1]$.

The closed curve γ **surrounds** $a \in \mathbb{C} \setminus \gamma([0,1])$ *if* $w(\gamma, a) \neq 0$.

Theorem 2.8

$$w(\gamma, a) = \frac{1}{2\pi i} \oint_\gamma \frac{1}{z-a}\, dz$$

for every continuous closed curve $\gamma\colon [0,1] \to \mathbb{C} \setminus \{a\}$.

Proof: We may assume $a = 0$ and $\gamma(0) = \gamma(1) = 1$,

$$\gamma(t) = |\gamma(t)|\, e^{i\vartheta(t)}, \vartheta(0) = 0 \ .$$

This curve is homotopic relative end points to one with constant modulus, via the homotopy $H\colon [0,1] \times [0,1] \to \mathbb{C} \setminus \{0\}$ with

$$H(s,t) = (1 - s + s\,|\gamma(t)|)e^{i\vartheta(t)} \ .$$

We may therefore also assume that

$$\gamma(t) = e^{i\vartheta(t)} \ .$$

Let

$$0 = \tau_0 < \tau_1 < \tau_2 < \cdots \tau_{n-1} < \tau_n = 1$$

be a subdivision of $[0,1]$ as in the definition of the complex line integral, so that we have a common antiderivative $F_j(z)$ of $\frac{1}{z}$ on $\gamma([\tau_{j-1}, \tau_j])$, hence

$$F_j(\gamma(\tau_j)) - F_j(\gamma(\tau_{j-1})) = F_j(e^{i\vartheta(\tau_j)}) - F_j(e^{i\vartheta(\tau_{j-1})}) = i\vartheta(\tau_j) - i\vartheta(\tau_{j-1})$$

because

$$\left.\frac{d}{dt}\right|_{t=u} F_j(e^{it}) = \frac{1}{e^{iu}} i e^{iu} = i \ .$$

Hence from the definition (2.2)

$$\oint_\gamma \frac{1}{z}\, dz = \sum_{i=1}^{n} i\vartheta(\tau_j) - i\vartheta(\tau_{j-1}) = i\vartheta(1) - i\vartheta(0) = 2\pi i w(\gamma, 0)$$

•

Theorem 2.9 (Homotopy Invariance of the Winding Number) *Let* $H\colon [0,1] \times [0,1] \to \mathbb{C} \setminus \{p\}$ *be continuous and so that* $H(s,0) = H(s,1)$ *for all* $s \in [0,1]$. *Let* γ_s, $s \in [0,1]$, *be the curves give by* $\gamma_s(t) = H(s,t)$. *Then* $w(\gamma_0, p) = w(\gamma_1, p)$.

Note that the curves γ_s need not have the same endpoints for different s, γ_0 is not necessarily homotopic to γ_1 relative endpoints. H is a homotopy of closed curves only.

Proof: If we had $H(s,0) = H(0,0)$ and $H(s,1) = H(0,1)$ for all s, i.e. if γ_0 were homotopic relative endpoint to γ_1 via H, then this theorem would be immediate from Theorem 2.8 and the homotopy invariance of the integral, Theorem 2.4.

Problem 2.10 *Fill in the gap in this proof of Theorem 2.9*

•

For $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, and $a \in \mathbb{C}$ we use the following notation for rays and sectors:

$$R(\omega, a) = \left\{ a + re^{i\omega} \,\middle|\, r \in \mathbb{R}^+ \right\}$$
$$S(\alpha, \beta) := \left\{ e^{it} \,\middle|\, \alpha < t < \beta \right\}$$
$$S(\alpha, \beta, a) := a + S(\alpha, \beta) = \left\{ a + e^{it} \,\middle|\, \alpha < t < \beta \right\} .$$

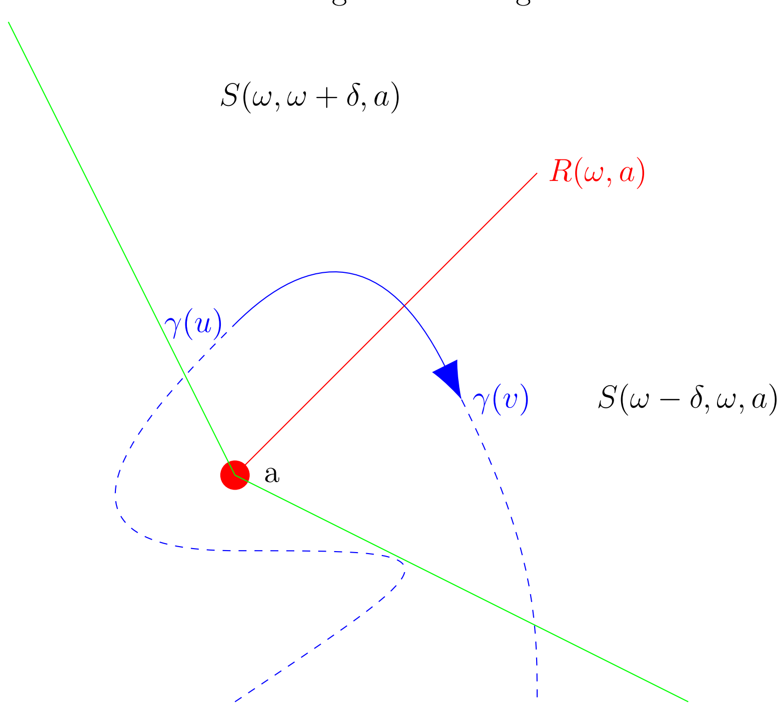
We say that a path $\gamma\colon [0,1] \to \mathbb{C}\setminus\{a\}$ crosses $R(\omega, a)$ positively (respectively negatively) in the subinterval $[u,v] \subset [0,1]$ if for some $\delta < \pi$, we have

$$\gamma([u,v]) \subset S(\omega - \delta, \omega + \delta, a)$$

and

$$\begin{array}{lll} \gamma(u) \in S(\omega-\delta, \omega, a) & \text{resp} & \gamma(v) \in S(\omega, \omega+\delta, a) \\ \gamma(v) \in S(\omega, \omega+\delta, a) & \text{resp} & \gamma(u) \in S(\omega, \omega+\delta, a) \end{array}$$

A negative crossing



Theorem 2.11 (Winding Number in Terms of Crossings) *Let* $a \in \mathbb{C}$ *and* $\gamma \colon [0,1] \to \mathbb{C} \setminus \{a\}$ *be a loop. Assume that*

$$0 = v_{-1} \leq u_0 \leq v_0 \leq u_1 \leq v_1 \ldots \leq u_{k-1} \leq v_{k-1} \leq u_k = 1$$

and that γ *intersects* $R(\omega, a)$ *only in intervals* (u_i, v_i), *i.e.*

$$\gamma(t) \in R(\omega, a) \Longrightarrow \exists i : t \in (u_i, v_i) \ .$$

Then

$$w(\gamma, a) = \#\textit{positive crossings} - \#\textit{negative crossings} \ .$$

Proof: Wlog we may assume that $a = 0$, $\omega = 0$ and that $\gamma(0) = \gamma(1) \notin \mathbb{R}^+ = R(0,0)$. We can choose an angle function θ for γ so that $\theta(0) \in (0, 2\pi)$. By definition of the winding number

$$\theta(1) = 2\pi w(\gamma, 0) + \theta(0) \ .$$

For any $j = -1, \ldots, k-1$ we have that $\theta(t) \notin 2\pi\mathbb{Z}$ if $t \in [v_j, u_{j+1}]$. Thus, by the intermediate value theorem, we have

$$\theta([v_j, u_{j+1}]) \subset (2\pi a(j), 2\pi(a(j)+1))$$

for some integers $a(j)$, $j = -1, \ldots, k-1$. By our assumptions we have

$$\begin{aligned} a(-1) &= 0 \\ w(\gamma, 0) &= a(k-1) \\ a(j+1) &= \begin{cases} a(j) + 1 \text{ if } \gamma \text{ crosses positively in } [u_j, v_j] \\ a(j) - 1 \text{ if } \gamma \text{ crosses negatively in } [u_j, v_j] \end{cases} \end{aligned}$$

•

2.3 The Cauchy Integral Theorem

Let $\Omega \subset \mathbb{C}$ be a domain, $\gamma \colon [0,1] \to \mathbb{C}$ a loop not surrounding any point outside Ω, i.e.

$$\forall a \notin \Omega : w(\gamma, a) = 0$$

and $f \in \mathcal{O}(\Omega)$. The function

$$\phi \colon \Omega \times \Omega \setminus \Delta\Omega \to \mathbb{C}$$

$$\phi(z, u) := \begin{cases} \dfrac{f(z) - f(u)}{z - u} & \text{if } z \neq u \\ f'(u) & \text{if } z = u \end{cases}$$

is continuous and $\phi(\cdot, u)$ and $\phi(z, \cdot)$ are holomorphic, by Riemann's Removable Singularities Theorem 12.12. Since $[0,1]$ is compact, the function

$$g_1(u) := \oint_\gamma \phi(z, u) \, dz$$

is holomorphic on Ω. For u not on γ we have

$$\begin{aligned} g_1(u) &= \oint_\gamma \frac{f(z)-f(u)}{z-u}\ dz \\ &= \oint_\gamma \frac{f(z)}{z-u}\ dz - \oint_\gamma \frac{f(u)}{z-u}\ dz \\ &= \oint_\gamma \frac{f(z)}{z-u}\ dz - 2\pi i\ w(\gamma,u)\ f(u) \end{aligned}$$

In particular, on the set of nonsurrounded points,

$$H = \{a \in \mathbb{C} \mid w(\gamma,a) = 0 \text{ and } a \notin \gamma([0,1])\}$$

the function g_1 is

$$g_1(a) = \oint_\gamma \frac{f(z)}{z-u}\ dz\ . \tag{2.12}$$

Now by assumption,

$$\mathbb{C} = \Omega \cup H\ .$$

We extend g_1 to a function on all of $\mathbb{C}$ by

$$g(u) := \begin{cases} g_1(u) & \text{if } u \in \Omega \\ \oint_\gamma \frac{f(z)}{z-u}\ dz & \text{if } u \in H \end{cases} \tag{2.13}$$

By (2.12) the two expressions for $g(u)$ in (2.13) coincide whenever $u \in H \cap \Omega$, $g(u)$ is well-defined. Since $H \cap \Omega$ is open, g is holomorphic an all of $\mathbb{C}$. There is $C \in \mathbb{R}$ so that

$$|g(u)| = \left| \oint_\gamma \frac{f(z)}{z-u}\ dz \right| \leq \frac{C}{|u|}\ . \tag{2.14}$$

Thus g is bounded and tends to 0 as $u \to \infty$. By Liouville's Theorem, $g = 0$.

We have proved

Theorem 2.15 (Cauchy Integral Formula) *Let $\Omega \subset \mathbb{C}$ be open, $f \in \mathcal{O}(\Omega)$, $\gamma\colon [0,1] \to \Omega$ a loop not surrounding any point outside Ω, i.e.*

$$\forall a \in \mathbb{C} \setminus \Omega : w(\gamma,a) = 0\ .$$

Then for all $u \in \Omega \setminus \gamma([0,1])$,

$$\oint_\gamma \frac{f(z)}{z-u}\ dz = 2\pi i\ w(\gamma,u)\ f(u)$$

Theorem 2.16 (Cauchy Integral Theorem) *Let Ω, γ and f be as in the previous theorem 11.2 Then*

$$\oint_\gamma f(z)\ dz = 0\ .$$

Proof: For $u \in \Omega \setminus \gamma([0,1])$ apply the Cauchy Integral Formula to the function $f(z)(z-u)$. •

2.4 The Residue Theorem

Assume that the function f has only isolated singularities on a domain G. Thus $f \in \mathcal{O}(G \setminus S)$, and for all $s \in S$, there is $r > 0$ so that f is holomorphic on $\mathrm{B}_r(s) \setminus \{s\}$. Assume that none of the singularities $s \in S$ is removable. If $p \in G \setminus S$, then f is complex differentiable at p and therefore bounded in a neighbourhood of p. By the Riemann Removable Singularity Theorem, only removable singularities can lie in this neighbourhood, hence no points from S. It follwos that this neighbourhood lies in $G \setminus S$, and S is closed in G.

Note that this does not mean that S needs to be a closed subset of $\mathbb{C}$. The set S can have limit points on the boundary of G.

Let $G \subset \mathbb{C}$ be a domain and $S \subset G$ be closed and discrete, i.e. for every $s \in S$ there is $r_s > 0$ so that $\mathrm{B}_{r(s)}(s) \cap G = \{s\}$. Let $f \in \mathcal{O}(G \setminus S)$ and let $\gamma\colon [0,1] \to G \setminus S$ be a closed continuous curve. Let

$$S_\gamma = \{s_1, s_2, \dots s_k\} = \{s \in S \,|\, w(\gamma, s) \neq 0\} \ .$$

This set is finite.

For each $s \in S$ choose a path $\beta_s\colon [0,1] \to G \setminus S$ so that $\beta(0) = \gamma(0) = \gamma(1)$, $\beta(1) = s + r(s)/2$ and let $\alpha_s(t) = s + \frac{r(s)}{2} e^{2\pi i t}$, $t \in [0,1]$. Let γ_s be the concatenation

$$\gamma_s = \beta_s * \alpha_s^{w(\gamma,s)} * \beta_s^{-1} \quad \text{hence} \quad w(\gamma_s, s) = w(\gamma, s) \ . \tag{2.17}$$

The loops γ_s does not surround any point outside $G \setminus S$ except s. By additivity, the concatenation

$$\gamma * \gamma_{s_1}^{-1} * \gamma_{s_2}^{-1} * \dots * \gamma_{s_k}^{-1}$$

does not surround any point outside $G \setminus S$. By the Cauchy Integral Theorem (and additivity of the integral)

$$\oint_\gamma f(z)\ dz = \sum_{i=1}^{k} \oint_{\gamma_{s_i}} f(z)\ dz = \tag{2.18}$$

The curves β_s do not contribute to this integral because they are involved twice in reverse direction in the concatenation (2.17). Thus the integral (2.18) becomes

$$= \sum_{i=1}^{k} w(\gamma, s_i) \oint_{\alpha_{s_i}} f(z)\ dz = \sum_{i=1}^{k} w(\gamma, s_i) \underbrace{\lim_{r \to 0} \oint_{|z - s_i| = r} f(z)\ dz}_{=:2\pi i \mathbf{Res}(f; s_i)} \ .$$

We therefore define

Definition 2.19 *Let $U \subset \mathbb{C}$, $p \in \overset{\circ}{U}$ and $f \in \mathcal{O}(U \setminus \{p\})$, p is an isolated singularity of f. The* **residue of f at p** *is*

$$\mathbf{Res}(f; p) = \mathbf{Res}(f(z); z = p) = \frac{1}{2\pi i} \lim_{r \to 0} \oint_{|z-p|=r} f(z)\ dz = \frac{1}{2\pi i} \oint_{|z-p|=R} f(z)\ dz$$

where R is any positive radius so that $\mathrm{B}_R(p) \subset U$.

The residue can be read off the Laurent series of f at p, which also explains the normalization. Let f be holomorphic on $\mathrm{B}_r(p) \setminus \{p\}$ for some $p \in \mathbb{C}$, $r > 0$, and let

$$f(z) = \sum_{n \in \mathbb{Z}} c_n (z-p)^n \quad , \quad 0 < |z-p| < r$$

be the Laurent series of f near p. Let $0 < R < r$. Then the residue is

$$\mathbf{Res}\,(f;p) = \frac{1}{2\pi i} \oint_{|z-p|=R} \sum_{n \in \mathbb{Z}} c_n (z-p)^n \, dz = \frac{1}{2\pi i} \sum_{n \in \mathbb{Z}} \oint_{|z-p|=R} c_n (z-p)^n \, dz = c_{-1}$$

since the Laurent series converges uniformly on compact sets, and

$$\oint_{|z-p|=R} (z-p)^n \, dz = \oint_{|z|=1} z^n \, dz = \begin{cases} 1 \text{ if } n = -1 \\ 0 \text{ if } n \neq -1 \end{cases} \quad .$$

Example 2.20 *Residues at poles of higher order can be computed by differentiation. As an example, we compute*

$$\mathbf{Res}\left(\frac{e^z}{\sin(z)^2}; z = 0 \right) \quad .$$

Since $\frac{e^z}{\sin(z)^2}$ *has a pole of order* 2 *at* $z = 0$*, we have*

$$\frac{e^z}{\sin(z)^2} = \frac{c_{-2}}{z^2} + \frac{c_{-1}}{z} + c_0 + \cdots \quad ,$$

hence

$$z^2 \frac{e^z}{\sin(z)^2} = c_{-2} + c_{-1} z + c_0 z^2 + \cdots \quad ,$$

$$\begin{aligned} \mathbf{Res}\left(\frac{e^z}{\sin(z)^2}; z = 0 \right) = c_{-1} = \left. \frac{d}{dz} \right|_{z=0} \frac{z^2 e^z}{\sin(z)^2} &= \left. \frac{2ze^z + z^2 e^z}{\sin(z)^2} - \frac{2z^2 e^z \cos(z)}{\sin(z)^3} \right|_{z=0} \\ &= \left. \frac{2e^z}{\sin(z)} + e^z - \frac{2e^z \cos(z)}{\sin(z)} \right|_{z=0} \\ &= 1 + e^z \left. \frac{2 - 2\cos(z)}{\sin(z)} \right|_{z=0} = 1 \end{aligned}$$

We compute the same residue by manipulating power series. Recall that

$$\frac{1}{1-q} = \sum_{n=0}^{\infty} q^n \quad \textit{for} \quad |q| < 1$$

and denote by h_i *any holomoprhic functions.*

$$\begin{aligned} \frac{e^z}{\sin(z)^2} &= \frac{e^z}{\left(z - \frac{z^3}{6} + z^5 h_1(z)\right)^2} = \frac{e^z}{z^2 - \frac{z^4}{3} + z^6 h_2(z)} \\ &= \frac{e^z}{z^2} \frac{1}{1 - \frac{z^2}{3} + z^4 h_2(z)} = \frac{e^z}{z^2} \left(1 + \frac{z^2}{3} - z^4 h_3(z) \right) \\ &= \frac{1}{z^2} \left(1 + z + \frac{z^2}{2} + z^3 h_4(z) \right) \left(1 + \frac{z^2}{3} - z^4 h_3(z) \right) \\ &= \frac{1}{z^2} \left(1 + z + \frac{5z^2}{6} + z^3 h_5(z) \right) = \frac{1}{z^2} + \frac{1}{z} + \frac{5}{6} + z h_5(z) \quad . \end{aligned}$$

Theorem 2.21 (Residue Theorem) *Let $G \subset \mathbb{C}$ be a domain and $S \subset G$ a closed discrete subset. Let $\gamma\colon [0,1] \to G \setminus S$ be closed, continuous and not surround any point outside G. Let $f \in \mathcal{O}(G \setminus S)$. Then*

$$\oint_\gamma f(z)\ dz = 2\pi i \sum_{s \in S} w(\gamma, s) \mathbf{Res}\,(f; s)\ .$$

2.4.1 Improper Integrals

A typical case is $\int_{-\infty}^{\infty} R(x)\ dx$ where $R \in \mathcal{O}\left(\overline{\mathcal{H}} \setminus S\right)$, $S \subset \mathcal{H}$ finite, such that

$$\lim_{z \to \infty} |z|\, R(z) = 0\ .$$

Then

$$\left| \int_{|z|=T, \Im(z) \geq 0} R(z)\ dz \right| = \left| \int_0^\pi R(Te^{it}) iTe^{it}\ dt \right| \leq \max \left\{ |R(z)| \,\middle|\, |z| = T, \Im(z) \geq 0 \right\} \pi T \xrightarrow{T \to \infty} 0\ .$$

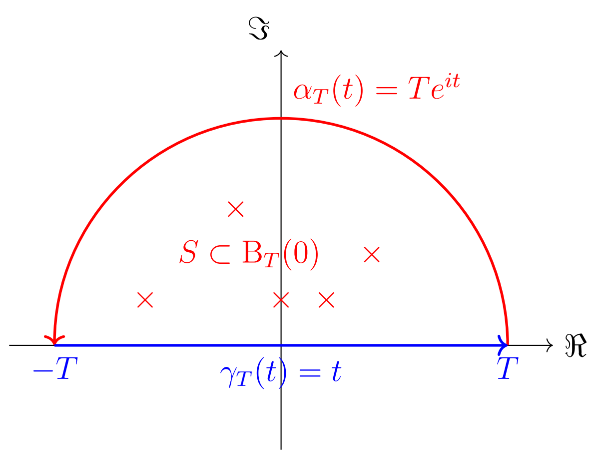
For $T \in \mathbb{R}^+$, let γ_T and α_T be the paths

$$\gamma_T(t) = t,\ t \in [-T, T] \quad \text{and} \quad \alpha_T(t) = Te^{it},\ t \in [0\pi]\ .$$

We can now compute the improper integral with the residue theorem,

$$\begin{aligned} \int_{-\infty}^{+\infty} R(x)\ dx &= \lim_{T \to \infty} \int_{-T}^{T} R(x)\ dx \\ &= \lim_{T \to \infty} \left(\int_{-T}^{T} R(x)\ dx + \int_{|z|=T, \Im(z) \geq 0} R(z)\ dz \right) \\ &= \oint_{\gamma_T * \alpha_T} R(z)\ dz \quad \text{if } T \text{ is so large that} \quad S \subset \mathrm{B}_T(0) \\ &= 2\pi i \sum_{s \in \mathcal{H}} \mathbf{Res}\,(R, s) \end{aligned}$$

because $w(\gamma_T * \alpha_T, s) = 1$ for all $s \in S$ and sufficiently large T.



$$\left|\int_{\alpha_T} R(z)dz\right| = \left|\int_0^{2\pi} iR(Te^{it})\ dt\right| \leq T \sup_{|z|=T} |R(z)| \xrightarrow{T\to\infty} 0$$

Example 2.22 *The* **Fourier transform** *of a function* $f\colon \mathbb{R} \to \mathbb{R}$ *with* $\int_{-\infty}^{\infty} |f(x)|^2\ dx < \infty$ *is the function* $\hat{f}\colon \mathbb{R} \to \mathbb{R}$ *with*

$$\hat{f}(s) = \int_{-\infty}^{\infty} e^{-ist} f(t)\ dt\ .$$

We compute the Fourier transform of $a(t) = \frac{1}{1+t^2}$*. Since* $|e^{-iz}| = e^{\Im(z)}$*, we can use the auxilliary path* α_T *as above in the upper half plane if* $s \leq 0$*. If* $s \geq 0$ *we need to close the path of integration in the lower half plane.*

$$\hat{a}(s) = \int_{-\infty}^{\infty} \frac{e^{-ist}}{1+t^2}\ dt = 2\pi i \begin{cases} \mathbf{Res}\left(\frac{e^{-isz}}{1+z^2}; z = i\right) & \text{if } s \leq 0 \\ -\mathbf{Res}\left(\frac{e^{-isz}}{1+z^2}; z = -i\right) & \text{if } s \geq 0 \end{cases} .$$

The winding numbers are $+1$ *in the upper half plane and* -1 *in the lower half plane. The residues are*

$$\mathbf{Res}\left(\frac{e^{-isz}}{1+z^2}; z = \pm i\right) = e^{\pm s}\mathbf{Res}\left(\frac{1}{1+z^2}; z = \pm i\right) = e^{\pm s}\mathbf{Res}\left(\frac{-i/2}{z-i} + \frac{i/2}{z+i}; z = \pm i\right)$$

$$= \frac{\mp i(\pm 1)e^{\pm s}}{2}$$

Thus

$$\hat{a}(s) = 2\pi i \frac{(\pm 1) \mp ie^{\pm s}}{2} = \pi e^{-|s|}\ .$$

Example 2.23 *In order to compute*

$$\int_0^{\infty} \frac{\sqrt[4]{x}}{1+x^4}\ dx$$

we exploit the symmetry of the integrand $f(z) = \frac{\sqrt[4]{z}}{1+z^4}$*,*

$$f(iz) = i^{1/4} f(z) \tag{2.24}$$

because $i^4 = 1$*. Of course there is a choice in the 4th root of* i *here. We will extend* $\sqrt[4]{\cdot}$ *from* $\mathbb{R}^+$ *holomorphically so that*

$$\sqrt[4]{e^{it}} = e^{it/4} \quad \text{for all} \quad t \in [-\pi/4, 3\pi/4]\ .$$

Thus $\sqrt[4]{i} = \sqrt[4]{e^{i\pi/2}} = e^{i\pi/8}$*, for instance. We now consider the paths*

$$\gamma_T(t) = t\ ,\ t \in [0, T]$$
$$\omega_T(t) = it\ ,\ t \in [0, T]$$
$$\alpha_T(t) = Te^{it}\ ,\ t \in [0, \pi/2]$$

Because of the symmetry (2.24),

$$\int_{\omega_T} f(z)\ dz = \int_0^T f(it)i\ dt = i^{5/4} \int_0^T f(t)\ dt = i^{5/4} \int_{\gamma_T} f(z)\ dz\ .$$

From the Residue Theorem, for $T > 1$,

$$
\begin{aligned}
2\pi i \mathbf{Res}(f,\epsilon) &= \oint_{\gamma_T * \alpha_T * \omega_T^{-1}} f(z)\ dz = \lim_{T\to\infty} \oint_{\gamma_T * \alpha_T * \omega_T^{-1}} f(z)\ dz \\
&= \lim_{T\to\infty} \int_{\gamma_T} f(z)\ dz + \underbrace{\lim_{T\to\infty} \int_{\alpha_T} f(z)\ dz}_{=0} - \underbrace{\lim_{T\to\infty} \int_{\omega_T} f(z)\ dz}_{=i^{5/4} \lim_{T\to\infty} \int_{\gamma_T} f(z)\ dz} \\
&= \int_0^\infty \frac{\sqrt[4]{x}}{1+x^4}\ dx \times (1 - i^{5/4})
\end{aligned}
$$

hence

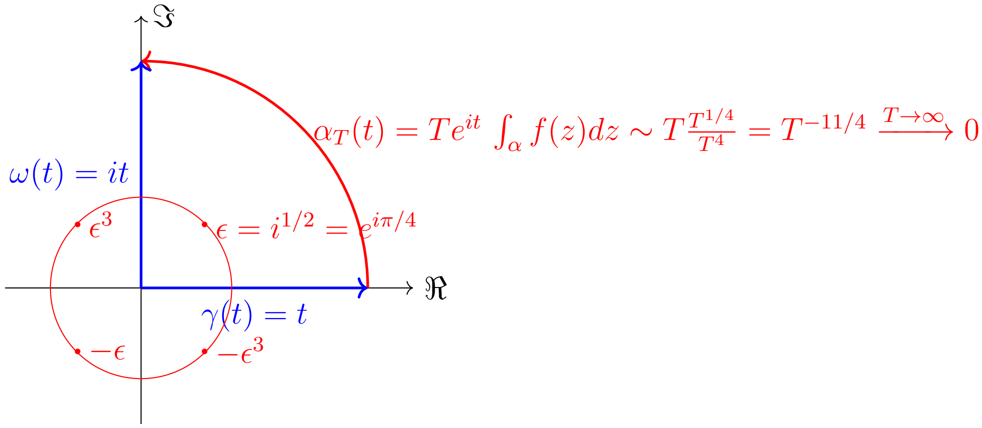
$$
\int_0^\infty \frac{\sqrt[4]{x}}{1+x^4}\ dx = \frac{2\pi i \mathbf{Res}(f,\epsilon)}{1 - i^{5/4}} .
$$

For the residue we get

$$
\frac{1}{1+z^4} = \frac{1}{(z-\epsilon)(z-\epsilon^3)(z+\epsilon)(z+\epsilon^3)} ,
$$

$$
\mathbf{Res}\left(\frac{\sqrt[4]{z}}{1+z^4}, \epsilon\right) = \frac{\epsilon^{1/4}}{(\epsilon-\epsilon^3)(\epsilon+\epsilon)(\epsilon+\epsilon^3)} = \frac{i^{1/8}}{\sqrt{2}2\epsilon i\sqrt{2}} = \frac{i^{-11/8}}{4} .
$$

$$
\int_0^\infty \frac{\sqrt[4]{x}}{1+x^4}\ dx = \frac{2\pi i\ i^{-11/8}}{4(1-i^{5/4})} = \frac{\pi}{2}\frac{i^{-3/8}}{1-i^{5/4}} = \frac{\pi}{2}\frac{1}{i^{3/8}+i^{-3/8}} = \frac{\pi}{4\cos(3\pi/16)} .
$$



2.4.2 Period Integrals of Trigonometric Rational Functions

In order to compute

$$
\int_0^{2\pi} \frac{1+\cos(t)}{2+\sin(t)}\ dt
$$

we first rewrite the integrand

$$
\frac{1+\cos(t)}{2+\sin(t)} = \frac{1+\frac{e^{it}+e^{-it}}{2}}{2+\frac{e^{it}-e^{-it}}{2i}} = \frac{1+\frac{z+\frac{1}{z}}{2}}{2+\frac{z-\frac{1}{z}}{2i}} = i\frac{z^2+2z+1}{z^2+4iz-1} =: f(z) \quad , \quad z = e^{it} .
$$

The integral then is the same as a complex line integral over the curve $\gamma\colon [0,2\pi] \to \mathbb{C}$, $\gamma(t) = e^{it}$,

$$\int_0^{2\pi} f(e^{it})\, dt = \int_0^{2\pi} f(e^{it}) \frac{1}{ie^{it}} \underbrace{ie^{it}}_{\gamma'(t)}\, dt = \oint_\gamma \frac{f(z)}{iz}\, dz = \oint_{|z|=1} \frac{f(z)}{iz}\, dz\ .$$

In the case at hand,

$$\int_0^{2\pi} \frac{1+\cos(t)}{2+\sin(t)}\, dt = \oint_{|z|=1} \frac{z^2+2z+1}{z^2+4iz-1}\frac{1}{z}\, dz = \oint_{|z|=1} \underbrace{\frac{(z+1)^2}{(z-i(-2-\sqrt{3})\ (z-i(-2+\sqrt{3})z}}_{J(z)}\, dz\ .$$

The integrand $J(z)$ has no singularities z with $|z|=1$. The winding number of γ is 1 around points in $\mathrm{B}_1(0)$ and 0 else. By the residue theorem, the integral is therefore equal to

$$2\pi i \sum_{|q|<1} \mathbf{Res}\,(J(z); z=p)\ .$$

There are two singularities of $J(z)$ in $\mathrm{B}_1(0)$, with residues

$$\mathbf{Res}\,(J(z); z=0) = -1$$

$$\mathbf{Res}\left(J(z); z=i(-2+\sqrt{3})\right) = \left.\frac{(z+1)^2}{(z-i(-2-\sqrt{3})z}\right|_{z=i(-2+\sqrt{3})} = \frac{(1-2i+\sqrt{3}i)^2}{(i(-2+\sqrt{3})-i(-2-\sqrt{3}))i(-2+\sqrt{3})}$$
$$= -\frac{-6+4\sqrt{3}-4i+2\sqrt{3}i}{2\sqrt{3}(-2+\sqrt{3})} = \frac{6-4\sqrt{3}+4i-2\sqrt{3}i}{6-4\sqrt{3}}\ .$$

The integral is

$$2\pi i\left(-1+\frac{6-4\sqrt{3}+4i-2\sqrt{3}i}{6-4\sqrt{3}}\right) = 2\pi\frac{4-2\sqrt{3}}{4\sqrt{3}-6} = \frac{8\sqrt{3}}{12}\pi = \frac{2\pi}{\sqrt{3}}\ .$$

2.5 Logarithmic Derivative

Let $U \subset \mathbb{C}$ be open, $p \in U$ and $f\colon U \to S$ be holomorphic. If f has a pole of order $-k$ or a zero of order k at p, then

$$f(z) = (z-p)^k g(z)$$

where $g(p) \neq 0, \infty$. Hence

$$f'(z) = k(z-p)^{k-1}g(z) + (z-p)^k g'(z)\ ,$$
$$\frac{f'}{f}(z) = \frac{k}{z-p} + \frac{g'}{g}\ .$$

Since $g(p) \neq 0, \infty$,

$$\operatorname{ord}(f;p) = k = \frac{1}{2\pi i}\oint_{|z-p|=r} \frac{f'(z)}{f(z)}\, dz = \frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}\, dz$$

provided that r is so small that there are no other poles/zeros of f in $\mathrm{B}_r(p)$, respectively that $w(\gamma,p)=1$ and γ does not surround any points outside U nor any other poles or zeros of f. We can further rewrite the integral

$$\oint_\gamma \frac{f'(z)}{f(z)}\, dz = \int \frac{1}{f(\gamma(t))} f'(\gamma(t)\gamma'(t)\, dt = \oint_{f\circ\gamma} \frac{1}{z}\, dz = 2\pi i w(f\circ\gamma, 0)\ .$$

Theorem 2.25 (Argument Principle, Pole-Zero-Counting integral, winding number) *Let $G \subset \mathbb{C}$ be a domain and f be meromorphic on G, i.e. $f\colon G \to S$ is holomorphic as defined in 12.15. Let $A \subset G$ be open and let γ be a boundary curve for A, i.e.*

$$\forall a \in A : w(\gamma, a) = 1 \quad , \quad \forall a \in \mathbb{C} \setminus \overline{A} : w(\gamma, a) = 0 \ .$$

Then the number of zeros minus the number of poles of f in A, both counted with multiplicity, is

$$\sum_{p \in A} \operatorname{ord}(f; p) = \frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(z)} \, dz = w(f \circ \gamma, 0) \ .$$

From the homotopy invariance of the winding number we immediately have

Corollary 2.26 (Rouché's Theorem) *Let $G \subset \mathbb{C}$ be a domain and $f, g \in \mathcal{O}(G)$. Let $A \subset G$ and $\gamma = \partial A$ a boundary curve of A. If*

$$\forall z \in \partial A : |g(z)| < |f(z)| \ ,$$

then f and $f + g$ have the same number of zeros, counted with multiplicity, in A, i.e.

$$\sum_{a \in A} \operatorname{ord}(f, a) = \sum_{a \in A} \operatorname{ord}(f + g, a) \ .$$

Proof: By the Pole-Zero counting winding number theorem, this follows once we have that

$$w(f \circ \gamma, 0) = w((f + g) \circ \gamma, 0) \ .$$

But this is a consequence of the homotopy invariance of the winding number. The curves

$$f \circ \gamma, (f + g) \circ \gamma \colon [0, 1] \to \mathbb{C} \setminus \{0\}$$

are homotopic. A homotopy is

$$H\colon [0, 1] \times [0, 1] \to \mathbb{C} \setminus \{0\} \quad , \quad H(s, t) = f(\gamma(t)) + sg(\gamma(t)) \ .$$

This is well-defined, because for $s, t \in [0, 1]$, we have

$$\begin{aligned} |H(s, t)| &= |f(\gamma(t)) + sg(\gamma(t))| \geq |f(\gamma(t))| - |sg(\gamma(t))| = |f(\gamma(t))| - s\,|g(\gamma(t))| \\ &\geq |f(\gamma(t))| - |g(\gamma(t))| > 0 \ . \end{aligned}$$

•

Definition 2.27 (Multiplicity) *Let $G \subset S = \mathbb{C} \cup \{\infty\}$ be a domain and $f \in \mathcal{O}(G, S)$. For $a \in S$ we define*

$$f_a(z) = \begin{cases} f(z) - a & \text{if } a \in \mathbb{C} \\ \dfrac{1}{f(z)} & \text{if } a = \infty \end{cases}$$

The multiplicity of a *as a value of f is*

$$m(f, a) := \sum_{u \in G} \operatorname{ord}(f_a, u) \ .$$

If $p(z) = \sum_{k=0}^{n} a_k z^k \in \mathbb{C}[z]$ has degree $n = \deg p$, $a_n \neq 0$, then the only zero of $\frac{1}{p(z)}$ is at ∞, and

$$\operatorname{ord}\left(\frac{1}{p(z)}, z = \infty\right) = \operatorname{ord}\left(\frac{1}{p(1/z)}, z = 0\right) = n$$

because

$$\frac{1}{p(1/z)} z^{-j} = \frac{1}{z^j \sum_{k=0}^{n} a_k z^{-k}} = \frac{1}{\sum_{k=0}^{n} a_k z^{j-k}} = \frac{1}{\sum_{k=0}^{n} a_k z^{j-k}} = \frac{1}{a_n z^{j-n} + a_{n-1} z^{j-n+1} + \cdots + a_0 z^j}$$

is holomoprhic at 0 exactly for $j \leq n$. Thus $m(p, \infty) = \deg p$. If $a \in \mathbb{C}$ then

$$p(z) - a = c \prod_{i=1}^{r} (z - \lambda_i)^{m_i} \quad , \quad \sum_{i=1}^{r} m_i = \deg p$$

and

$$m(p, a) = \sum_{z \in S} \operatorname{ord}\left(p(z) - a, z\right) = \sum_{i=1}^{r} \operatorname{ord}\left(p(z) - a, z = \lambda_i\right) = \sum_{i=1}^{r} m_i = \deg p \ .$$

Thus a polynomial assumes all values with the same multiplicity equal to its degree.

A rational function is a quotient $R(z) = \frac{p(z)}{q(z)}$ of two coprime polynomials (i.e. having no common zero).

Theorem 2.28 *A rational function assumes all values with equal multiplicity.*

Proof: If $\phi, \psi \colon S \to S$ are biholomorphic, then for all $f \in \mathcal{O}(S, S)$, $a \in S$ we have

$$m(f, a) = m(\phi \circ f \circ \psi, \phi(a)) \ .$$

Let R be a rational function and $a \in S$ finite i.e. $a \in \mathbb{C}$. We will show that $m(R, a) = m(R, \infty)$. If $R - a$ has a zero or a pole at ∞ we replace $R(z)$ with $R(b + 1/z)$ where $b \in \mathbb{C}$ and $R(b) \neq a, \infty$, without changing the multiplicity:

$$m(R, a) = m(R - a, 0) = m(R(b + 1/z) - a, 0) \ .$$

Since $\tilde{R}(z) = R(b + 1/z) - a$ has only finitely many zeros and poles and none of them at ∞, there is T so that $\tilde{R}^{-1}\{0, \infty\} \subset \mathrm{B}_T(0)$. Let α_T be a boundary curve of $\mathrm{B}_T(0)$, e.g. $\alpha_T(t) = Te^{it}$ for $t \in [0, 2\pi]$. By the theorem on the counting integral,

$$m(R, a) - m(R, \infty) = m(\tilde{R}, 0) - m(\tilde{R}, \infty) = \sum_{z \in R^{-1}(\{0 \infty\})} \operatorname{ord}\left(\tilde{R}, z\right) = \frac{1}{2\pi i} w(\tilde{R} \circ \alpha_T, 0) \xrightarrow{T \to \infty} 0$$

because $\tilde{R}(\alpha_T(t)) \xrightarrow{T \to \infty} \tilde{R}(\infty) \in \mathbb{C}$, i.e. $\tilde{R} \circ \alpha_T$ converges to a constant curve, and constant curves have zero winding numbers. •

3 Compact Convergence

3.1 The basic complete functions spaces and the Arzela Ascoli Theorem

If E is a set and (M, d) is a metric space, then on $M^E = \{f: E \to M \mid f \text{ a map}\}$ we consider the almost distance

$$d_\infty: M^E \times M^E \to \mathbb{R}_0^+ \cup \{\infty\} \quad \text{with}$$
$$d(f,g) = \sup\{d(f(e), g(e)) \mid e \in E\} \in \mathbb{R}_0^+ \cup \{\infty\} \ .$$

If (M, d) is complete, then (M^E, d_∞) is complete, i.e. every Cauchy sequence converges. Convergence with respect to d_∞ is called **uniform convergence**.

If T is a topology on E, then

$$C(E, M) = \{f: E \to M \mid f \text{ continuous }\} \subset M^E$$

is closed with respect to d_∞ and therefore complete if (M, d) is complete. If E is compact then d_∞ is finite on $C(E, M)$. Thus if (M, d) is complete, then $(C(E, M), d_\infty)$ is a complete metric space as well.

Definition 3.1 *Let (E, T) be a compact topological space and (M, d) be a metric space. A set $H \subset \mathbb{C}((E, T), (M, d))$ of continuous functions is* **pointwise totally bounded** *if for each $e \in E$, the set $H(e) = \{h(e) \mid h \in H\}$ is totally bounded.*

The set H is **equicontinuous** *if*

$$\forall e \in E, \epsilon > 0\ \exists U_e \in T, e \in U_e \forall h \in H : h(U_e) \subset \mathrm{B}_\epsilon(h(e)) \ .$$

Recall that a metric space (M, d) is **totally bounded** if

$$\forall r > 0 \exists F \subset M \text{ finite } : M = \bigcup_{p \in F} \mathrm{B}_r(p) \ .$$

Theorem 3.2 (Arzela-Ascoli) *Let (E, T) be a compact topological space and (M, d) be a metric space. A subset $H \subset C(E, M)$ is totally bounded if and only if*

1. *H is pointwise totally bounded and*
2. *H is equicontinuous.*

A subset $A \subset X$ of a topological space X is **precompact** (also **relatively compact**) if its closure is compact. If the X is a complete metric space then a subset of X is precompact if and only if it is totally bounded.

If H consists of differentiable functions $\mathbb{R}^n \overset{\text{open}}{\supset} U \subset \overline{U} \xrightarrow{h} \mathbb{R}^k$ and there is a common bound on the functions in H as well as one on their derivatives, then H is pointwise (totally) bounded and equicontinuous.

We will use the following immediate consequence of the Arzela Ascoli-Theorem in this case:

Corollary 3.3 *Let $H \subset C^1(\overline{\mathrm{B}_1^{\mathbb{R}^n}(0)}, \mathbb{R}^k)$ be a set of continuously differentiable functions*

$$h\colon \overline{\mathrm{B}_1^{\mathbb{R}^n}(0)} = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\} \to \mathbb{R}^k \ .$$

If H and $dH = \{dh \mid h \in H\}$ are locally bounded, then H is pointwise totally bounded and equicontinuous. In particular, any sequence $(h_n)_{n\in\mathbb{N}} \in H^{\mathbb{N}}$ has a uniformly convergent subsequence.

Note that the subsequence need not converge to a differentiable function. Since all norms on a finitely dimensional real vector space are equivalent, boundedness of dH does not depend on the norm we choose to specify this on $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^k)$.

Proof: The differential of a function $h\colon \overline{\mathrm{B}_1^{\mathbb{R}^n}(0)} \to \mathbb{R}^k$ is the map

$$\mathrm{d}h\colon \overline{\mathrm{B}_1^{\mathbb{R}^n}(0)} \to \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^k) \quad , \quad p \mapsto \mathrm{d}_p h$$

so that for all $p, q \in \overline{\mathrm{B}_1^{\mathbb{R}^n}(0)}$,

$$h(q) = h(p) + \mathrm{d}_p h(q - p) + R(q, p)$$

with a function R so that

$$\lim_{q \to p} \frac{R(q,p)}{|q - p|} = 0 \ .$$

The linear function $\mathrm{d}_p h$ is uniquely determined by this, even if $p \in \partial\overline{\mathrm{B}_1^{\mathbb{R}^n}(0)}$.

We now assume H and dH locally bounded, and let $e \in \overline{\mathrm{B}_1^{\mathbb{R}^n}(0)}$. Then there is $U \overset{\text{open}}{\subset} \overline{\mathrm{B}_1^{\mathbb{R}^n}(0)}$ and $C \in \mathbb{R}$ so that

$$\forall x \in U, h \in H : \|h(x)\| < C, \ \|\mathrm{d}_x h\|_{\mathrm{op}} < C \ .$$

Thus $H(e) \subset \mathrm{B}_C^{\mathbb{R}^k}(0)$ is bounded and by the Heine-Borel theorem totally bounded. For equicontinuity, we estimate

$$\|h(x) - h(e)\| \leq \max_{u \in U} \|\mathrm{d}_u h\|_{\mathrm{op}} \|x - e\| \leq C \operatorname{diam} U \leq 2C \quad \text{hence} \quad h(U) \subset \mathrm{B}_{2C}^{\mathbb{R}^k}(h(e))$$

for all $h \in H$ and $x \in U$. •

3.2 Compact Convergence and Normal Families

The topology of uniform convergence, i.e. the metric topology of the (almost) distance d_∞ has too many open sets, and thus too few convergent sequences for most of the sequel. Thus, for instance a power series almost never converges uniformly on its ball of convergence. We therefore will almost exclusively work with a "weaker" topology, that can be obtained by requiring convergence to be uniform only on compact sets. To define this, for $f \in C(E, M)$, $K \subset E$ compact and $r > 0$, let

$$\mathrm{B}_{K,r}(f) := \{h \in C(E, M) \,|\, \forall k \in K : \mathrm{d}(h(k), f(k) < r\}$$

and define the topology of **compact convergence** to be the smallest topology on M^E in which all the sets $\mathrm{B}_{K,r}(f)$ are open.

Definition 3.4 *Let (E, T) be a topological space and (M, d) be a metric space. A sequence $(f_n)_{n\in\mathbb{N}} \in \left(M^E\right)^{\mathbb{N}}$* **converges compactly** *if for every compact $K \subset E$ the sequence $(f_n|_K)_{n\in\mathbb{N}} \in \left(M^K\right)^{\mathbb{N}}$ of the restrictions converges uniformly: For $(f_n)_{n\in\mathbb{N}} \in (M^E)^{\mathbb{N}}$,*

$$f_n \xrightarrow{n\to\infty} f \text{ compact} \iff \forall K \subset E \text{ compact}, \epsilon > 0 \exists n_{K,\epsilon} \forall n > n_{K,\epsilon} \forall k \in K : \mathrm{d}(f_n(k), f(k)) < \epsilon\ .$$

A pre-compact subset subset $H \subset M^E$ is also called **normal**. *Thus if M is complete, a subset $H \subset M^E$ is normal if every sequence $(h_n)_{n\in\mathbb{N}} \in H^{\mathbb{N}}$ contains a compactly convergent subsequence (with limit not necessarily in H).*

Power series converge compactly on their open disc of convergence: If r is the radius of convergence of a power series $p = \sum_{k=0}^{\infty} a_n(z-p)^k$, then the series converges uniformly on $\mathrm{B}_\rho(p)$ whenever $\rho < r$. If $K \subset E = \mathrm{B}_r(p)$ is compact then $K \subset \mathrm{B}_{\rho_K}(p)$ for some $\rho_K < r$. Thus the series converges compactly.

Theorem 3.5 *Let $E \subset \mathbb{C}$ be a domain. Then $\mathcal{O}(E) \subset C(E, C)$ is closed (with respect to the topology of compact convergence).*

Proof: By Morera's Theorem 11.19, a continuous functions $f : E \to \mathbb{C}$ is holomorphic if $J\gamma(f) := \int_\gamma f(z)\, dz = 0$ for all boundary curves γ of triangles in E. Since these curves are compact Since

$$|J_\gamma(f) - J_\gamma(h)| \leq \textbf{length}\,(\gamma)\mathrm{d}_{K,\infty}(f, h)$$

the maps $J_\gamma : C(E, \mathbb{C}) \to \mathbb{C}$ are continuous. In particular, the sets $J_\gamma^{-1}(0)$ are closed. Thus

$$\mathcal{O}(E) = \bigcap_\gamma J_\gamma^{-1}(0)$$

is the intersection of closed sets. •

Since every point in $\mathbb{C}$ has a compact neighbourhood, compact convergence for functions on domains in $\mathbb{C}$ is the same as locally uniform convergence.

A sequence $(f_n)_{n\in\mathbb{N}} \in (M^E)^{\mathbb{N}}$ converges locally uniformly, if every point $e \in E$ has a neighourhood U_e so that the sequence of the restrictions $(f_n|_{U_e})_{n\in\mathbb{N}} \in (M^{U_e})^{\mathbb{N}}$ converges uniformly on U_e.

Theorem 3.5 says that a locally uniform limit of holomorphic functions is holomorphic.

A subset $H \subset M^E$ is **locally bounded** if

$$\forall e \in E \; \exists U \in T, e \in U, m \in M, r \in \mathbb{R}^+ : H(U) \subset \mathrm{B}_r(m) \; ,$$

where $H(U) = \{h(u) \,|\, u \in U, h \in H\}$.

Theorem 3.6 (Montel) *Let $G \subset \mathbb{C}$ be a domain and let $H \subset \mathcal{O}(G)$ be locally bounded. Then H is normal.*

Proof: For $e \in G$, let $e \in U \overset{\text{open}}{\subset} G$ and $C \in \mathbb{R}$ be so that $H(U) \subset \mathrm{B}_C(0)$. Choose $r > 0$ so that $B_{3r} \subset U$. Then for $q \in U$, $h \in H$ we have $h(q) \in \mathrm{B}_C(0)$, hence H is pointwise bounded, and by the Cauchy Integral Formula, for $q \in \mathrm{B}_r(e)$, we have

$$|h'(q)| = \left| \frac{1}{2\pi i} \oint_{|z-p|=2r} \frac{h(z)}{(z-q)^2} \; dz \right| \le \frac{1}{2\pi} 2\pi 2r C \frac{1}{r^2} = \frac{2C}{r}$$

because $|q-p| < r$, $|z-p| = 2r$ imply $|z-q| > r$, hence $\frac{1}{|z-q|^2} < r^2$. This shows that the set of the derivatives of the functions in H is also locally bounded. By the corrollary 3.3 to the Arzela-Ascoli Theorem, H is normal. •

Theorem 3.7 (Compact Convergence of the Derivative) *Let $G \subset \mathbb{C}$ be a domain and let $(h_n)_{n\in\mathbb{N}} \in \mathcal{O}(G)^{\mathbb{N}}$ converge locally uniformly to h. Then $h \in \mathcal{O}(G)$. Show that the sequence of derivatives $(h'_n)_{n\in\mathbb{N}} \in \mathcal{O}(G)^{\mathbb{N}}$ converges locally uniformly to h'.*

Proof: This follows from the Cauchy integral theorem. Since $(h_n)_{n\in\mathbb{N}}$ converges locally uniformly to h, every point $e \in G$ has a neighbourhood, wlog $\mathrm{B}_{3r}(e)$, so that $(h_n)_{n\in\mathbb{N}}$ converges uniformly on $\mathrm{B}_{3r}(e)$. Thus for every $\epsilon > 0$ there is $N_\epsilon \in \mathbb{N}$, so that

$$\forall n > N_\epsilon, x \in \mathrm{B}_{3r}(e) : |h_n(x) - h(x)| < \epsilon \; .$$

The Cauchy integral formula now gives an estimate for the derivatives at q with $|q-e| < r$,

$$\begin{aligned} |h'_n(q) - h'(q)| &= \left| \frac{1}{2\pi i} \oint_{|z-e|=2r} \frac{h_n(z) - h(z)}{(z-q)^2} \; dz \right| \\ &< \frac{1}{2\pi} 2\pi r \frac{\epsilon}{r^2} = \frac{\epsilon}{r} \end{aligned}$$

Thus $(h'_n)_{n\in\mathbb{N}}$ converges uniformly on $\mathrm{B}_r(e)$. •

4 The Riemann Mapping Theorem

4.1 Holomorphic maps of the unit disc

Theorem 4.1 (Schwarz Lemma) *Let $f \colon \mathrm{B}_1(0) \to \mathrm{B}_1(0)$ be holomorphic with $f(0) = 0$. Then $|f'(0)| \le 1$ and for all $z \in \mathrm{B}_1(0)$ we have $|f(z)| \le |z|$.*

If $|f'(0)| = 1$ *or if there is* $p \in \mathrm{B}_1(0) \setminus \{0\}$ *with* $|f(p)| = |p|$*, then there is* $m \in \mathrm{S}^1$ *(*$m = e^i\theta$ *for some* $\theta \in \mathbb{R}$*) so that for all* z *we have* $f(z) = mz$*. Thus* f *is rotation by the angle* θ*.*

Proof: By Theorem 11.5 on the Taylor series of a holomorphic function, f is given by a power series converging on all of $\mathrm{B}_0(0)$ and starting with z, because $f(0) = 0$. Therefore $f(z) = zg(z)$ with $g \in \mathcal{O}(\mathrm{B}_1(0))$. In particular we have

$$|g(z)| < \frac{1}{|z|} \quad \text{for all} \quad z \in \mathrm{B}_1(0)\ .$$

By the Maximum Principle 11.18, for any r, $0 < r < 1$ the function $|g|$ assumes its maximum on $\overline{\mathrm{B}}_r(0)$ on the boundary, hence

$$\forall 0 < r < 1, |z| < r : |g(z)| \le \max_{|u|=r} |g(u)| \le \frac{1}{r}\ .$$

But this implies

$$\forall z \in \mathrm{B}_1(0) : |g(z)| \le 1 \text{ hence } |f(z)| \le |z|\ .$$

If $|f(p)| = |p|$ for some $p \in \mathcal{B}_1(0) \setminus \{0\}$ or if $p = 0$ and $1 = |f'(0)| = |g(0)|$, then $|g(p)| = 1$ and p is a maximum of $|g|$, which must be on the boundary of $\mathrm{B}_1(0)$ if g is not constant. Hence g is constant. •

Corollary 4.2 *If* $f\colon \mathcal{H} \to \mathcal{H}$ *is biholomorphic, then*

$$f(z) = \frac{az+b}{cz+d}$$

for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}^+(2,\mathbb{R})$*.*

Proof: The **Cayley transform** is the biholomorphic map

$$c\colon \mathcal{H} \to \mathrm{B}_1(0) \quad , \quad c(z) = \frac{z-i}{z+i} \quad , \quad c(i) = 0,\ c(0) = -1,\ c(\infty) = 1\ . \tag{4.3}$$

It inverse is given by

$$c^{-1}(y) = \frac{iy+i}{-y+1}\ .$$

By the Schwarz Lemma, every biholomorphic map of the disc fixing 0 is a rotation, $R_t\colon z \mapsto e^{it}z$, $t \in \mathbb{R}$. Therefore a biholomorhic map of the upper half plane fixing i must be of the form

$$c^{-1} \circ R_t \circ c = \mu\left[\begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix}\begin{pmatrix} e^{it} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\right] = \mu\left[\begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix}\begin{pmatrix} e^{it} & -ie^{it} \\ 1 & i \end{pmatrix}\right]$$

$$= \mu\begin{pmatrix} i(e^{it}+1) & e^{it}-1 \\ -e^{it}+1 & i(e^{it}+1) \end{pmatrix} = \mu\left[\frac{1}{2ie^{it/2}}\begin{pmatrix} i(e^{it}+1) & e^{it}-1 \\ -e^{it}+1 & i(e^{it}+1) \end{pmatrix}\right]$$

$$= \mu \underbrace{\begin{pmatrix} \cos(t/2) & \sin(t/2) \\ -\sin(t/2) & \cos(t/2) \end{pmatrix}}_{\in \mathrm{GL}^+(2,\mathbb{R})}$$

where μA denotes the Möbius transform of the matrix A. •

Lemma 4.4 (Transitivity of Möbius transformations) *For $p \in \mathbb{E}$ let $w_p \in \mathcal{O}(\mathbb{E})$ be the function with*

$$w_p(z) = \frac{z - p}{-\overline{p}z + 1} \ . \tag{4.5}$$

Then w_p is a biholomorhic map $\mathbb{E} \to \mathbb{E}$, $w_p(p) = 0$, $w_p(0) = -p$.

Proof: The function w_p defines a biholomorphic map of the Riemann sphere, its inverse is given by

$$w_p^{-1} = w_{-p} \quad , \quad w_p^{-1}(y) = \frac{y + p}{\overline{p}y + 1} \ .$$

We only need to check $w_p(\mathbb{E}) \subset \mathbb{E}$, i.e.

$$|z - p| < |1 - \overline{p}z|$$

$$|z|^2 + |p|^2 - z\overline{p} - \overline{z}p < 1 + |p|^2 |z|^2 - \overline{p}z - p\overline{z} \ .$$

•

Some simple domains can be mapped to $\mathbb{E}$ via Möbius transforms and powers. For instance the quarter disc is biholomorphically equivalent to the disc,

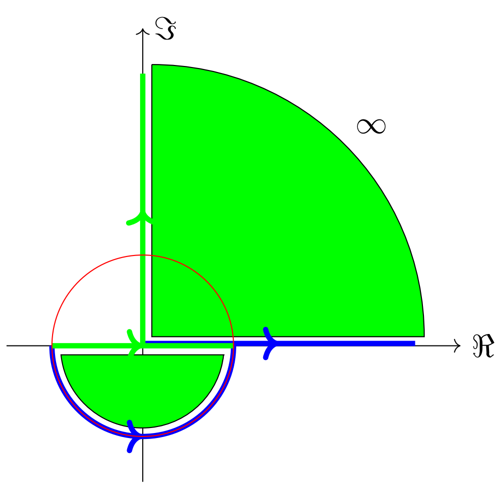
$$Q = \{z \in \mathbb{C} \,\big|\, |z| < 1,\ \Re(z) > 0,\ \Im(z) > 0\} = \left\{re^{it} \,\Big|\, 0 < r < 1,\ 0 < t < \frac{\pi}{2}\right\} \cong \mathbb{E} \ .$$

To construct such a map, recall the Cayley transform (4.3)

$$c \colon \mathcal{H} \to \mathbb{E} \quad , \quad z \mapsto \frac{z - i}{z + i} \ .$$

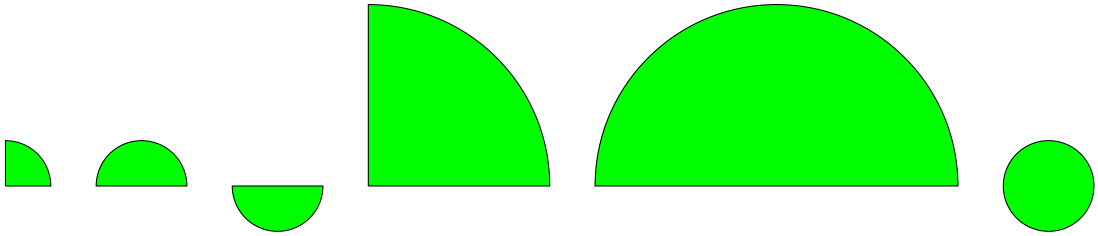
This maps

$$i \mapsto 0 \mapsto -1 \mapsto i \quad , \quad 1 \mapsto -i \quad , \quad \infty \mapsto 1$$



An explicit biholomorphic map $Q \to \mathbb{E}$ is the composition

$$Q \xrightarrow{\text{square}} \mathcal{H} \cap \mathbb{E} \xrightarrow{z \mapsto -z} -\mathcal{H} \cap \mathbb{E} \xrightarrow{c^{-1}} \mathcal{H} \cap -i\mathcal{H} \xrightarrow{\text{square}} \mathcal{H} \xrightarrow{c} \mathbb{E} \ .$$



4.2 The Riemann Mapping Theorem

Theorem 4.6 *Let $\emptyset \neq G \underset{\neq}{\subset} \mathbb{C}$ be a simply connected domain different from $\mathbb{C}$. Then there is a biholomorphic map*

$$\phi\colon G \to \mathbb{E} = \mathrm{B}_1(0) \ .$$

Proof: The proof proceeds in three steps.

1. *There is an injective holomorphic map $\phi\colon G \to \mathbb{E}$.* Such a map ϕ is then biholomorphic onto its image.

$$\phi \xrightarrow{\cong} \phi(G) \subset \mathbb{E} \ .$$

Let $p \in \mathbb{C} \setminus G$. Since G is simply connected, there is a function $g \in \mathcal{O}(G)$ so that $g(z)^2 = z - p$, thus "$z - p$ has a square root on G". The function g is injective because g has a left inverse. g is not constant, hence open. If $x \in g(G)$, then $x \neq 0$ and $-x \notin g(G)$. Also, for some $r > 0$, $\mathrm{B}_r(x) \subset g(G)$, becaues g is open. But then

$$\mathrm{B}_r(-x) \cap g(G) = \emptyset \ .$$

The map

$$\phi\colon G \to \mathbb{E} \quad , \quad \phi(z) = \frac{r}{g(z) + x}$$

is biholomorphic $G \cong \mathbb{E}$ and By composing ϕ with a Moebius transformation $w_{\phi(x)}$ as defined in (4.5) we can get 0 in the image. Thus the map

$$w_{\phi(x)} \circ \phi\colon G \to \mathbb{E}$$

is biholomorhic and $0 \in w_{\phi(x)} \circ \phi(G)$. Thus for the remainder of the proof we may assume

$$0 \in G \subset \mathbb{E} \ .$$

2. *Surjectivity and maximal derivative at 0:* Let $0 \in G \subset \mathbb{E}$, $f\colon G \to \mathbb{E}$ holomorphic, $f(0) = 0$, injective, **not surjective**. Then there is an injective holomorphic function $F\colon G \to \mathbb{E}$ with $F(0) = 0$ and

$$|F'(0)| > |f'(0)| \ .$$

To see this, let $q \in \mathbb{E} \setminus f(G)$ and let w_q be the Moebius transformation defined in (4.5). Then $0 \notin w_q(f(G))$. Since $w_q(f(G))$ is simply connected, we have a holomorphic square root

$$\mathrm{sqrt}\colon w_q(f(G)) \to \mathbb{E} \quad , \quad (\mathrm{sqrt}(w_q(f(z)))^2 = w_q(f(z)) \quad \text{for all} \quad z \in G \ .$$

Abbreviate $y = \text{sqrt}(w_q(f(0))) = \text{sqrt}(w_q(0)) = \text{sqrt}(-q)$ and let h and F be the compositions

$$h = w_y \circ \text{sqrt} \circ w_q \colon f(G) \to \mathbb{E} \ ,$$
$$F = h \circ f \colon G \to \mathbb{E} \ .$$

Both are biholomorphic and $h(0) = 0$. The inverse of h is

$$h^{-1}(z) = w_q^{-1}((w_y^{-1}(z))^2)$$

and therefore extends holomorphically to a map

$$h^{-1}(\mathbb{E}) \to \mathbb{E} \ .$$

This is not biholomorphic, since the Moebius transformations are but the square is not. In particular, h^{-1} is not a rotation. By the Schwarz Lemma we must have $(h^{-1})'(0) < 1$, hence $h'(0) > 1$ and

$$|F'(0)| = |h'(0)|\,|f'(0)| > |f'(0)| \ .$$

3. *Maximizers for the derivative at 0 exist and are biholomorphic:* Among the injective functions $f \colon G \to E$ with $f(0) = 0$ there is one with maximal $|f'(0)|$ and this is biholomorhic.

   Crucial for this step is the functional

$$\mathcal{O}\,(G, \mathbb{E}, \text{inj}) := \{f \in \mathcal{O}\,(G, E) \mid f(0) = 0, f \text{ injective}\} \xrightarrow[E]{f \mapsto |f'(0)|} \mathbb{R} \ .$$

   By the Cauchy Integral formula, E is bounded, because for $f \in \mathcal{O}\,(G, \mathbb{E}, \text{inj})$ we can estimate

$$|f'(0)| = \left| \frac{1}{2\pi i} \oint_{|z|=r} \frac{f(z)}{z^2}\, dz \right| \leq \frac{1}{r^2} \quad \text{if} \quad \mathrm{B}_r(0) \subset G \ .$$

   Thus $\sup_{f \in \mathcal{O}(G,\mathbb{E},\text{inj})} |f'(0)| = s < \infty$. Since $\mathcal{O}\,(G, \mathbb{E}, \text{inj})$ is locally bounded, it is normal by Montel's Theorem. Thus there is a compactly convergent sequence

$$(f_n)_{n \in \mathbb{N}} \in \mathcal{O}\,(G, \mathbb{E}, \text{inj})^{\mathbb{N}} \quad , \quad \lim_{n \to \infty} f_n =: f \quad , \quad \lim_{n \to \infty} |f_n'(0)| = |f'(0)| = s \ .$$

   By step 2, the function f is surjective. If f were not injective, then there would be two points $x, y \in G$ so that $f(x) = f(y) =: w$. The zeros of the function $f(z) - w$ must be isolated since f can not be constant. We can therefore find a closed continuous path γ in G avoiding all zeros of $f(z) - w$ surrounding both x, y once. Since $f_n - w$ converges uniformly on γ to $f - w$ there is $N \in \mathbb{N}$ so that for all t,

$$|f(\gamma(t)) - f_N(\gamma(t))| < \min_t |f(\gamma(t)) - w| \ .$$

   By Rouché's Theorem, f and f_N have the same number of zeros surrounded by γ, a contradiction.

•

At the end of the proof of the Riemann Mapping Theorem we used a special case of lower semicontinuity of the number of leaves, i.e. the multiplicity, i.e. of the following Theorem.

Theorem 4.7 *Let $G \subset \mathbb{C}$ be a domain and $(f_n)_{n\in\mathbb{N}} \in \mathcal{O}(G)^{\mathbb{N}}$, $f_n \xrightarrow[compact]{n\to\infty} f$. Let $a \in \mathbb{C}$, $M \in \mathbb{N}$ be so that*

$$\forall n \in \mathbb{N} : m(f_n, a) \leq M \ .$$

Then

$$m(f, a) \leq M \ \text{or} \ f = a \ .$$

Proof: Assume that f is not constant and $m(f, a) > M$. Then $f^{-1}(a) \subset G$ is discrete. There is a finite subset $P \subset f^{-1}(a)$ and $r > 0$ so that

$$\forall p \in P : \{p\} = \mathrm{B}_r(p) \cap f^{-1}(a), \ \ P \cap \mathcal{S}_r(p) = \emptyset \quad \text{and} \quad \sum_{p\in P} \operatorname{ord}\left(f(z) - a; z = p\right) > M$$

The union $\mathcal{S} = \bigcup_{p\in P} \mathcal{S}_r(p)$ of the boundary circles is compact and $f(z) - a$ is nowhere zero there. Thus

$$\min_{z\in\mathcal{S}} |f(z) - a| > 0$$

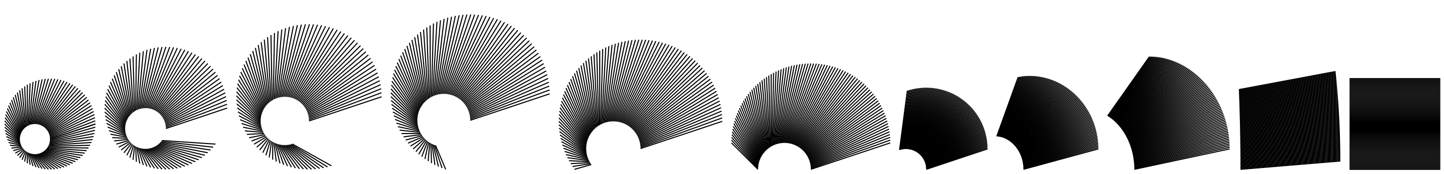
exists. Since $f_n \xrightarrow[\text{compact}]{n\to\infty} f$ there is $N \in \mathbb{N}$ so that

$$\forall z \in \mathcal{S} : |f_N(z) - f(z)| < \min_{\zeta\in\mathcal{S}} |f(\zeta) - a|$$

By Rouché's Theorem, $f - a$ and $f_N - a$ have the same number of zeros in $\mathrm{B} = \bigcup_{p\in P} \mathrm{B}_r(p)$, counting multiplicities, i.e.

$$M = m(f_N, a) = \sum_{q\in G} \operatorname{ord}\left(f_N(z) - a; z = q\right) \geq \sum_{q\in \mathrm{B}} \operatorname{ord}\left(f_N(z) - a; z = q\right) = \sum_{q\in \mathrm{B}} \operatorname{ord}\left(f(z) - a; z = q\right) > M$$

a contradiction. •



Complex Analysis I - MT333P

5 Homework

1. Compute real and imaginary part of all $z \in \mathbb{C}$ with $z^2 = -8 + 6i$.

   Solution: We need to compute $a, b \in \mathbb{R}$ so that

$$(a+ib)^2 = a^2 - b^2 + 2iab = -8 + 6i \ ,$$

$$a^2 - b^2 = -8 \quad \text{and} \quad ab = 3 \ ,$$

$$a^2 - \frac{9}{a^2} = -8 \ ,$$

   hence $a^2 = 1$ (the other solution, $a^2 = -9$ does not give $a \in \mathbb{R}$). Thus the square roots of $-8 + 6i$ are

$$1 + 3i \quad \text{and} \quad -1 - 3i$$

2. Compute real and imaginary part of

$$\sum_{k=0}^{123} (1+i)^k \ .$$

   Solution: The key to this is that

$$1 + i = \sqrt{2}\epsilon \quad \text{where} \quad \epsilon^4 = -1 \ .$$

   together with the formula for the geometric series

$$\sum_{j=0}^{k} q^j = \frac{1 - q^{k+1}}{1 - q} \quad \text{if} \quad q \neq 1 \ .$$

   Thus

$$\sum_{k=0}^{123} (1+i)^k = \sum_{k=0}^{123} (\sqrt{2}\epsilon)^k = \frac{1 - (\sqrt{2}\epsilon)^{124}}{1 - \sqrt{2}\epsilon} = \frac{1 - 2^{62}(-1)^{31}}{-i} = (2^{62} + 1)i \ .$$

   Thus the real part is 0 and the imaginary part is $2^{62} + 1$.

3. Recall the **standard scalar product** on $\mathbb{R}^n$: For $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ we define

$$\langle x \mid y \rangle = \sum_{i=1}^{n} x_i y_i \ .$$

   The euclidean norm on $\mathbb{R}^n$ is given by

$$\|x\| = \sqrt{\langle x \mid x \rangle}$$

   and the cosine of the **angle between** x **and** y, $x, y \neq 0$, is $\angle(x, y) \in [0, \pi]$,

$$\cos \angle(x, y) := \frac{\langle x \mid y \rangle}{\|x\| \, \|y\|} \ .$$

This is well defined because by the Cauchy-Schwarz inequality, $|\langle x \mid y\rangle| \leq \|x\| \, \|y\|$, and $[0, \pi] \xrightarrow{\cos} [-1, 1]$ is bijective .

Determine all matrices $A \in \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}\,(2, \mathbb{R})$ preserving angles, i.e. so that for all $x, y \in \mathbb{R}^2$ we have

$$\angle(Ax, Ay) = \angle(x, y) \tag{5.1}$$

whenever both sides are defined (i.e. $x \neq 0 \neq y$, $Ax \neq 0 \neq Ay$).

Solution: Since the cosine is injective in the range referred to in the definition of the angle, (5.1) means that

$$\langle Ax \mid Ay\rangle = \frac{\langle x \mid y\rangle \, \|Ax\| \, \|Ay\|}{\|x\| \, \|y\|} \ . \tag{5.2}$$

Thus if $\{b_1, \ldots, b_n\}$ is an orthonormal basis for $\mathbb{R}^n$, then the vectors $Ab_1, \ldots, Ab_n$ are pairwise orthogonal. Thus there is an orthonormal basis $\{c_1, \ldots, c_n\}$ and $\lambda_i \in \mathbb{R}$ so that

$$Ab_i = \lambda_i c_i \quad , \quad i = 1, \ldots, n \ .$$

Let $T \in \mathrm{O}\,(n)$ be the orthogonal transformation of $\mathbb{R}^n$ with $Tc_i = b_i$ for all i. Then

$$TAb_i = \lambda_i b_i \quad , \quad i = 1, \ldots, n \ .$$

We may also assume that $\lambda_i \geq 0$ because the sign of λ_i can also be absorbed in the isometry T. We now show that $\lambda_i = \lambda_j$ for all $i, j \in \{1, \ldots, n\}$. To this end we only need to look at one other angle, for instance the one between $x = b_i + b_j$ and $y = b_i - b_j$. Clearly x, y are perpendicular and both have norm $\sqrt{2}$. Hence from (5.2),

$$0 = \langle Ax \mid Ay\rangle = \langle \lambda_i b_i + \lambda_j b_j \mid \lambda_i b_i - \lambda_j b_j\rangle = \lambda_i^2 - \lambda_j^2 \quad \text{hence} \quad \lambda_i = \lambda_j \ .$$

We thus have shown that for some $\lambda \in \mathbb{R}_0^+$ we have

$$TA = \lambda \mathrm{id}_{\mathbb{R}^n} \quad \text{hence} \quad A = \lambda T^{-1} \ .$$

In the special case $n = 2$, this gives

$$A = \lambda \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \quad \text{or} \quad A = \lambda \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ \sin(\alpha) & -\cos(\alpha) \end{pmatrix} \ .$$

because every orthogonal (2×2)-matrix is of the form

$$\begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ \sin(\alpha) & -\cos(\alpha) \end{pmatrix} \ .$$

depending on whether it preservers or reverses orientation. Thus the conformal linear automorphisms of $\mathbb{R}^2$ are of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \quad , \quad a, b \in \mathbb{R}, (a, b) \neq (0, 0) \ .$$

4. Sketch the subsets of $\mathbb{C} = \mathbb{R}^2$ given below.

   (a) $X_a = \{z \in \mathbb{C} \mid \Re(z) + 1 = |z|\}$
   Solution: In cartesian coordinates, this is the set $\{(x, y) \mid y^2 = (x + 1)^2 - x^2\} = \{(x, y) \mid y^2 = 2x + 1\}$, a parabola.

(b) $X_b = \left\{z \in \mathbb{C} \,\middle|\, |z|^2 = \Im(z)\right\}$
Solution: In cartesian coordinates, this is the set $\{(x,y) \,|\, y^2 - y + x^2 = 0\} = \left\{(x,y) \,\middle|\, \left(y-\frac{1}{2}\right)^2 - \frac{1}{4} + x^2 = 0\right\}$, a circle.

(c) $X_c = \{z \in \mathbb{C} \,|\, |z-i| + |z+i| = 4\Im(z)\}$

5. Stereographic projection is the map

$$\mathcal{S}^2 = \left\{(x,y,z) \in \mathbb{R}^3 \,\middle|\, x^2+y^2+z^2 = 1\right\} \xrightarrow{\phi} \mathbb{R}^2 \cup \{\infty\} = \mathbb{C} \cup \{\infty\} =: \hat{\mathbb{C}}$$

which takes $(0,0,1) \in \mathrm{S}^2$ to ∞ and so that for $p \in \mathrm{S}^2 \setminus \{(0,0,1)\}$ the points

$$(0,0,1) \quad , \quad p \quad , \quad (\phi(p), 0)$$

lie on a common line in $\mathbb{R}^3$. Thus $\phi(x,y,0) = (x,y)$, for instance. Stereographic projection is bijective, you can compute explicit formulas for ϕ and ϕ^{-1}.
Let $\mathrm{inv}\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the involution of $\hat{\mathbb{C}}$ mapping

$$z \leftrightarrow \frac{1}{z} \quad \text{if} \quad z \neq 0, \infty \quad \text{and} \quad 0 \leftrightarrow \infty \;.$$

Find the conjugate $\phi^{-1} \circ \mathrm{inv} \circ \phi$, i.e. for $x,y,z \in \mathrm{S}^2$, what is

$$\phi^{-1}(\mathrm{inv}(\phi(x,y,z))) \;?$$

Solution:

$$\begin{aligned}
\phi(x,y,t) &= \left(\frac{x}{1-t}, \frac{y}{1-t}\right) \;, \\
\phi(z,t) &= \frac{z}{1-t} \quad \text{if we identify} \quad \mathbb{C} \times \mathbb{R} \cong \mathbb{R}^3 \;,\; ((x+iy),t) = (x,y,t), \\
\mathrm{inv}(u+iv) &= \frac{u-iv}{u^2+v^2} \\
\phi^{-1}(u,v) &= \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right) \\
\phi^{-1}(z) &= N(2z, |z|^2 - 1) \;,
\end{aligned}$$

where we denote by N the map $\mathbb{R}^3 \setminus \{0\} \to \mathrm{S}^2$, $N(v) = \frac{x}{\|v\|}$. For $(x,y,t) = (z,t) \in \mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$, we get

$$\phi^{-1}(\phi(x,y,t)^{-1}) =$$

$$\begin{aligned}
\phi^{-1}(\phi(z,t)^{-1}) = \phi^{-1}\left(\frac{\overline{z}(1-t)}{|z|^2}\right) &= N\left(2\frac{\overline{z}(1-t)}{|z|^2}, \left|\frac{\overline{z}(1-t)}{|z|^2}\right|^2 - 1\right) \\
&= N\left(\overline{z}, \frac{\left|\frac{\overline{z}(1-t)}{|z|^2}\right|^2 - 1}{2\frac{1-t}{|z|^2}}\right) = N\left(\overline{z}, \frac{|z|^2(1-t) - |z|^2}{2|z|^2}\right) \\
&= N\left(\overline{z}, \frac{(1-t)}{2} - \frac{|z|^2}{2(1-t)}\right) \\
&= N\left(\overline{z}, \frac{(1-t)}{2} - \frac{1-t^2}{2(1-t)}\right) \quad \text{because} \quad 1 = |z|^2 + t^2 \\
&= N\left(\overline{z}, \frac{(1-t)}{2} - \frac{1+t}{2}\right) \\
&= N(\overline{z}, -t) = (\overline{z}, -t) \\
&= (x, -y, -t) \ .
\end{aligned}$$

Thus inversion corresponds to reflection at the real line,

$$\phi^{-1} \circ \operatorname{inv} \circ \phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in \mathrm{SO}\,(3)$$

on the sphere.

6. Find a formula for $\sum_{n=0}^{\infty} nq^n$, for $|q| < 1$.

Solution: For $|z| < 1$, if we set

$$f(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

then

$$\begin{aligned}
f'(z) &= \sum_{n=1}^{\infty} nz^{n-1} \\
zf'(z) &= \sum_{n=0}^{\infty} nz^n
\end{aligned}$$

but this can also be computed directly,

$$zf'(z) = z\left(\frac{1}{1-z}\right)' = \frac{z}{(1-z)^2} \ .$$

Alternatively:

$$\sum_{n=0}^{\infty} nq^n = \sum_{k=1}^{\infty}\sum_{n=k}^{\infty} q^n = \sum_{k=1}^{\infty} q^k \sum_{n=0}^{\infty} q^n = \frac{q}{1-q}\frac{1}{1-q} \ .$$

7. Find a function $v \colon \mathbb{C} \to \mathbb{R}$ so that the function $f \colon \mathbb{C} \to \mathbb{C}$ given by

$$f(x+iy) = xy^2 - \frac{1}{3}x^3 + x^2 - y^2 + iv(x+iy)$$

for $x, y \in \mathbb{R}$ is complex differentiable.

Solution: By the Cauchy-Riemann equations we must have

$$y^2 - x^2 + 2x = v_y \quad \text{and} \quad 2xy - 2y = -v_x \ ,$$

which is satisfied by

$$v(x+iy) = \frac{1}{3}y^3 - x^2 y + 2xy \ .$$

8. What is the radius of convergence of

$$\sum_{n=0}^{\infty} \frac{z^{n!}}{n+1} \ ?$$

Solution: The kth coefficient of this series is $\frac{1}{n+1}$ if $k = n!$ for some $n \in \mathbb{N}_0$ and 0 otherwise. The formula for the radius of convergence ρ gives

$$\rho = \frac{1}{\limsup_{k\to\infty} \sqrt[k!]{\frac{1}{k+1}}} = 1 \ ,$$

because

$$1 \geq \sqrt[k!]{\frac{1}{k+1}} \geq \sqrt[k]{\frac{1}{k+1}} \xrightarrow{k\to\infty} 1$$

9. What is the radius of convergence of the series

$$\sum_{n=0}^{\infty} 2^{n^2} z^n \quad , \quad \sum_{n=0}^{\infty} \frac{z^{n^3}}{2^{n^2}} \ ?$$

Prove your result!

Hint: Trying some values for z may be easier than using Hadamard's formula for the radius of convergence.

Solution: The series $\sum_{n=0}^{\infty} 2^{n^2} z^n$ converges for no $z \neq 0$. To see this, assume $2^k < z \in \mathbb{R}^+$ for some $k \in \mathbb{Z}$. Then

$$2^{n^2} z^n > 2^{n^2+k} \xrightarrow{n\to\infty} \infty \ .$$

It follows that the radius of convergence of this series is 0.

The series $\sum_{n=0}^{\infty} \frac{z^{n^3}}{2^{n^2}}$ converges for $z = 1$. If $z \in \mathbb{R}$, $z > 1$ then there is $\epsilon > 0$ so that

$$1 < 2^{\epsilon} < z \ .$$

Hence

$$\frac{z^{n^3}}{2^{n^2}} > 2^{\epsilon n^3 - n^2} \xrightarrow{n\to\infty} \infty$$

and the series does not converge. It follows that the radius of convergence of this series is 1.

10. Find $\sin^{-1}(2)$, i.e. the set

$$\sin^{-1}(2) = \{z \in \mathbb{C} \mid \sin(z) = 2\} \ .$$

Solution: If z is in this set, then $4i = 2i\sin(z) = e^{iz} - e^{-iz}$, hence

$$\left(e^{iz}\right)^2 - 4ie^{iz} - 1 = 0$$

$$e^{iz} = 2i \pm \sqrt{-4+1} = (2 \pm \sqrt{3})i$$

11. Prove that three complex numbers a_1, a_2, a_3 are the vertices of a equilateral triangle if and only if

$$a_1^2 + a_2^2 + a_3^2 = a_1a_2 + a_2a_3 + a_3a_1 \ .$$

Solution: For the polynomial

$$p(z) = (z - a_1)(z - a_2)(z - a_3)$$

with roots the given a_i, the equation means that with $A = a_1 + a_2 + a_3$, $p(z)$ is of the form

$$p(z) = z^3 - Az^2 + \frac{A^2}{3}z + B$$

$$= \left(z - \frac{A}{3}\right)^3 + B + \frac{A^3}{27}$$

12. Let $f\colon \mathbb{C} \to \mathbb{C}$ be the function with $f(z) = \overline{z} + z + z^3$ and let $\gamma\colon [0,1]$ be the curve with $\gamma(t) = t^3 + it^2$. Compute $\int_\gamma f(z)\ dz$.

Hint: You can compute this directly or use that the function "almost" has an antiderivative.

Solution: Since the function $g(z) = z + z^3 = G'(z)$, $G(z) = \frac{z^2}{2} + \frac{z^4}{4}$ is holomorphic, we can compute its integral from the endpoints,

$$\int_\gamma g(z)\ dz = G(\gamma(1)) - G(\gamma(0)) = G(1+i) - G(0) = \frac{(1+i)^4}{4} + \frac{(1+i)^2}{2} = -1 + i \ .$$

It remains to compute

$$\int_\gamma \overline{z}\ dz = \int_0^1 \overline{\gamma(t)}\gamma'(t)\ dt = \int_0^1 (t^3 - it^2)(3t^2 + 2it)\ dt$$

$$= \int_0^1 3t^5 + 2t^3 - it^4\ dt = \frac{1}{2} + \frac{1}{2} - \frac{i}{5} \ .$$

Thus

$$\int_\gamma \overline{z} + z + z^3\ dz = \frac{4}{5}i$$

13. Let $\gamma\colon [-1,2] \to \mathbb{C}$ be the curve with

$$\gamma(t) = 2t + i\sin(\pi t) \ .$$

Compute

$$f(z) = \int_\gamma \frac{z}{(1 - z^2)^2}\ dz \ .$$

Solution: The function f is the derivative of

$$F(z) = \frac{1/2}{1-z^2} \ .$$

Hence

$$\int_\gamma f(z)\ dz = F(\gamma(2)) - F(\gamma(-1)) = F(4) - F(-2) = \frac{-1}{30} - \frac{-1}{6} = \frac{4}{30} = \frac{2}{15} \ .$$

14. Let f be complex differentiable at $p \in \mathbb{C}$. Show that the function g with

$$g(z) = \overline{f(\overline{z})}$$

is complex differentiable at $\overline{p}$.

Hint: $\overline{a+ib} = a - ib$, the complex conjugate

Solution:

$$\lim_{h\to 0} \frac{g(\overline{p}+h) - g(\overline{p})}{h} = \lim_{h\to 0} \frac{g(\overline{p+\overline{h}}) - g(\overline{p})}{h} = \lim_{h\to 0} \frac{\overline{f(p+\overline{h})} - \overline{f(p)}}{h} = \lim_{h\to 0} \frac{f(p+h) - f(p)}{h} \ .$$

15. Let $R > 0$ and let f be complex differentiable on B_R with $f(0) = 0$, $f'(0) \neq 0$ and $f(z) \neq 0$ for all $z \in \mathrm{B}_R \setminus \{0\}$. Prove that

$$\oint_{|z|=r} \frac{dz}{f(z)} = \frac{2\pi i}{f'(0)} \quad \text{for all} \quad r,\ 0 < r < R \ . \tag{5.3}$$

Hint: By "$\oint_{|z|=r} \ldots$" we mean "$\oint_\omega \ldots$" where $\omega\colon [0,1] \to \mathbb{C}$ is the curve $\omega(t) = re^{2\pi i t}$.

Apply the Cauchy Integral Theorem to the annulus $\{z \mid r_1 < |z| < r_2\}$ to show that the integral in (5.3) does not depend on r.

Solution: Since $f'(0) \neq 0$ there is $\epsilon > 0$ so that

$$f^{-1}(0) \cap \mathrm{B}_\epsilon = \{0\} \ .$$

Hence $\frac{1}{f(z)}$ is holomoprhic on $\mathrm{B}_\epsilon \setminus \mathrm{B}_r$ if $0 < r < \epsilon$, and therefore

$$\oint_{|z|=r} \frac{dz}{f(z)} = \oint_{|z|=\epsilon} \frac{dz}{f(z)}$$

for all r, $0 < r < \epsilon$. Now

$$f(z) = f'(0)z + R(z) \quad \text{for} \quad z \in \mathrm{B}_\epsilon$$

and $\lim_{z\to 0} R(z)z = 0$, hence

$$\frac{1}{f(z)} = \frac{1}{f'(0)z}$$

16. Extension of ln to $\mathbb{C} \setminus \mathbb{R}^-$. We define

$$\ln\colon \mathbb{C} \setminus \mathbb{R}^- = \{a + ib \mid a, b \in \mathbb{R}, b \neq 0 \text{ if } a < 0\} \to \mathbb{C}$$

by

$$\ln(z) = \int_{\gamma_z} \frac{1}{z}\ dz \tag{5.4}$$

where $\gamma_z\colon [0,1] \to \mathbb{C} \setminus \mathbb{R}^-$ is **any** C^1 curve with $\gamma_z(0) = 1$ and $\gamma_z(1) = z$. Show that the integral in (5.4) is the same for all such curves.

Solution: The map

$$\mathbb{R}^2 \to \mathbb{R}^2 \quad , \quad (x, y) \mapsto (x^2 - y^2, 2xy)$$

is C^1 and its restriction to the right half plane

$$\phi\colon \mathbb{R}^+ \times \mathbb{R} \to \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, y \neq 0 \text{ or } x > 0\right\}$$

is a diffeomorphism.

This is the map $\{z \in \mathbb{C} \,|\, \Re(z) > 0\} \xrightarrow{z \mapsto z^2} \mathbb{C} \setminus \mathbb{R}^-$. *The inverse of* ϕ *is*

$$\phi^{-1}\colon \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, y \neq 0 \textit{ or } x > 0\right\} \to \mathbb{R}^+ \times \mathbb{R}$$

$$\phi^{-1}(a, b) = \left(\sqrt{\frac{a}{2} + \sqrt{\frac{a^2 + b^2}{4}}}, \frac{b}{2\sqrt{\frac{a}{2} + \sqrt{\frac{a^2+b^2}{4}}}} \right)$$

Let $\gamma, \mu\colon [0,1] \to \mathbb{C} \setminus \mathbb{R}^-$ be C^1-curves with $\gamma(0) = \mu(0) = 1$ and $\gamma(1) = \mu(1) = z$.

Then

$$H(t, s) := \phi\left(s\phi^{-1}(\gamma(t)) + (1 - s)\phi^{-1}(\mu(t))\right) \in \mathbb{C} \setminus \mathbb{R}^-$$

for all $(t, s) \in [0,1] \times [0,1]$ and

$$H(0, s) = 1 \quad , \quad H(1, s) = z \quad , \quad H(t, 0) = \mu(t) \quad , \quad H(t, 1) = \gamma(t)\ .$$

Thus μ and γ are homotopic relative endpoint in $\mathbb{C} \setminus \mathbb{R}^-$. Since $\frac{1}{z}$ is holomorphic in $\mathbb{C} \setminus \mathbb{R}^-$, the complex line integrals over γ and μ coincide.

17. For $n \in \mathbb{Z}$, compute

$$\oint_{|z|=5} \frac{e^z - e^{-z}}{z^n}\, dz\ .$$

Hint: Recall that $\oint_{|z|=5} \ldots$ denotes the integral $\int_\mu \ldots$ where μ is the oriented boundary curve of B_5, $\mu\colon [0,1] \to \mathbb{C}$, $\mu(t) = 5e^{2\pi t}$.

Solution: The power series for the numerator of the integrand,

$$e^z - e^{-z} = 2\sinh(z) = 2\sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!}$$

hence the integrand is

$$2\sum_{k=0}^{\infty} \frac{z^{2k+1-n}}{(2k+1)!} = 2\sum_{k<\frac{n-1}{2}} \frac{z^{2k+1-n}}{(2k+1)!} + \underbrace{2\sum_{k\geq\frac{n-1}{2}} \frac{z^{2k+1-n}}{(2k+1)!}}_{\text{entire}}$$

The closed line integral over the entire part vanishes by the Cauchy integral theorem and so we are left with

$$\oint_{|z|=5} 2\sum_{k<\frac{n-1}{2}} \frac{z^{2k+1-n}}{(2k+1)!}\, dz = 2\sum_{k<\frac{n-1}{2}} \frac{1}{(2k+1)!} \oint_{|z|=5} z^{2k+1-n}\, dz = \begin{cases} 0 & \text{if } n \text{ odd} \\ \frac{4\pi i}{(n-1)!} & \text{if } n \text{ even} \end{cases}$$

because for all $q \in \mathbb{Z}$, $r > 0$, we have

$$\oint_{|z|=r} z^q \, dz = \begin{cases} 2\pi i \text{ if } q = -1 \\ 0 \quad \text{if } \text{ else} \end{cases}$$

18. (a) Find the power series expansion of $f(z) = \dfrac{e^z - 1}{z}$.

Hint: Start with

$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \cdots$$

This will also "define" $f(0)$.

Solution:

$$\frac{e^z - 1}{z} = 1 + \frac{z}{2} + \frac{z^2}{3!} + \cdots + \frac{z^{k-1}}{k!} + \cdots = \sum_{k=0}^{\infty} \frac{z^k}{(k+1)!} \tag{5.5}$$

(b) Compute the first four terms a_0, a_1, a_2, a_3 of the power series for

$$g(z) = \frac{1}{f(z)} = \frac{z}{e^z - 1} = \sum_{k=0}^{\infty} a_k z^k \ .$$

Show that for $n > 0$,

$$\sum_{k=0}^{n} \binom{n+1}{k} k! a_k = 0 \ .$$

Solution: We must have $g(z)\dfrac{e^z - 1}{z} = 1$ for all z. By (5.5) this leads to

$$1 = a_0 \quad , \quad 0 = \frac{1}{2}a_0 + a_1 \quad , \quad \ldots \quad ,$$

$$0 = \sum_{k=0}^{n} \frac{a_k}{(n-k+1)!} = \frac{1}{(n+1)!} \sum_{k=0}^{n} \binom{n+1}{k} k! a_k$$

19. Exploit the differential equation "$\tan' = 1 + \tan^2$" to derive a recursive formula for the coefficients of the power series for tan around 0.

Solution: If $\sum_{k=0}^{\infty} a_k z^k = \tan(z)$ is the Taylor series of tan around 0, then

$$\tan'(z) = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n \ .$$

$$\tan(z)^2 = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} a_j a_{n-j} \right) z^n$$

"Comparing coefficients", i.e. by uniqueness of the Taylor coefficients, this gives

$$(n+1) a_{n+1} = \sum_{j=0}^{n} a_j a_{n-j} \ .$$

20. Let $(a_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}_0}$ be so that for all $z \in \mathrm{B}_{1/237}$ we have that

$$\sum_{k=0}^{\infty} a_n z^n = \frac{1}{\cos((1+i)z)} \ .$$

Compute $\limsup_{n\to\infty} \sqrt[n]{|a_n|}$.

Hint: This is an attempted obfuscation. Do not compute the Taylor series of the right hand side!

Solution: The zeros of the denominator of the right hand side form the set

$$H = \left\{ \frac{\pi(2k+1)}{2(1+i)} \,\middle|\, k \in \mathbb{Z} \right\} \ .$$

The points closest to 0 in H are $\pm q := \pm\frac{\pi}{2(1+i)}$ and their distance from 0 is

$$r := \left| \frac{\pi}{2(1+i)} \right| = \frac{\pi}{2\sqrt{2}} \ .$$

By the corollary of the Cauchy integral theorem for the Taylor series of a holomorphic function, the series on the left hand side must be the Taylor series of the function f on the right hand side and its radius of convergence is at least r.

On the other hand, the radius of convergence can not be bigger than r because then f would extend holomorphically to q. But

$$\lim_{z\to q} f(z) = \lim_{z\to\pi/2} \frac{1}{\cos(z)} = \infty \ .$$

21. Let $p \in U \overset{\text{open}}{\subset} \mathbb{C}$ so that

$$\forall u \in U,\ t \in [0,1] : tu + (1-t)p \in U \ .$$

Let $f \in \mathcal{O}(U)$ and $\gamma\colon [0,1] \to U$ be a closed C^1-curve. Show that $\oint_\gamma f(z)\ dz = 0$.

Hint: γ is the main part of a rectangular domain. As in the case of the annulus, the contribution of the auxilliary curve cancels.

Solution: Let $q = \gamma(0) = \gamma(1)$. The map

$$\phi\colon Q \to U \quad , \quad \phi(t,s) = s\gamma(t) + (1-s)p$$

is C^1 and if α denotes the boundary curve of Q, then $\phi\circ\alpha$ is a reparametrization of the concatenation

$$\beta * \gamma^{-1} * \beta^{-1}$$

where β is the curve

$$\beta\colon [0,1] \to U \quad , \quad \beta(t) = tq + (1-t)p$$

lies in U. By the Cauchy Integral Theorem for rectangular domains,

$$0 = \oint_{\phi\circ\alpha} f(z)\ dz = \int_\beta f(z)\ dz - \int_\gamma f(z)\ dz - \int_\beta f(z)\ dz$$

22. Let f be an entire function and assume that for all $z \in \mathbb{C}$ we have

$$|f(z)| \leq \ln(1+|z|) \ .$$

Show that $f(z) = 0$ for all $z \in \mathbb{C}$.

Hint: Look at the proof of Liouville's Theorem.

Solution: Since f is entire, we have

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

for all $z \in \mathbb{C}$ and some sequence $(c_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}_0}$. By the estimate of the Taylor coefficients c_n, we have

$$|c_n| \leq \frac{\max\{|f(z)| \mid |z| = r\}}{r^n} \leq \frac{\ln(1+r)}{r^n} \xrightarrow{r\to\infty} 0$$

if $n > 0$. It follows that $f(z) = c_0$. Since $\ln(1) = 0$ we must have $c_0 = 0$ as well.

23. Let $\gamma\colon [-\pi, \pi] \to \mathbb{C}$ be the curve with

$$\gamma(t) = \cos(t) + i\left(\sin(t) + \frac{e^{-\sqrt{t^2+1}}}{16}\sin(43t)\right) .$$

Compute

$$\int_\gamma \frac{1}{\sin(z)}\ dz .$$

Hint: Sketch the curve. It is homotopic relative endpoint to a simpler one. Then use (5.3).

Solution: The integrand

$$f(z) = \frac{1}{\sin(z)}$$

is holomorphic on $\mathbb{C}\backslash\pi\mathbb{Z}$. The curve γ is homotopic to the unit circle in this domain via the homotopy

$$H(t,s) = e^{it} + si\frac{e^{-\sqrt{t^2+1}}}{16}\sin(43t) \quad , \quad (t,s) \in [-\pi,\pi]\times[0,1] .$$

The function H is C^1 and

$$H(t,0) = e^{it} \quad , \quad H(t,1) = \gamma(t) \quad , \quad H(-\pi,s) = H(\pi,s) = -1 ,$$

H is a homotopy rel endpoint. To show that H is a homotopy in the domain where f is holomoprhic, we estimate

$$1 - \frac{1}{16} \leq |H(t,s)| \leq 1 + \frac{1}{16}$$

for all $(t,s) \in [-\pi,\pi]\times[0,1]$ and therefore $H(t,s) \notin \pi\mathbb{Z}$ for all such (t,s). It thus remains to compute

$$\oint_{|z|=1} \frac{1}{\sin(z)}\ dz = \frac{2\pi i}{\sin'(0)} = 2\pi i$$

by (5.3).

24. Let $U, V \subset \mathbb{C}$ be open and let $f\colon U \to V$ and $g\colon V \to \mathbb{C}$ be holomorphic. Let $p \in U$. Show that

$$\operatorname{ord}(g\circ f; p) = \operatorname{ord}(g; f(p))\operatorname{ord}(f; p)$$

Hint: Look at (11.12) or the local form theorem 11.13.

Solution: By (11.12) there are functions $\tilde{f}$ and $\tilde{g}$ holomorphic near p respectively $f(p)$ with $\tilde{f}(p) \neq 0$ and $\tilde{g}(f(p)) \neq 0$) so that

$$\begin{aligned} f(z) &= f(p) + (z-p)^{\operatorname{ord}(f;p)} \tilde{f}(z) \\ g(y) &= g(f(p)) + (y - f(p))^{\operatorname{ord}(g;f(p))} \ \tilde{g}(y) \end{aligned}$$

for z, y in sufficiently small neighbourhoods of p and $f(p)$ respectively. Thus

$$\begin{aligned} g(f(z)) &= g(f(p)) + \left((z-p)^{\operatorname{ord}(f;p)} \tilde{f}(z)\right)^{\operatorname{ord}(g;f(p))} \ \tilde{g}\left(f(p) + (z-p)^{\operatorname{ord}(f;p)} \tilde{f}(z)\right) \\ &= g(f(p)) + (z-p)^{\operatorname{ord}(f;p)\operatorname{ord}(g;f(p))} \ \underbrace{\tilde{f}(z)^{\operatorname{ord}(g;f(p))} \ \tilde{g}\left(f(p) + (z-p)^{\operatorname{ord}(f;p)} \tilde{f}(z)\right)}_{Q(z)} \end{aligned}$$

Clearly

$$Q(p) = \tilde{f}(p)^{\operatorname{ord}(g;f(p))} \ \tilde{g}(f(p)) \neq 0 \ .$$

25. Let $f \in \mathcal{O}(\mathrm{B}_1)$ so that for all $z \in \mathrm{B}_1$ we have

$$\Re(f(z)) = \Im(f(z))^2 \ .$$

Prove that f is constant.

Hint: Open Mapping Theorem

Solution: The assumption on f says that

$$f(\mathrm{B}_1) \subset \left\{x + iy \,\middle|\, x, y \in \mathbb{R}, y^2 = x\right\} =: P$$

But the set P is a parabola in $\mathbb{R}^2 = \mathbb{C}$ and does not contain any nonempty open sets. Hence $f(\mathrm{B}_1)$ does not contain any nonempty open sets and therefore f can not be open. By the Open Mapping Theorem, f must be constant.

26. Classify the isolated singularities $p \in S_i$ of the following functions $f_i \in \mathcal{O}(\mathbb{C} \setminus S_i)$, $S_i \subset \mathbb{C}$ discrete, as removable, pole, essential. In the case of poles determine the order.

(a) $f_a(z) = \dfrac{e^z - 1 - z}{z^3(z-1)}$, $S_a = \{0, 1, 2\}$

(b) $f_b(z) = \dfrac{z^2 - 1}{\sin(\pi z)^2}$, $S_b = \mathbb{Z}$

(c) $f_c(z) = \dfrac{z^2 + 1}{1 + z + z^2 + z^3}$, $S_c = \{1, i, -1, -i\}$

27. Assume that for all $z \in \mathrm{B}_{\frac{1}{345}}(2)$ we have

$$\sum_{n=-\infty}^{\infty} c_n z^n = \frac{\sin(z)}{z^2 + \frac{1}{4}} + \frac{e^z}{1 + 5i - z} \ .$$

Compute $\limsup\limits_{n\to\infty} \sqrt[n]{|c_n|}$ and $\limsup\limits_{n\to\infty} \sqrt[n]{|c_{-n}|}$.

Solution: The function

$$f(z) = \frac{\sin(z)}{z^2 + \frac{1}{4}} + \frac{e^z}{1 + 5i - z}$$

has poles of order 1 at $i/2, -i/2, 1+5i$ and no other singularities. Therefore the Laurent series around 0 converging at 2 converges on the largest annulus $A_{r,R} \subset \mathbb{C} \setminus \{i/2, -i/2, 1+5i\}$. Hence $r = \frac{1}{2}$ and $R = \sqrt{26}$. The inner radius is the inverse of the radius of convergence of the principal part, hence

$$\limsup_{n\to\infty} \sqrt[n]{|c_{-n}|} = \frac{1}{2} \ .$$

The outer radius is the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n z^n$, hence

$$\limsup_{n\to\infty} \sqrt[n]{|c_n|} = \frac{1}{\sqrt{26}} \ .$$

Some problems for the study week

1. Determine the type of the singularity 0 of

$$f(z) = \frac{z(e^z-1)^2}{(\cos(z)-1)^2 \sin(z)^4 \tan(z)} \ .$$

Solution: In the sequel h_i will denote functions holomorphic in a neighbourhood of 0 with $h_i(0) \neq 0$. Inserting Taylor series, we can rewrite $f(z)$ as

$$f(z) = \frac{z \left(\sum_{k=1}^{\infty} \frac{z^k}{k!}\right)^2 \left(\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!}\right)}{\left(\sum_{k=1}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!}\right)^2 \left(\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!}\right)^5}$$

$$= \frac{z(zh_1(z))^2 h_2(z)}{(z^2 h_3(z))^2 (zh_5(z))^5} = \frac{z^{1+2}}{z^{2\times 2+5}} \frac{h_1(z)^2 h_2(z)}{h_3(z)^2 h_5(z)^5} = z^{-6} h_6(z)$$

hence 0 is a pole of order 6 of this function.

2. Determine the type of the singularity 0 of

$$f(z) = \frac{e^{1/z}}{\cosh(1/z)} \ .$$

Solution: Recall that $\cosh(z) = \frac{e^z + e^{-z}}{2}$, hence

$$f(z) = \frac{2e^{1/z}}{e^{1/z} + e^{-1/z}} = \frac{2e^{2/z}}{e^{2/z} - 1} \ .$$

The limit of this as $z \to 0$ does not exist,

$$\lim_{z\to 0, z\in\mathbb{R}^+} f(z) = 2 \qquad \neq \qquad \lim_{z\to 0, z\in\mathbb{R}^-} f(z) = 0 \ ,$$

hence the singularity is essential.

3. Compute the complex line integral

$$\int_\gamma \frac{1}{(z-1)(z+3)}\, dz \quad \text{where} \quad \gamma\colon [0,1] \to \mathbb{C}\ ,\ \gamma(t) = 2e^{it + i\sin(\pi t)}\ .$$

Solution: The integrand

$$f(z) := \frac{1}{(z-1)(z+3)} = \frac{1}{4}\left(\frac{1}{z-1} - \frac{1}{z+3}\right)$$

is holomorphic for $z \in \mathbb{C} \setminus \{1,-3\}$. Let $H\colon [0,1]\times[0,1] \to \mathbb{C}\setminus\{1,-3\}$,

$$H(s,t) := 2e^{it + is\sin(\pi t)}\ .$$

This maps to $\mathbb{C}\setminus\{1,-3\}$, because $|H(s,t)| = 2$ for all s,t. Then $H(1,t) = \gamma(t)$ and $H(0,t) = 2e^{it} = \mu(t)$, μ the boundary curve of B_2. The second summand, $\frac{1}{z-3}$ is holomorphic on B_2, hence contributes 0 to the closed line integral, by the Cauchy Integral Theorem. Thus

$$\int_\gamma \frac{1}{(z-1)(z+3)}\, dz = \oint_\gamma \frac{1}{4}\left(\frac{1}{z-1} - \frac{1}{z+3}\right)\, dz = \oint_\mu \frac{1}{4}\left(\frac{1}{z-1} - \frac{1}{z+3}\right)\, dz$$

$$= \oint_\mu \frac{1}{4}\frac{1}{z-1}\, dz = \oint_{|z|=2} \frac{1}{4}\frac{1}{z-1}\, dz = \frac{2\pi i}{4} = \frac{\pi i}{2}$$

4. Let $f \in \mathcal{O}(\mathbb{C})$ be so that $f(1/z)$ has a removable singularity at $z=0$. Show that f is constant.

Solution: Since 0 is a removable singularity of $f(1/z)$, the limit $\lim_{z\to 0} f(1/z)$ exists. Thus there are $L \in \mathbb{C}$ and $\delta > 0$ so that for all z with $|z| < \delta$ we have $|f(1/z) - L| < 1$, in particular,

$$|f(1/z)| < |L| + 1\ .$$

Since $|z| < \delta$ if and only if $|1/z| > 1/\delta$ we thus have

$$|f(z)| < |L| + 1 \quad \text{for all} \quad z\ ,\ |z| > \frac{1}{\delta}\ .$$

Since $\overline{\mathrm{B}_{1/\delta}}$ is compact and f entire, in particular continuous, the function f must be bounded. By Liouville's Theorem, f is constant.

5. Let $f, g \in \mathcal{O}(\mathbb{C})$ so that $f(g(z)) = 0$ for all $z \in \mathbb{C}$. Show that f or g is constant.

Solution: If f and g are both not constant, then both are open maps. Hence so is their composition.

6. Let f, g be entire functions so that $f(g(z)) = p(z) \in \mathbb{C}[z]$ and assume that $\deg p > 0$. Show that $f(z), g(z) \in \mathbb{C}[z]$. What follows for the degrees of these polynomials?

Solution: Neither of the functions is constant since p is not constant. By Liouville's Theorem neither of the two functions can be bounded. Since p has no essential singularity at ∞ neither f nor g have an essential singularity and thus must be polynomial and we have $\deg p = \deg f \times \deg g$.

7. Show that the function

$$f(z) := \begin{cases} e^{-1/z^4} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}\ .$$

satisfies the Cauchy-Riemann equations (9.7) on all of $\mathbb{C}$ but is not complex differentiable at $z=0$.

Solution: We have

$$\lim_{t\to 0, t>0} \frac{f(tz)-f(0)}{t} = 0 \quad \text{for} \quad z = 1, i, -1. - i$$

hence the partial derivatives in (9.7) are all 0. But f is not even continuous in 0. For instance,

$$\lim_{t\to 0, t>0} f((1+i)t) = \lim_{t\to 0, t>0} e^{+1/4t^4} = \infty$$

8. For $n \in \mathbb{N}$, compute $\displaystyle\oint_{|z-1|=1} \left(\frac{z}{z-1}\right)^n \, dz$.

   Solution: Substituting z by $z+1$ gives

$$\oint_{|z-1|=1} \left(\frac{z}{z-1}\right)^n dz = \oint_{|z|=1} \left(\frac{z+1}{z}\right)^n dz = \oint_{|z|=1} 1 + \frac{n}{z} + \binom{n}{2}\frac{1}{z^2} + \cdots + \frac{1}{z^n}\, dz = 2\pi n$$

   because $\displaystyle\oint_{|z|=1} z^k \, dz = \begin{cases} 0 & \text{if } k \neq -1 \\ 2\pi i & \text{if } k = -1 \end{cases}$.

9. Determine all functions $f \in \mathcal{O}(\mathbb{C} \setminus \{0\})$ for which 0 is not an essential singularity and which satisfy

$$f\left(\frac{1}{n+1}\right) = \frac{n+1}{n} f\left(\frac{1}{n}\right) \quad \text{for all} \quad n \in \mathbb{N}\ . \tag{5.6}$$

   Solution: If f is such a function, then for some k, 0 is a removable singularity of $z^k f(z)$. By the recursion formula (5.6), for $n \in \mathbb{N}$,

$$f\left(\frac{1}{n}\right) = n f(1)$$

   hence

$$z^k f(z)\big|_{z=n} = n^{1-k}$$

   which is bounded only for $k = 1$. Thus the singularity at 0 must be a pole of order 1 and $zf(z)$ is (extends to) an entire function with

$$zf(z) = f(1) \quad \text{for all} \quad z \in \frac{1}{\mathbb{N}} \cup \{0\}\ .$$

   The set $\frac{1}{\mathbb{N}} \cup \{0\}$ contains an accumulation point and the two entire functions $zf(z)$ and $f(1)$ coincide on this set. By the identity theorem 11.10 the two functions must be equal. Thus $zf(z) = f(1)$ for all $z \in \mathbb{C}$. It follows that the set of functions the problem asks for is

$$\{f_\lambda \mid \lambda \in \mathbb{C}\} \quad \text{where} \quad f_\lambda(z) = \frac{\lambda}{z}\ .$$

10. Find all Laurent series centered at 2 of the function

$$f(z) = \frac{1}{z^2+z-12}\ .$$

   Solution: The partial fraction decomposition

$$f(z) = \frac{1}{z^2+z-12} = \frac{1}{(z-3)(z+4)} = \frac{1/7}{z-3} - \frac{1/7}{z+4} \tag{5.7}$$

shows that f has poles of order 1 at $z = 3$ and $z = -4$. The distances of these singularities from the center of the Laurent series we seek are 1 and 6. To get the Laurent series centered at 2 we rewrite (5.7) in terms of $z - 2$ and use the geometric series

$$\frac{1}{1-q} = \frac{-1}{q}\frac{1}{1-\frac{1}{q}} = \begin{cases} \displaystyle\sum_{k=0}^{\infty} q^k & \text{if } |q| < 1 \\ \displaystyle -\sum_{k=1}^{\infty} q^{-k} & \text{if } |q| > 1 \end{cases}.$$

This gives

$$\begin{aligned} f(z) &= \frac{1/7}{(z-2)-1} - \frac{1/7}{(z-2)+6} \\ &= \frac{-1/7}{1-(z-2)} - \frac{1/42}{1-(-(z-2)/6)} \end{aligned}$$

$$= \begin{cases} \displaystyle\sum_{k=0}^{\infty}\left(\frac{-1}{7}\right)(z-2)^k - \sum_{k=0}^{\infty}\frac{1}{42}\left(\frac{-1}{6}\right)^k(z-2)^k & \text{if } |z-2| < 1 \\ \displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{7}\right)(z-2)^{-k} - \sum_{k=0}^{\infty}\frac{1}{42}\left(\frac{-1}{6}\right)^k(z-2)^k & \text{if } 1 < |z-2| < 6 \\ \displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{7}\right)(z-2)^{-k} + \sum_{k=1}^{\infty}\frac{1}{42}(-6)^k(z-2)^{-k} & \text{if } |z-2| > 6 \end{cases}$$

$$= \begin{cases} \displaystyle\sum_{k=0}^{\infty}\left(\frac{-1}{7} - \frac{1}{42}\left(\frac{-1}{6}\right)^k\right)(z-2)^k & \text{if } |z-2| < 1 \\ \displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{7}\right)(z-2)^{-k} - \sum_{k=0}^{\infty}\frac{1}{42}\left(\frac{-1}{6}\right)^k(z-2)^k & \text{if } 1 < |z-2| < 6 \\ \displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{7} + \frac{1}{42}(-6)^k\right)(z-2)^{-k} & \text{if } |z-2| > 6 \end{cases}$$

References

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6 Complex Numbers

We are looking for a field $\mathbb{C}$ "containing" the field $\mathbb{R}$ of real numbers (i.e. with an injective homomorhism of fields $\mathbb{R} \hookrightarrow \mathbb{C}$), so that the equation "$z^2+1=0$" has a solution. There are (at least) three constructions of such a field $\mathbb{C}$ of complex numbers:

1. ***Gauss Plane:*** $\mathbb{C}_1 = \mathbb{R}^2 = \{x, y \mid x, y \in \mathbb{R}\}$ with componentwise addition, $(x,y)+(x',y') = (x+x', y+y')$ and multiplication given by
$$(x,y)\cdot(x',y') = (xx'-yy', xy-+x'y)\ .$$

2. ***Field extension:*** $\mathbb{C}_2 := \mathbb{R}[i]/\langle i^2+1\rangle$, i.e. the quotient of the ring of real polynomials in the variable i by the ideal generated by the polynomial i^2+1. Since this is irreducible, the quotient is a field.

3. ***2×2 matrices, orientation and angle preserving linear maps of $\mathbb{R}^2$:*** $\mathbb{C}_3 = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}$.

All three $\mathbb{C}_1$, $\mathbb{C}_2$, $\mathbb{C}_3$ are fields and there are natural injective field homomorhisms $\iota_i \colon \mathbb{R} \hookrightarrow \mathbb{C}_i$ given by

$$\mathbb{R} \ni x \mapsto (x,0) \in \mathbb{R}^2 \quad , \quad \mathbb{R} \ni x \mapsto x \in \mathbb{R}[i] \text{ the constant polynomial} \quad , \quad \mathbb{R} \ni x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$$

respectively. The maps
$$(x,y) \leftrightarrow x+iy \leftrightarrow \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$
are isomorphisms between the $\mathbb{C}_i$ compatible with the embeddings ι_i.

Examples

$$\begin{aligned}(2+3i)^3 &= 8+4\cdot(3i)+2\cdot(3i)^2+(3i)^3\\ &= 8+12i+12i^2+27i^3\\ &= 8+12i-12-27i\\ &= -4-15i\end{aligned}$$

$$\begin{aligned}\frac{1}{a+bi} &= \frac{a-bi}{(a+bi)(a-bi)}\\ &= \frac{a-bi}{a^2-b^2} \quad \text{denominator real!}\end{aligned}$$

Compare this with matrix inversion:

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}^{-1} = \frac{1}{\det\begin{pmatrix} a & -b \\ b & a \end{pmatrix}} \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

6.1 Real Part, Imaginary Part, Complex Conjugate, Polar Coordinates, Modulus, Argument

The cartesian coordinates of a complex number

$$z = (a,b) = a+ib = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

are called **real part** and **imaginary part** of z. If $a, b \in \mathbb{R}$ we write

$$\Re(a+ib) = a \quad , \quad \Im(a+ib) = b \ .$$

The **modulus** of a complex number is its eulidean norm and the square root of its determinant,

$$|a+ib| = \sqrt{a^2+b^2} = \sqrt{\det\begin{pmatrix} a & -b \\ b & a \end{pmatrix}} \ .$$

The **complex conjugate** $\overline{z}$ of a complex number z is the its mirror image under reflection at the real axis, or its transpose, when viewed as a matrix,

$$\overline{a+ib} = a-ib \ .$$

From this it is immediate that for complex numbers u, v we have

$$|uv| = |u|\,|v| \ ,$$

$$|u| = \sqrt{u\overline{u}} \ ,$$

$$\begin{aligned} u^{-1} &= \frac{1}{u} = \frac{\overline{u}}{|u|^2} \ , \\ \Re(u) &= \frac{u+\overline{u}}{2} \ , \\ \Im(u) &= \frac{u-\overline{u}}{2i} \ , \\ \Re(uv) &= \langle u \,|\, \overline{v} \rangle \ , \\ \Im(uv) = \Re(-iuv) &= \left\langle u \,\middle|\, \overline{iv} \right\rangle \ , \end{aligned}$$

where $\langle \cdot \,|\, \cdot \rangle$ denotes the standard scalar product on $\mathbb{R}^2$.

We will use the following fact about the trigonometric functions cos and sin:

Theorem 6.1 *The functions* $\cos, \sin \colon \mathbb{R} \to \mathbb{R}$ *are* 2π*-periodic. For every pair*

$$(a,b) \in \mathrm{S}^1 := \{ z \in \mathbb{C} \,|\, |z| = 1 \} = \left\{ (a,b) \in \mathbb{R}^2 \,\middle|\, a^2 + b^2 = 1 \right\}$$

there is $t \in \mathbb{R}$ *so that*

$$\cos(t) = a \quad \textit{and} \quad \sin(t) = b$$

and t *is determined by* a, b *up to an integer multiple of* 2π*. Thus the map*

$$\mathbb{R}/2\pi\mathbb{Z} \xrightarrow{\ t \mapsto \cos(t) + i\sin(t)\ } \mathrm{S}^1$$

is bijective.

We write $e^{it} := \cos(t) + i\sin(t)$ (in [5] this is called $\mathrm{cis}(t)$). Thus every complex number z can be written as

$$z = re^{it} \quad \text{with} \quad r = |z| \in \mathbb{R}_0^+ \quad \text{and} \quad t \in \mathbb{R} \ .$$

A real number t so that $z = re^{it}$ is called an **argument** for z, denoted by $\arg(z) \in \mathbb{R}/2\pi\mathbb{Z}$.

The exponential notation is justified by the functional equation: The addition theomrems for cos and sin turn the map

$$\mathbb{R} \to \mathrm{S}^1 \quad , \quad t \mapsto e^{it} = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$$

into a group homomorphism,

$$e^{i(t+s)} = e^{it} e^{is} \ .$$

7 Series

If $A = (a_n)_{n \in \mathbb{N}_0} \in \mathbb{C}^{\mathbb{N}_0}$ or $\mathbb{R}^{\mathbb{N}_0}$ is a sequence of real or complex numbers, then the sequence S of its partial sums is the sequence

$$S = \left(\sum_{n=0}^{k} a_n \right)_{k \in \mathbb{N}_0} \qquad \text{i.e. the sequence}$$

$$(a_0,\ a_0+a_1,\ a_0+a_1+a_2,\ a_0+a_1+a_2+a_3,\ a_0+a_1+a_2+a_3+a_4,\ldots)$$

If S converges we say "the series $\sum_{n=0}^{\infty} a_n$ converges", and write

$$\sum_{n=0}^{\infty} a_n := \lim_{k\to\infty} \sum_{n=0}^{k} a_n\ .$$

If this limit does not exist, then we say "the series $\sum_{n=0}^{\infty} a_n$ diverges".

A power series $\sum_{n=0}^{\infty} a_n$ **converges absolutely** if the power seris of the moduli $\sum_{n=0}^{\infty} |a_n|$ converges.

Examples

The **geometric series** $\sum_{n=1}^{\infty} q^n$ converges for $|q|<1$ and diverges if $|q|\geq 1$. To see this, first note that the series diverges if $|q|\geq 0$ because then the sequence does not converge to 0.

For $q\neq 1$ we have the formula

$$\sum_{n=0}^{k} q^k = \frac{1-q^{k+1}}{1-q}\ .$$

In case $|q|<1$ we get

$$\sum_{n=0}^{k} q^n = \frac{1-q^{k+1}}{1-q} \xrightarrow{k\to\infty} \frac{1}{1-q}$$

Hence

$$\sum_{n=0}^{\infty} q^n = \lim_{k\to\infty} \sum_{n=0}^{k} q^n = \lim_{k\to\infty} \frac{1-q^{k+1}}{1-q} = \frac{1}{1-q}$$

The **Euler number** is

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{\infty}\frac{1}{k!}$$

The **harmonic series** does not converge,

$$\sum_{n=1}^{\infty}\frac{1}{n} = 1+\frac{1}{2}+\underbrace{\frac{1}{3}+\frac{1}{4}}_{>\frac{1}{2}}+\underbrace{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}}_{>\frac{1}{2}}+\cdots = +\infty\ .$$

The **alternating harmonic series** converges but not absolutely, see below.

7.1 Convergence Criteria:

Let $(a_n)_{n\in\mathbb{N}_0}$ be a sequence of real or complex numbers. Then the series $\sum\limits_{n=0}^{\infty} a_n$ converges any of the following criteria holds:

1. **Comparison Test:** There is a **convergent** series $\sum\limits_{n=0}^{\infty} b_n$, with

$$|a_n| \leq b_n \in \mathbb{R} \quad \text{for all} \quad n \in \mathbb{N} \ .$$

<u>**Example:**</u> Since $\left|\cos(n^3)e^{-n^2+in^3}\right| \leq 1$, we have

$$\left|\frac{\cos(n^3)e^{-n^2+in^3}}{2^n}\right| \leq \frac{1}{2^n} \quad \text{for all} \quad n \in \mathbb{N} \ .$$

By comparison with the geometric series $\sum_{n=0}^{\infty} \frac{1}{2^n}$, the series

$$\sum_{n=0}^{\infty} \frac{\cos(n^3)e^{-n^2+in^3}}{2^n} \ .$$

converges.

2. **Quotient Test:** There is $q \in [0,1)$ and $N \in \mathbb{N}$ so that for all $n \geq N$ we have

$$\left|\frac{a_{n+1}}{a_n}\right| \leq q \ .$$

<u>**Example:**</u> If the quotient is small the series converges fast. For example, for any $q > 0$, we can choose $N \in \mathbb{N}$ so that $Nq > 1$. Then for all $n > N$,

$$\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \frac{1}{n+1} < q \ .$$

By the Quotient Test, the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} =: e \quad \text{converges, giving one definition of the Euler number.}$$

3. **Root Test:** There is $q \in [0,1)$ and $N \in \mathbb{N}$ so that for all $n \geq N$ we have

$$\sqrt[n]{|a_n|} \leq q \ .$$

4. **Leibniz Criterion, Alternating Series Test:** We have $\lim\limits_{n\to\infty} a_n = 0$ and there is $N \in \mathbb{N}$ so that for all $n \geq N$, $a_n \in \mathbb{R}$ and

$$\mathrm{sgn} a_{n+1} = -\mathrm{sgn} a_n \ .$$

The sign of a real number λ is

$$\mathrm{sgn}\lambda = \begin{cases} +1 & \textit{if } \lambda > 0 \\ 0 & \textit{if } \lambda = 0 \\ -1 & \textit{if } \lambda < 0 \end{cases}$$

Example: Recall that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. However, since $\frac{1}{n} \xrightarrow{n\to\infty} 0$, the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \quad \text{converges.}$$

5. **Integral Test:** There is a decreasing Riemann integrable function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $a_n = f(n)$ for all $n \in \mathbb{N}$, and for some $x \in \mathbb{R}^+$ the integral

$$\int_x^{\infty} f = \lim_{y\to\infty} \int_x^y f \quad \text{exists.}$$

A function $f\colon \mathbb{R} \to \mathbb{R}$ is decreasing if $f(x) \geq f(y)$ for all $x, y \in \mathbb{R}$ with $x \leq y$.

Example: Let $s \in \mathbb{R}$. Recall that for any positive real number n, the power n^s is defined as

$$n^s := e^{s\ln(n)} \ .$$

Consider the function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$, $f(x) = x^s$, and its integral for $0 < x < y$,

$$\int_x^y n^s \, ds = \begin{cases} \ln(y) - \ln(x) & \text{if } s = -1 \\ \dfrac{y^{s+1} - x^{s+1}}{s+1} & \text{if } s \neq 1 \end{cases} \xrightarrow{y\to\infty} \begin{cases} \infty & \text{if } s \geq -1 \\ \dfrac{-x^{s+1}}{s+1} & \text{if } s \leq -1 \end{cases} .$$

By the Integral Test, the series

$$\sum_{n=1}^{\infty} n^s \quad \text{converges for real} \quad s < -1$$

A necessary condition for the convergence of a series $\sum_{n=1}^{\infty} a_n$ is that the sequence $(a_n)_{n\in\mathbb{N}}$ converges to 0. This rules out convergence of many sequences! Thus none of the series

$$\sum_{n=1}^{\infty} \frac{n^2+4}{3n+2n^2} \quad , \quad \sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n \quad , \quad \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{17}\right)$$

converge because

$$\lim_{n\to\infty} \frac{n^2+4}{3n+2n^2} = \frac{1}{2} \neq 0 \ ,$$

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e \neq 0 \ ,$$

$$\lim_{n\to\infty} \sin\left(\frac{n\pi}{17}\right) \quad \text{does not exist.}$$

8 Power Series

A power series is (function in z given by) a series with a parameter, say z, of the form

$$f(z) = \sum_{k=0}^{\infty} a_n z^n \ .$$

This concept generalizes that of polynomial

$$p(z) = \sum_{k=0}^{N} a_n z^n \quad , \quad N = \text{degree}\, p \text{ if } a_N \neq 0 \ .$$

Theorem 8.1 (Convergence of Power Series) *Let $(a_n)_{n\in\mathbb{N}_0} \in \mathbb{C}^{\mathbb{N}_0}$. The series $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely if $|z| < \dfrac{1}{\limsup \sqrt[n]{a_n}} =: \rho$ and diverges if $|z| > \rho$.*

Note that this says nothing about convergence on the circle $|z| = \rho$. The number ρ above is called **radius of convergence** of the series.

Proof: If $\sum_{n=0}^{\infty} a_n z^n$ converges, then we must have

$$\lim_{n\to\infty} a_n z^n = 0 \quad \Longrightarrow \quad \limsup_{n\to\infty} |a_n z^n| = 0 \quad \Longrightarrow \quad \limsup_{n\to\infty} \sqrt[n]{|a_n z^n|} \leq 1 \quad \Longrightarrow \quad |z| \leq \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}} =: \rho$$

which must be read as ∞ if the denominator vanishes. Thus the series diverges if $|z| > \rho$.

Conversly if $|z| < \rho$, then

$$\limsup_{n\to\infty} \sqrt[n]{|a_n z^n|} = |z| \limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1$$

and there is $q \in [0, 1)$ and $N \in \mathbb{N}$ so that

$$\forall n > N : |a_n z^n| < q^n \ .$$

Since $0 \leq q < 1$ the geoemtric series $\sum_{j=0}^{\infty} q^j$ converges absolutely. The comparison test yields absolute convergence of $\sum_{n=0}^{\infty} a_n z^n$. •

If M is a set and X a metric space with distance function $\mathrm{d}\colon X \times X \to \mathbb{R}_0^+$, then we have (and will usually refer to) the almost distance d_∞ on the set X^M of functions $f\colon M \to X$ given by

$$\mathrm{d}_\infty(f, g) = \sup\{\mathrm{d}(f(m), g(m)) \mid m \in M\} \in \mathbb{R}_0^+ \cup \{\infty\} \ .$$

Convergence with respect to d_∞ is called **uniform convergence**.

Definition 8.2 *A sequence* $(f_n)_{n\in\mathbb{N}_0} \in \left(X^M\right)^{\mathbb{N}_0}$ *of functions* $f_n\colon M \to X$ *from a set* M *to a metric space* (X, d) **converges uniformly** *to* $g \in X^M$ *if*

$$\lim_{n\to\infty} \mathrm{d}_\infty(f_n, g) = 0 \ ,$$

i.e. more explicitly

$$\forall \epsilon > 0 \exists N_\epsilon \in \mathbb{N} \forall n \in \mathbb{N}, n > N, m \in M : \mathrm{d}(f_n(m), g(m)) < \epsilon \ .$$

We also recall that this construction gives rise to two basic complete metric spaces:

Theorem 8.3 *If* (X, d) *is complete, then* (X^M, d_∞) *is complete.*

If M *is a topological space, then the subspace* $C(M, X) = \{f\colon M \to X \mid f \,\text{continuous}\}$ *is complete with respect to* d_∞.

In particular, a locally uniformly convergent sequence of continuous functions from a topological space to $\mathbb{R}$ or $\mathbb{C}$ has a continuous limit.

We apply this to power series.

Theorem 8.4 *A power series* $\sum_{n=0}^{\infty} a_n z^n$ *with radius of convergence* ρ *converges uniformly on* $\mathrm{B}_\tau = \{z \in \mathbb{C} \mid |z < \tau|\}$ *for every* $\tau < \rho$. *In particular, the function* f *defined by* $f(z) := \sum_{n=0}^{\infty} a_n z^n$ *is continuous.*

Proof: For $|z| < \tau < \rho$, we have $\limsup_{n\to\infty} |a_n z^n| \leq \limsup_{n\to\infty} |a_n| \tau^n < q^n$ for some $q \in [0, 1)$. Hence there is $N \in \mathbb{N}$ so that for all $n > N$ we have $|a_n z^n| < q^n$. If g denotes the limit and f_n the partial sums of the series,

$$g(z) = \sum_{n=0}^{\infty} a_n z^n \quad , \quad f_k(z) = \sum_{n=0}^{k} a_n z^n \quad \text{on} \quad \mathrm{B}_\tau$$

then

$$\mathrm{d}_\infty(g, f_k) = \sup_{|z|<\tau} \left| \sum_{n=0}^{\infty} a_n z^n - \sum_{n=0}^{k} a_n z^n \right| \leq \sup_{|z|<\tau} \sum_{n=k+1}^{\infty} |a_n z^n| \ .$$

By the formula for the geometric series we can estimate this,

$$\sup_{|z|<\tau} \sum_{n=k}^{\infty} |a_n z^n| \leq \sum_{n=k+1}^{\infty} q^n = q^{k+1} \sum_{n=0}^{\infty} q^n = \frac{q^{k+1}}{1-q} \xrightarrow{k\to\infty} 0 \ .$$

•

9 Complex Differentiability

Definition 9.1 *Let $U \subset \mathbb{C}$ be an open subset, $p \in U$ and $f\colon U \to \mathbb{C}$ a function. Then f is* **differentiable at** *p if the limit*

$$f'(p) := \lim_{z \to p, z \in U} \frac{f(z) - f(p)}{z - p} \tag{9.2}$$

exists. If f is complex differentiable at all $p \in U$, then f is called holomorphic on U. f' the **derivatative** *of f and f an* **antiderivative** *of f'.*

If U is not open, then we will say that a function f is holomorphic on U if it has a holomorphic extension to some open neighbourhood of U.

We denote the set of all holomorphic functions $U \to \mathbb{C}$ by $\mathcal{O}(U)$.

Note that the quotient in (9.2) is with respect to complex multiplication.

Theorem 9.3 (Sums, products, quotients, compositions) *If $f, g \in \mathcal{O}(U)$ then the sum $f + g$, the product fg and, if g has no zeros on U, also the quotient f/g are complex differentiable and their derivatives are given by the formulae*

$$\begin{aligned} (f+g)' &= f' + g' \\ (fg)' &= f'g + fg' \\ \left(\frac{f}{g}\right)' &= \frac{f'g - fg'}{g^2} \end{aligned}$$

If $V \subset \mathbb{C}$ is open, $h \in \mathcal{O}(() V)$ and $f(U) \subset V$, then the composition $g \circ f$ is holomorpic and we have the **Chain Rule**

$$(g \circ f)' = g' \circ f \cdot f'$$

i.e. for all $u \in U$, $(g \circ f)'(u) = g'(f(u))f'(u)$.

Examples

Obviously, the constant functions are holomorphic, with derivative 0. The function f with $f(z) = z$ has derivative $f'(z) = 1$. More generally, every polynomial $p \in \mathbb{C}[z]$ with complex coefficients is holomorphic. If

$$p(z) = \sum_{i=0}^{n} a_n z^n \quad \text{for all} \quad z \in \mathbb{C}$$

then

$$p'(z) = \sum_{i=1}^{n} n a_n z^{n-1} \quad \text{for all} \quad z \in \mathbb{C}\ .$$

Problem 9.4 *Show that this follows from Theorem 9.3, you do not need to go back to the definition.*

Complex conjugation, i.e. the function c with $c(z) = \overline{z}$ is nowhere complex differentiable because for all $p \in \mathbb{C}$ the limit

$$\lim_{z\to p} \frac{\overline{z} - \overline{p}}{z - p} = \lim_{z\to 0} \frac{\overline{z}}{z}$$

does not exist, for instance because

$$\lim_{\substack{t\to 0\\ t\in\mathbb{R}}} \frac{\overline{tv}}{tv} = \frac{\overline{v}}{v}$$

depends on v.

9.1 Comparison with Real Differentiation, Cauchy-Riemann Equations

The derivative of a function $f\colon \mathbb{R}^m \to \mathbb{R}^k$ at $p \in \mathbb{R}^m$ is the linear map

$$\mathrm{d}_p f\colon \mathbb{R}^m \to \mathbb{R}^k \quad \text{so that} \quad f(p+h) = f(p) + \mathrm{d}_p f \cdot h + R_p(h)$$

holds for all $h \in \mathbb{R}^m$ with some function $R_p\colon \mathbb{R}^m \to \mathbb{R}^k$ such that

$$\lim_{h\to 0} \frac{R_p(h)}{\|h\|} = 0 \ .$$

Recall that the matrix coefficients of the differential $\mathrm{d}_p f$ are the partial derivatives $\frac{\partial f_i}{\partial x_j}$. If the partial derivatives exist in a neihgbourhood of p and are continuous at p, then f is differentiable at p. The existence of the partial derivatives at p only does not suffice.

Let $U \subset C$ be an open set, $p \in U$ and $f\colon U \to \mathbb{C}$ a function. Under the identification $\mathbb{C}$ as $\mathbb{R}^2$,

$$\mathbb{C} \ni z = x + iy \leftrightarrow (\Re z, \Im z) = (x, y) \in \mathbb{R}^2$$

we can write f as a pair of real valued functions $u, v\colon \mathbb{R}^2 \to \mathbb{R}$, defined by

$$f(x+iy) = u(x+iy) + iv(x+iy) = (u(x,y), v(x,y)) \quad \text{for} \quad x, y \in \mathbb{R} \ .$$

The derivative of f at a point p is now (given by) the matrix of partial derivatives

$$\mathrm{d}_p f = \begin{pmatrix} \frac{\partial}{\partial u} x(p) & \frac{\partial}{\partial u} y(p) \\ \frac{\partial}{\partial v} x(p) & \frac{\partial}{\partial v} y(p) \end{pmatrix} =: \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \ . \tag{9.5}$$

The function is complex differentiable at p if the matrix in (9.5) represents a complex number, i.e. is of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. Thus we have proved:

Theorem 9.6 (Cauchy-Riemann Equations) *Let $U \subset \mathbb{C}$ be open and $u, v\colon U \to \mathbb{R}$ and let $f\colon U \to \mathbb{C}$ be given by*

$$f(x+iy) = u(x+iy) + iv(x+iy) \quad \textit{for all} \quad x + iy \in U \ .$$

Then f is complex differentiable on U if and only if the component functions u, v of f are continuously partially differentiable on U and satisfy the partial differential equations

$$u_x = v_y \quad \textit{and} \quad u_y = -v_x \ . \tag{9.7}$$

The pointwise version of this is false, see problem 7.

9.2 Harmonic Functions

If the second partial derivatives are continuous, we can eliminate one of the component functions of f by inserting one of the equations in (9.7) in the other and interchanging the order of differentiation,

$$u_{xx} = v_{yx} = v_{xy} = -u_y y \quad , \quad v_{xx} = -u_{yx} = -u_{xy} = -v_{yy}$$

hence

$$\Delta u = \Delta v = 0 \ ,$$

where $\Delta \colon C^2\mathbb{R}^2) \to C(\mathbb{R}^2)$ is the Laplace operator on $\mathbb{R}^2$,

$$\Delta u = \operatorname{trace} \mathrm{d}^2 u = u_{xx} + u_{yy} \ .$$

Functions in the kernel of Δ are called **harmonic**.

9.3 The Derivative of Power Series

Differentiation of power series is formal,

Theorem 9.8 *Let $f(z) = \displaystyle\sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence ρ, i.e. convergent on B_ρ. Then the derivative of f exists on B_ρ and is given there by*

$$f'(z) = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n \ . \tag{9.9}$$

The power series for f' has the same radius of convergence ρ.

Proof: The power series (9.9) has the same radius of convergence because $\lim_{n\to\infty} \sqrt[n]{n} = 1$, hence

$$\limsup_{n\to\infty} \sqrt[n]{|na_n|} = \lim_{n\to\infty} \sqrt[n]{n} \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

For $p \in \mathrm{B}_\rho$ and $h \in \mathbb{C}$ so that $|p| + |h| \in \mathrm{B}_\rho$ we will show that

$$\lim_{h\to 0} \frac{\displaystyle\sum_{n=0}^{\infty} a_n (p+h)^n - a_n p^n}{h} - \sum_{n=1}^{\infty} n a_n p^{n-1} = 0 \ .$$

To this end we estimate

$$\frac{\sum_{n=0}^{\infty} a_n(p+h)^n - a_n p^n}{h} - \sum_{n=1}^{\infty} n a_n p^n = \frac{\sum_{n=1}^{\infty} a_n(p+h)^n - a_n p^n}{h} - \sum_{n=1}^{\infty} n a_n p^{n-1}$$

$$= \frac{\sum_{n=1}^{\infty} a_n(p+h)^n - a_n p^n - n a_n p^{n-1} h}{h} = \frac{\sum_{n=1}^{\infty} a_n(p+h)^n - a_n p^n - n a_n p^{n-1} h}{h}$$

$$= \frac{\sum_{n=2}^{\infty} a_n(p+h)^n - a_n p^n - n a_n p^{n-1} h}{h} = \frac{\sum_{n=2}^{\infty} a_n \sum_{j=2}^{n} \binom{n}{j} p^{n-j} h^j}{h}$$

$$= \sum_{n=2}^{\infty} a_n \sum_{j=2}^{n} \binom{n}{j} p^{n-j} h^{j-1} = h \sum_{n=2}^{\infty} a_n \sum_{j=2}^{n} \binom{n}{j} p^{n-j} h^{j-2} = h \sum_{n=0}^{\infty} a_{n+2} \sum_{j=2}^{n+2} \binom{n+2}{j} p^{n+2-j} h^{j-2}$$

$$= h \underbrace{\sum_{n=0}^{\infty} a_{n+2} \underbrace{\sum_{j=0}^{n} \binom{n+2}{j+2} p^{n-j} h^j}_{S}}_{P(h)} = hP(h) \xrightarrow{h \to 0} 0$$

In order to show this convergence we now estimate the binomial coefficient

$$\binom{n+2}{j+2} = (n+2)(n+1)\frac{n!}{(j+2)!(n-j)!} \leq (n+2)(n+1)\frac{n!}{j!(n-j)!} = (n+2)(n+1)\binom{n}{j}$$

and thus

$$|S| \leq \sum_{j=0}^{n} \binom{n+2}{j+2} |p|^{n-j} |h|^j \leq \sum_{j=0}^{n} (n+2)(n+1) \binom{n}{j} |p|^{n-j} |h|^j = (n+2)(n+1)(|p|+|h|)^n \ .$$

For P above this gives

$$|P(h)| = \left| \sum_{n=0}^{\infty} a_{n+2} \sum_{j=0}^{n} \binom{n+2}{j+2} p^{n-j} h^j \right| \leq \sum_{n=0}^{\infty} |a_{n+2}| \, (n+2)(n+1)(|p|+|h|)^n = g(|p|+|h|) \ ,$$

where

$$g(z) = \sum_{n=0}^{\infty} (n+2)(n+1) \, |a_n| \, z^n.$$

This power series also has radius of convergence ρ. By Theorem 8.4, g is continuous on B_ρ, hence P is bounded and the limit above vanishes. •

Examples

1. Exponential Function

$$\exp(z) = e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

2. Trigonomoetric Functions

$$\sin(z) := \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \quad \text{and} \quad \cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

proves immediately that

$$e^{iz} = \cos(z) + i\sin(z) \quad \text{for all} \quad z \in \mathbb{C} \ .$$

$$\exp' = \exp$$

$$\cos' = -\sin \quad , \quad \sin' = \cos$$

3. Hyperbolic Functions

$$\sinh(z) := \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \quad \text{and} \quad \cosh(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$$

proves immediately that

$$e^{z} = \cosh(z) + \sin(z) \quad \text{for all} \quad z \in \mathbb{C} \ .$$

$$\cosh' = \sinh \quad , \quad \sinh' = \cosh$$

4. Logarithm: On $\mathbb{R}^+$ this is defined by $\ln'(z) = \frac{1}{z}$ and $\ln(1) = 0$. Since

$$\frac{1}{1-h} = \sum_{n=0}^{\infty} h^n \text{ for } |h| < 1$$

we have

$$\frac{1}{z} = \sum_{n=0}^{\infty} (1-z)^n = \sum_{n=0}^{\infty} (-1)^n (z-1)^n \text{ for } |z| < 1 \ .$$

Integration gives

$$\ln(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (z-1)^{n+1} \ . \tag{9.10}$$

For positive real z, a we get

$$\ln(z) - \ln(a) = \ln\left(\frac{z}{a}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \left(\frac{z}{a} - 1\right)^{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)a^{n+1}} (z-a)^{n+1}$$

which converges for all z, $|z| < a$. Since any $z \in \mathbb{C}$ with $\Re(z) > 0$ lies in $\mathrm{B}_a(a)$ for $a \in \mathbb{R}^+$ sufficiently large, this formula defines an extension of ln from $\mathbb{R}^+$ to the right half plane $\{z \in \mathbb{C} \mid \Re z > 0\}$.

5. For $a \in \mathbb{C}$ we can define

$$z^a = e^{a \ln(z)}$$

where ln is defined, for instance on the right half plane. Thus, for $|z| < 1$,

$$(1+z)^a = \exp\left(-a \sum_{k=1}^{\infty} \frac{(-z)^k}{k}\right) = \sum_{n=0}^{\infty} \frac{(-a)^n}{n!} \left(\sum_{k=1}^{\infty} \frac{(-z)^k}{k}\right)^n$$

10 The Cauchy Integral Theorem

Definition 10.1 *A* **curve** *or* **path** *(in $\mathbb{C}$) is a continuous map $\gamma\colon [a,b] \to \mathbb{C}$ from an interval $[a,b] \subset \mathbb{R}$ to $\mathbb{C}$, $a, b \in \mathbb{R} \cup \{\pm\infty\}$, $a \le b$.*

A **loop** *or* **closed path/curve** *is a curve $\gamma\colon [a,b] \to \mathbb{C}$, $a,b \in \mathbb{R}$, with $\gamma(a) = \gamma(b)$.*

The loop γ is called a **simple closed curve** *if γ is injective execept at the endpoints, i.e. if $\gamma(t) = \gamma(t')$ only for $t = t'$ or $\{t, t'\} = \{a, b\}$.*

A curve $\gamma\colon [a,b] \to \mathbb{C}$, $a,b \in \mathbb{R}$, is **piecewise continuously differentiable** *if there are $t_i \in [a,b]$, $i = 1, \dots, n$,*

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b \quad \textit{so that}$$

1. *the restrictions $\gamma|_{[t_{i-1}, t_i]}$ are continuously differentiable, and*
2. *γ is continuous.*

If $\gamma([a,b]) \subset U \subset \mathbb{C}$ and $f \in C(U, \mathbb{C})$, then the **complex line integral of f over γ** *is the Riemann integral*

$$\int_\gamma f(z)dz := \int_a^b f(\gamma(t))\gamma'(t)\ dt\ .$$

By the transformation formula the complex line integral is invariant under orientation preserving reparametrisation of the curve. Thus, if $\phi\colon [u,v] \to [a,b]$ is a diffeomorphism, then

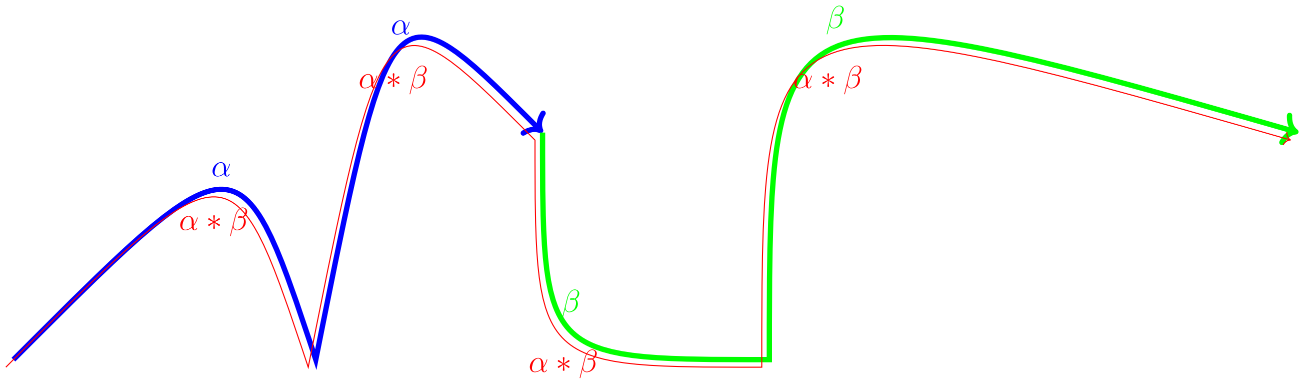
$$\int_{\gamma\circ\phi} f(z)\ dz = \int_u^v f(\gamma(\phi(s)))(\gamma\circ\phi)'(s)\ ds = \int_u^v f(\gamma(\phi(s)))\gamma'(\phi(s))\phi'(s)\ ds$$

$$= \int_{\phi(u)}^{\phi(v)} f(\gamma(t))\gamma'(t)\ dt = \pm \int_a^b f(\gamma(t))\gamma'(t)\ dt = \pm \int_\gamma f(z)\ dz$$

depending on whether ϕ preserves the orientation, i.e. $\phi' > 0$ and $\phi(u) = a, \phi(v) = b$, or else ϕ reverses the orientation, $\phi' < 0$ and $\phi(u) = b, \phi(v) = a$.

The complex line integral is additive. Thus if $\alpha\colon [a,b] \to \mathbb{C}$, $\beta\colon [c,d] \to \mathbb{C}$ are piecewise C^1 curves so that $\alpha(b) = \beta(c)$, then the **concatenation** of the two curves is the curve

$$\alpha * \beta\colon [a, b+d-c] \to \mathbb{C} \quad \text{with} \quad \alpha * \beta(t) = \begin{cases} \alpha(t) & \text{if } t \in [a,b] \\ \beta(c+t-b) & \text{if } t \in [b, b+d-c] \end{cases}$$



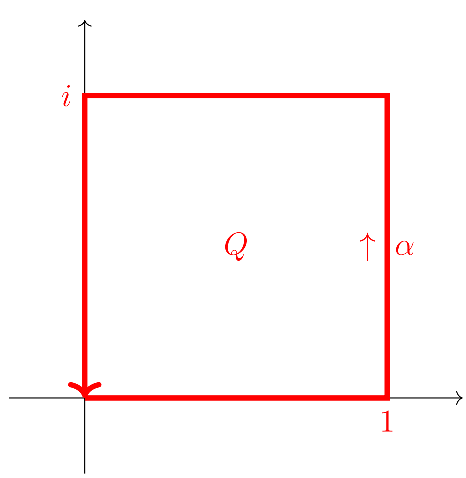
Concatenation of Curves

If $f\colon \mathbb{C} \to \mathbb{C}$ continuous, then

$$\int_{\alpha * \beta} f(z)\ dz = \int_{\alpha} f(z)\ dz + \int_{\beta} f(z)\ dz\ .$$

We will often need the boundary curve of a rectangle, here the unit square:

$$\alpha\colon [0,1] \to \mathbb{C} \quad , \quad \alpha(t) = \begin{cases} 4t & \text{if } 0 \le t \le \frac{1}{4} \\ 1 + i(4t-1) & \text{if } \frac{1}{4} \le t \le \frac{1}{2} \\ 3 - 4t + i & \text{if } \frac{1}{2} \le t \le \frac{3}{4} \\ i(4-4t) & \text{if } \frac{3}{4} \le t \le 1 \end{cases} \tag{10.2}$$



Standard Square

We will first prove the

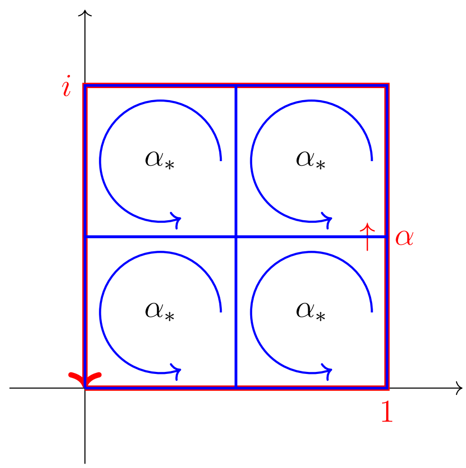
Theorem 10.3 (Cauchy Integral Theorem for Rectangles) *Let* α *be the curve given in* (10.2) *and let* $U \subset \mathbb{C}$ *be an open set containing the square* $Q = [0,1] + i[0,1]$ *surrounded by* α*. Then for all* $f \in \mathcal{O}(U)$

$$\oint_\alpha f(z)\ dz = 0\ .$$

Proof: For $z \in \mathbb{C}$ and $r \in \mathbb{R}_0^+$ let $\alpha_{z,r}(t) = z + r\alpha(t)$. Subdividing the square Q in four subsquares

$$\frac{1}{2}Q \quad , \quad \frac{1}{2} + \frac{1}{2}Q \quad , \quad \frac{i}{2} + \frac{1}{2}Q \quad , \quad \frac{1+i}{2} + \frac{1}{2}Q \tag{10.4}$$

with anti clockwise boundary curves $\alpha_{0,1/2}$, $\alpha_{1/2,1/2}$, $\alpha_{(1+i)/2,1/2}$ and $\alpha_{i/2,1/2}$



Interior arcs cancel

we see that the inner arcs of the α_* cancel. Thus

$$\oint_\alpha f(z)\ dz = \oint_{\alpha_{0,1/2}} f(z)\ dz + \oint_{\alpha_{1/2,1/2}} f(z)\ dz + \oint_{\alpha_{i/2,1/2}} f(z)\ dz + \oint_{\alpha_{(1+i)/2,1/2}} f(z)\ dz\ .$$

Among the four subsquares in (10.4) let Q' be the one with maximal absolute value of the boundary integral.

Iterating this subdivision/choice gives a sequence $(Q_n)_{n\in\mathbb{N}}$ of subsquares of Q starting with $Q_0 = Q$ and so that

$$Q_n = Q'_{n-1} \quad \text{for all} \quad n \in \mathbb{N}$$

is a subsquare with maximal boundary integral.

We then have

$$\left| \oint_{\partial Q_{n-1}} f(z)\ dz \right| \leq 4 \left| \oint_{\partial Q_n} f(z)\ dz \right|$$

hence

$$\left| \oint_\alpha f(z)\ dz \right| = \left| \oint_{\partial Q} f(z)\ dz \right| \leq 4^n \left| \oint_{\partial Q_n} f(z)\ dz \right| \quad \text{for all} \quad n \in \mathbb{N}\ .$$

For each $n \in \mathbb{N}$ let z_n be the lower left corner of Q_n. Then the sequence $(z_n)_{n\in\mathbb{N}}$ is a Cauchy sequence because

$$|z_n - z_{n-1}| \leq \frac{1}{2}\operatorname{diam} Q_{n-1} = \operatorname{diam} Q_n = \frac{\sqrt{2}}{2^n}.$$

Let $p = \lim_{n\to\infty} z_n$ be the limit.

Then $p \in Q_n$ for all n and

$$\forall x \in Q_n : |x - p| \leq \operatorname{diam} Q_n \ .$$

Since f is complex differentiable, we have

$$f(p+h) = f(p) + f'(p)h + R(h)$$

with a function R so that $\lim_{h\to 0} \frac{R(h)}{h} = 0$.

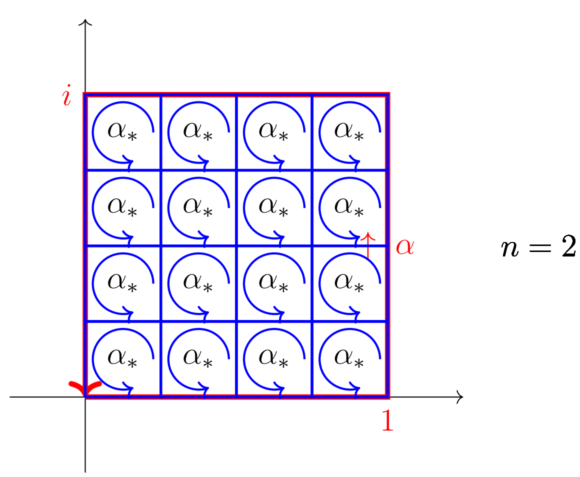
For any $\epsilon > 0$ there is therefore $\delta_\epsilon > 0$ so that for all h with $|h| < \delta_\epsilon$ we have $|R(h)| < |h|\,\epsilon$.

We now choose n so large that

$$\operatorname{diam} Q_n < \delta_\epsilon \ .$$

Then

$$\begin{aligned}\left|\oint_\alpha f(z)\ dz\right| &= \left|\oint_{\partial Q} f(z)\ dz\right| \leq 4^n \left|\oint_{\partial Q_n} f(z)\ dz\right| \\ &= \left|\oint_{\partial Q_n} R(z)\ dz\right| \leq 4^n \times \textbf{length}\,\partial Q_n \times \epsilon \times \operatorname{diam} Q_n \\ &= 4^n \times 4 \times 2^{-n} \times \epsilon \times \sqrt{2} \times 2^{-n} = 4\sqrt{2}\epsilon\ .\end{aligned} \tag{10.5}$$



Subdivision

•

We will now prove the version we will mainly use for now.

Theorem 10.6 (Cauchy Integral Theorem for Rectangular Domains) *Let $U \subset \mathbb{C}$ be open and let $\phi\colon Q \to U$ be a C^1-map. Let α be the boundary curve of Q as given in* (10.2). *Then for all $f \in \mathcal{O}(U)$*

$$\oint_{\phi\circ\alpha} f(z)\; dz = 0\ .$$

Proof: As in the previous proof we subdivide the square Q to obtain a sequence of subsquares Q_n with boundary curves α_n so that

$$\left|\oint_{\phi\circ\alpha} f(z)\; dz\right| \le 4^n \left|\oint_{\phi\circ\alpha_n} f(z)\; dz\right|$$

and so that the lower left corners z_n of the squares Q_n converge to some p, $p \in Q_n$ for all n. As before,

$$f(\phi(p+h)) = f(\phi(p)) + \mathrm{d}_p(f\circ\phi)h + R(h) = f(\phi(p)) + \mathrm{d}_{\phi(p)}f\mathrm{d}_p\phi h + R(h)$$

and for any $\epsilon > 0$ we have $\delta_\epsilon > 0$ so that

$$|R(h)| \le \epsilon\,|h| \quad \text{if} \quad |h| < \delta_\epsilon\ .$$

The assumption that ϕ is continuously differentiable means that the map

$$\mathrm{d}\phi\colon Q \to \mathrm{Hom}(\mathbb{R}^2,\mathbb{R}^2) \quad , \quad z \mapsto \mathrm{d}_z\phi$$

is continuous, with respect to any norm on $\mathrm{Hom}(\mathbb{R}^2,\mathbb{R}^2)$ because this is a finite dimensional real vector space and all norms on finite dimensional vector spaces are equivalent, i.e. induce the same topology. We choose the operator norm on $\mathrm{Hom}(\mathbb{R}^2,\mathbb{R}^2)$. Recall that the operator norm of a linear map $T\colon V \to W$ of normed vector spaces is

$$\begin{aligned}\|T\|_{\mathrm{op}} &= \sup\left\{\frac{\|Tv\|}{\|v\|}\,\middle|\, v \in V\setminus\{0\}\right\}\\ &= \sup\left\{\|Tv\| \,\middle|\, v\in V, \|v\| = 1\right\}\\ &= \inf\left\{\lambda \in \mathbb{R}\,\middle|\, \forall v \in V : \|Tv\| \le \lambda\,\|v\|\right\} \in \mathbb{R}_0^+ \cup \{\infty\}\end{aligned}$$

where $\inf\emptyset = \infty$. Since continuous maps map compact sets to compact sets, the function $Q \to \mathbb{R}$, $z \mapsto \|\mathrm{d}_z\phi\|_{\mathrm{op}}$, has a maximum M. Hence

$$\forall z \in Q : \|\mathrm{d}_z\phi\|_{\mathrm{op}} \le M \quad \text{i.e.} \quad \forall z \in Q, h \in \mathbb{R}^2 = \mathbb{C} : |\mathrm{d}_z\phi h| \le M\,|h|\ .$$

We now can modify the estimate (10.5).

$$\begin{aligned}\left|\oint_{\phi\circ\alpha} f(z)\; dz\right| &\le 4^n \left|\oint_{\phi\circ\alpha_n} f(z)\; dz\right|\\ &= 4^n \left|\oint_{\phi\circ\alpha_n} R(z)\; dz\right|\\ &\le 4^n \times \textbf{length}\,(\phi\circ\alpha_n) \times \max_t |R(\alpha_n(t) - p)| \qquad (10.7)\\ &\le 4^n \times M \times \textbf{length}\,(\alpha_n) \times \max_t |R(\alpha_n(t) - p)|\\ &\le 4^n \times M \times 4 \times 2^{-n} \times \epsilon \times \sqrt{2} \times 2^{-n}\\ &= 4\sqrt{2}M\epsilon\end{aligned}$$

which holds for all $\epsilon > 0$, hence the integral vanishes. •

There is already an abundance of rectangular domains:

1. **Triangles** For the convex hull
$$\operatorname{conv}\{a,b,c\} = \left\{\alpha a + \beta b + \gamma c \,\middle|\, \alpha,\beta,\gamma \in \mathbb{R}_0^+, \alpha+\beta+\gamma = 1\right\}$$
of $\{a,b,c\} \subset \mathbb{C}$ we can use the parametriztion
$$\phi\colon Q \to \Delta(a,b,c) \quad , \quad \phi(s,t) = sa + tb + (1-s-t)c\ .$$

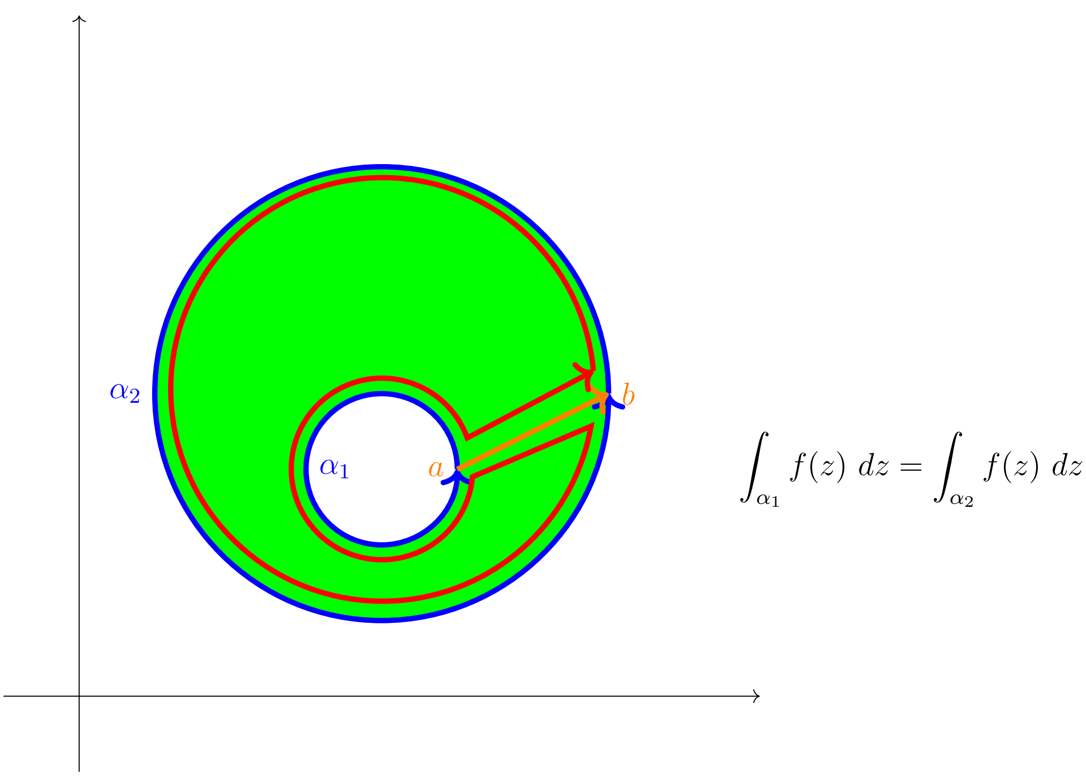
2. **Homotopy of curves relative endpoint** This describes a deformation Γ from one curve $\gamma_0\colon [0,1] \to \mathbb{C}$ to another $\gamma_1\colon [0,1] \to \mathbb{C}$, keeping the endpoints fixed. Differentiability of this deformation is expressed as the differentiability of the map Γ on Q:

   Definition 10.8 *Let $U \subset \mathbb{C}$ and let $\gamma_0, \gamma_1\colon [0,1] \to U$ be C^1 curves. Then the two curves are* **C^1-homotopic relative endpoint**, *we write $\gamma_0 \simeq \gamma_1 \operatorname{rel}\{0,1\}$, if there is a C^1 map*
$$\Gamma\colon [0,1]\times[0,1] \to U$$
   so that

   (a) $\forall t \in [0,1], s \in \{0,1\} : \Gamma(s,t) = \gamma_s(t)$

   (b) $\forall t \in \{0,1\}, s \in [0,1] : \Gamma(s,t) = \gamma_0(t) = \gamma_1(t)$

3. **Annuli**



The segment $[a,b]$ does not contribute to the line integral over the red curve

Problem 10.9 *Find the formula for a C^1-function $\phi\colon Q \to \mathbb{C}$ mapping the square Q to the annulus and the edges of Q to the circles and the segment $[a,b]$ shown in the picture. The curve $\phi \circ \alpha$ moves twice through the segment $[a,b]$, but in opposite directions .*

An **annulus** is a subset of $\mathbb{C}$ of the form $\overline{\mathrm{B}}_R(P) \setminus \mathrm{B}_r(p)$, where $P, p \in \mathbb{C}$ and $R > r > 0$ so that $|P-p| + r < R$. If $f \in \mathcal{O}\left(\left(\right) U\right)$ with $U \subset \mathbb{C}$ open and $\overline{\mathrm{B}}_R(P) \setminus \mathrm{B}_r(p) \subset U$, then f has the same complex line integral over both boundary curves of the annulus. To see this, we consider the curve $\alpha_1 * \beta * \alpha_2^{-1} * \beta^{-1}$. By additivity,

$$\begin{aligned}\oint_{\alpha_1 * \beta * \alpha_2^{-1} * \beta^{-1}} f(z)\, dz &= \int_{\alpha_1} f(z)\, dz + \int_{\beta} f(z)\, dz - \int_{\alpha_2} f(z)\, dz - \int_{\beta} f(z)\, dz \\ &= \int_{\alpha_1} f(z)\, dz - \int_{\alpha_2} f(z)\, dz\end{aligned}$$

But since the annulus is a rectangular region and $\alpha_1 * \beta * \alpha_2^{-1} * \beta^{-1}$ its boundary curve, this vanishes by Cauchy's Integral Theorem.

A subset Ω of a real vector space is **convex** if with any two points, Ω also contains the segment between these points:

$$p, q \in \Omega, t \in [0,1] \Longrightarrow tp + (1-t)q \in \Omega \ .$$

Problem 10.10 *Let $\Omega \subset \mathbb{C}$ be a convex domain, $f \in \mathcal{O}(\Omega)$ and $\gamma\colon [0,1] \to \Omega$ be a closed, piecewise continuously differentiable curve. Prove that* $\displaystyle\oint_{\gamma} f(z)\, dz = 0$.

Proof: We may assume that $\gamma(0) = \gamma(1) = 0$. Let $0 = t_0 < t_1 < \cdots < t_n = 1$ be the only non-differentiability points of γ. Then $\phi\colon [t_{i-1}, t_i] \times [0,1] \to \mathbb{C}$, $\phi(t,s) = s\gamma(t)$ is continuously differentiable and maps $[t_{i-1}, t_i] \times [0,1]$ in Ω. The boundary curve of this rectangular region is the concatenation $\beta_i := \ell_{\gamma(t_{i-1})} * \gamma|[t_{i-1}, t_i] * \ell_{\gamma t_i}^{-1} * c_0$, where c_p is the constant curve at p and ℓ_x is the segment from 0 to x. Now $\oint_\gamma f(z)\, dz = \sum_{i=1}^n \int_{\gamma|_{[t_{i-1}, t_i]}} f(z)\, dz = \sum_{i=1}^n \oint_{\beta_i} f(z)\, dz$, because the integrals over the ℓ_{γ_i} in this sum cancel (and the integrals over the constant path are 0). By Cauchy's Integral Theorem, $\oint_{\beta_i} f(z)\, dz = 0$ for all i. •

11 Immediate Consequences of the Cauchy Integral Theorem

Theorem 11.1 (Cauchy Integral Formula on the disc) *Let $r > 0$ and $p \in \mathbb{C}$, $f \in \mathcal{O}\left(\overline{\mathrm{B}}_r(p)\right)$. Then for all $a \in \mathrm{B}_r(p)$ we have*

$$f(a) = \frac{1}{2\pi i} \oint_{|z-p|=r} \frac{f(z)}{z-a}\, dz \ . \tag{11.2}$$

Proof: We may assume $p = 0$ and $r = 1$. Let $a \in \mathrm{B}_1(0)$. For every ϵ, $0 < \epsilon < 1 - |a|$, the function

$$\frac{f(z)}{z-a} \quad \text{is holomorphic on the annulus} \quad \overline{\mathrm{B}}_1(0) \setminus \mathrm{B}_\epsilon(a) \ .$$

Hence by the previous corollary of the Cauchy Integral Theorem

$$\begin{aligned} \oint_{|z|=1} \frac{f(z)}{z-a}\, dz = \oint_{|z-a|=\epsilon} \frac{f(z)}{z-a}\, dz = \lim_{\epsilon' \to 0} \oint_{|z-a|=\epsilon'} \frac{f(z)}{z-a}\, dz \\ = \lim_{\epsilon' \to 0} \oint_{|h|=\epsilon'} \frac{f(a+h)}{h}\, dz \ . \end{aligned}$$

To compute this limit we expand

$$f(a+h) = f(a) + f'(a)h + R(h) \quad , \quad \frac{R(h)}{|h|} \xrightarrow{h \to 0} 0 \ .$$

This leads to

$$\begin{aligned} \left| \oint_{|h|=\epsilon} \frac{R(h)}{h}\, dz \right| \leq 2\pi\epsilon \sup \left\{ \left| \frac{R(h)}{h} \right| \,\middle|\, |h| \leq \epsilon \right\} \xrightarrow{\epsilon \to 0} 0 \ , \\ \left| \oint_{|h|=\epsilon} \frac{f'(a)h}{h}\, dz \right| = \left| \oint_{|h|=\epsilon} f'(a)\, dz \right| = 0 \ , \\ \oint_{|h|=\epsilon'} \frac{f(a)}{h}\, dz = \int_0^{2\pi} \frac{f(a)}{e^{it}} ie^{it}\, dt = 2\pi i f(a) \end{aligned}$$

•

Corollary 11.3 (Mean Value Property of Holomorphic Functions) *If $f \in \mathcal{O}\left(\overline{\mathrm{B}}_r(p)\right)$ then $f(p)$ is the average of f over the boundary circle,*

$$f(p) = \frac{1}{\textbf{length}\, \mathrm{S}_r(p)} \int_{\mathrm{S}_r(p)} f = \frac{1}{2\pi r} \int_0^{2\pi} f(p + e^{it})\, dt \ .$$

Problem 11.4 *Deduce the mean value property for harmonic functions on $\mathbb{R}^2$, i.e. if $f \in C^2(\mathbb{R}^2, \mathbb{R})$ is harmonic, $\Delta f = 0$, then for all $p \in \mathbb{R}^2$ and $r > 0$ we have*

$$f(p) = \frac{1}{\textbf{length}\, \mathrm{S}_r(p)} \int_{\mathrm{S}_r(p)} f \ .$$

Proof: We first extend f to a holomorphic function $f + iv$, where v needs to satisfy $v_x = -f_y$ and $v_y = f_x$, i.e. $\mathrm{grad} v = (-f_y, f_x)$. The vector field $(-f_y, f_x)$ on $\mathbb{R}^2$ has a potential v because its curl vanishes,

$$\mathrm{curl}(-f_y, f_x) = f_{xx} + f_{yy} = \Delta f = 0 \ .$$

•

Theorem 11.5 (Power Series, Taylor Series of a Holomorphic Function) *Let $U \subset \mathbb{C}$ be open and $f \in \mathcal{O}(U)$. Let $p \in U$, $r > 0$ and $\overline{\mathrm{B}}_r(p) \subset U$. Then*

1. *there is a unique power series* $\sum_{n=0}^{\infty} c_n(z-p)^n$ *with positive radius of convergence ρ representing f near p, i.e.*
$$\forall z \in \mathrm{B}_\rho(p) : \sum_{n=0}^{\infty} c_n(z-p)^n = f(z) \ ,$$
2. *The radius of convergence of this power series is the radius of the largest ball around p contained in U,*
$$\rho = \sup \left\{ \rho' \mid \mathrm{B}_{\rho'}(p) \subset U \right\} \ ,$$
3. *This power series is the Taylor series of f at p and its coefficients are given by*
$$c_n = \frac{f^{(n)}(p)}{n!} = \frac{1}{2\pi i} \int_{|z-p|=\rho'} \frac{f(z)}{(z-p)^{n+1}} \, dz \ .$$
4. **(Cauchy Estimate of the Taylor Coefficients)**
$$|c_n| \leq \sup_{|z-p|=r} \frac{|f(z)|}{r^n}$$

Problem 11.6 *Compute the Taylor series of* arctan *around* 0.

From the definition, $\arctan(z)$ is the antiderivative of $f(z) = \frac{1}{1+z^2}$ with $\arctan(0) = 0$. We can thus integrate the geometric series "formally",

$$\frac{1}{1+z^2} = \sum_{k=0}^{\infty} (-z^2)^k = \sum_{k=0}^{\infty} (-1)^k z^{2k} \quad \text{for} \quad |z| < 1 \ ,$$
$$\arctan(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} z^{2k+1} \quad \text{for} \quad |z| < 1 \ .$$

Proof: A power series representing f near p is unique if it exists, because by Theorem 9.8 it must be the Taylor series of f.

For the existence of such a power series, we use the Cauchy Integral Formula and the local uniform convergence of
$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \quad \text{for} \quad |a| < 1$$

To simplify notation, we will assume $p = 0$ and $r = 1$. Then for all $q \in \mathrm{B}_1(0)$,

$$\begin{aligned} f(q) &= \frac{1}{2\pi i} \oint_{|z|=1} \frac{f(z)}{z-q}\, dz = \frac{1}{2\pi i} \oint_{|z|=1} \frac{f(z)}{z} \frac{1}{1-\frac{q}{z}}\, dz \\ &= \frac{1}{2\pi i} \oint_{|z|=1} \frac{f(z)}{z} \sum_{n=0}^{\infty} \frac{q^n}{z^n}\, dz \\ &= \sum_{n=0}^{\infty} \underbrace{\left(\frac{1}{2\pi i} \oint_{|z|=1} \frac{f(z)}{z^{n+1}}\, dz \right)}_{c_n} q^n \end{aligned}$$

•

As an immediate corollaries we have

Theorem 11.7 (Goursat's Theorem) *A holomorphic function on domain in $\mathbb{C}$ is infinitely often complex differentiable, in particular (as function $\mathbb{R}^2 \to \mathbb{R}^2$) of class C^∞, smooth.*

Functions $f \in \mathcal{O}(\mathbb{C})$ are called **entire**. If such a function is bounded, say $|f(z)| < M$ for all $z \in \mathbb{C}$, then at all $p \in \mathbb{C}$ the nth Taylor coefficient c_n satisfies

$$|c_n| \le \sup_{|z-p|=r} \frac{|f(z)|}{r^n} \le \frac{M}{r^n} \quad \text{for all} \quad r > 0\ ,$$

hence $c_n = 0$ for all $n > 0$. We have proved

Theorem 11.8 (Liouville's Theorem) *An bounded entire function is constant.*

This also gives the probably shortest proof of the

Theorem 11.9 (Fundamental Theorem of Algebra) *Every complex polynomial of positive degree has at least one zero.*

Proof: Let $f \in \mathbb{C}[z]$, $f(z) = a_n z^n + \cdots + a_0$ with $n > 0$, $a_n \ne 0$ and assume that f has no zero, i.e. $f\colon \mathbb{C} \to \mathbb{C} \setminus \{0\}$. Then $\frac{1}{f}$ is holomorphic. But

$$|f(z)| = \left| z^n \left(a_n + \frac{a_{n-1}}{z} + \cdots \frac{a_0}{z^n} \right) \right| \ge |z|^n \left(|a_n| - \left| \frac{a_{n-1}}{z} \right| - \cdots - \left| \frac{a_0}{z^n} \right| \right) \xrightarrow{z \to \infty} \infty\ .$$

In particular, there is $R > 0$ so that $|f(z)| > 1$ whenever $|z| > R$. On the other hand, $\overline{\mathrm{B}}_R(0) = \{z \mid |z| \le R\}$ is compact, hence $m := \min_{z \le R} |f(z)|$ exists and is positive. Now

$$\min_{z \in \mathbb{C}} |f(z)| = \min \left\{ \min_{z \le R} |f(z)|, \min_{z > R} |f(z)| \right\} \ge \min\{m, 1\} > 0$$

and

$$\max_{z\in\mathbb{C}} \left|\frac{1}{f(z)}\right| = \frac{1}{\min_{z\in\mathbb{C}} |f(z)|}$$

is finite, $\frac{1}{f}$ is bounded.

By Liouville's Theorem $\frac{1}{f}$ is constant, but this implies that f is constant and therefore not of positive degree. •

Theorem 11.10 (Identity Theorem) *Let $U \subset \mathbb{C}$ be a domain (an open and connected subset), $p \in U$, and $f, g \in \mathcal{O}(U)$. Let $A \subset U$ be a subset with an accumulation point* **in** *U and $f|_A = g|_A$. Then $f = g$.*

A topological space X is **connected** if the only open and closed subsets of X are $\emptyset$ and X, or, equivalently, if $X = G \cup H$ with G, H open and disjoint can only be if one of G, H is empty.

For open subsets of $\mathbb{R}^n$ this is equivalent to being **path connected**: A topological space X is **path connected** if for every two points $p, q \in X$ there is a continuous map $c\colon [0,1] \to X$ (a "path"), joining p to q, $p \overset{c}{\rightsquigarrow} q$, i.e. $c(0) = p$, $c(1) = q$.

Proof: The Identity Theorem now follows from the power series expansion of holomorphic functions. In the setting of the theorem, consider the difference $h = f - g \in \mathcal{O}(U)$ and the set

$$G = \left\{ u \in U \,\middle|\, \forall k \in \mathbb{N}_0 : h^{(k)}(u) = 0 \right\} .$$

The set G is closed because all the derivatives $h^{(k)}$ are continuous. On the other hand, if $p \in G$, then all coefficients of the Taylor series of h at p vanish. But then h vanishes with all its derivatives on the ball of convergence of this power series. Thus p is an interior point of G, and we have shown that all points of G are interior, G is open.

By assumption h vanishes on a sequence $(a_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} a_n = q \in U$, $a_n \neq q$ for all n, and, by continuity, $h(q) = 0$ also. Assume there were $k \in \mathbb{N}$ with $h^{(k)}(q) \neq 0$. We may assume that k is minimal with this property and that $q = 0$. Then for some $r > 0$, for all $z \in \overline{\mathrm{B}}_r(0)$ we have,

$$h(z) = z^k \left(c_k + z \sum_{j=0}^{\infty} c_{k+1+j} z^j \right) = z^k (c_k + z g(z)) \neq 0$$

if $z \neq 0$ and $|z| \max_{|\zeta|\le r} g(\zeta) < |c_k|$. Therefore there is a neighbourhood of q where h has no zero other than q. But this contradicts the assumption that q is an acccumulation point of the set of zeros of h. •

Definition 11.11 *Let $U \subset \mathbb{C}$ be open, $p \in U$ and $f \in \mathcal{O}(U)$. Then the order of f at p is*

$$\operatorname{ord}(f, p) = \min \left\{ k \in \mathbb{N} \,\middle|\, f^{(k)}(p) \neq 0 \right\}$$

(with the convention that $\min \emptyset = +\infty$*).*

If the order is finite, then there is $r > 0$ so that $\mathrm{B}_r(p) \subset U$ and for all $z \in \mathrm{B}_r(p)$ we have

$$f(z) = f(p) + (z-p)^{\mathrm{ord}(f,p)} g(z) \tag{11.12}$$

with some $g \in \mathcal{O}(U)$ with $g(p) \neq 0$. Equivalently, the order is

$$\begin{aligned} \mathrm{ord}\,(f,p) &= \max\left\{ k \in \mathbb{N} \,\middle|\, \frac{f(z)-f(p)}{(z-p)^k} \text{ has holomorphic extension to } p \right\} \\ &= \max\left\{ k \in \mathbb{N} \,\middle|\, \exists g \in \mathcal{O}(\mathrm{B}_r(p))\, \forall z \in \mathrm{B}_r(p) : f(z) = f(p) + (z-p)^k g(z) \right\} \end{aligned}$$

Theorem 11.13 (Local form of a holomorphic function) *Let $U \subset \mathbb{C}$ be a domain and $f \in \mathcal{O}(U)$ be not constant. Then for any $p \in U$ there is a neighbourhood $V \subset U$ of p and a biholomorphic map*

$$\phi\colon \mathrm{B}_r(0) \to V \quad \textit{with} \quad \phi(0) = p, \quad \textit{for some} \quad r > 0 \quad \textit{so that}$$

$$f(\phi(z)) = f(p) + z^{\mathrm{ord}(f;p)} \quad \textit{for all} \quad z \in \mathrm{B}_r(0)$$

Proof: Since f is not constant on the domain U, it must have finite order at each point. W.l.o.g. $p = 0 = f(p)$ and $k = \mathrm{ord}(f,0)$, thus

$$f(z) = c_k z^k + \sum_{j=1}^{\infty} c_{k+j} z^{k+j} \quad \text{for all} \quad z \in \mathrm{B}_{r'}(0)$$

with some c_j, $c_k \neq 0$ and $r' > 0$. We may further assume $c_k = 1$ so that

$$f(z) = z^k + \sum_{j=1}^{\infty} c_{k+j} z^{k+j} = z^k g(z)$$

for all $z \in \mathrm{B}_r(0)$ and some $g \in \mathcal{O}(\mathrm{B}_{r'}(0))$ with $g(0) = 1$. By continuity of g there is $r'' \leq r'$ so that $g(\mathrm{B}_{r''}(0)) \subset \mathrm{B}_1(1)$. Since ln is given by the power series (9.10) on $\mathrm{B}_1(1)$, ln is holomoprhic there and we have

$$f(z) = \left(z e^{\frac{1}{k}\ln(g(z))} \right)^k = (h(z))^k \quad \text{for all} \quad |z| < r''$$

with some function $h \in \mathcal{O}(\mathrm{B}_{r''}(0))$, $h(0) = 0$, $h'(0) = 1$.

By the Inverse Function Theorem the function h is a local diffeomorphism near 0. Thus, there is an open neighbourhood $V \subset \mathrm{B}_{r''}(0)$, $r > 0$ and a (real) differentiable map $\phi\colon \mathrm{B}_r(0) \to V$. The derivative of ϕ is the inverse of the derivative of h, hence complex multiplication

$$\mathrm{d}_y \phi = \frac{1}{h'(\phi(y))}$$

hence $\phi\colon \mathrm{B}_r(0) \to V$ is biholomorphic and we have

$$f(\phi(z)) = z^k \quad \text{for all} \quad z \in \mathrm{B}_r(0)\ .$$

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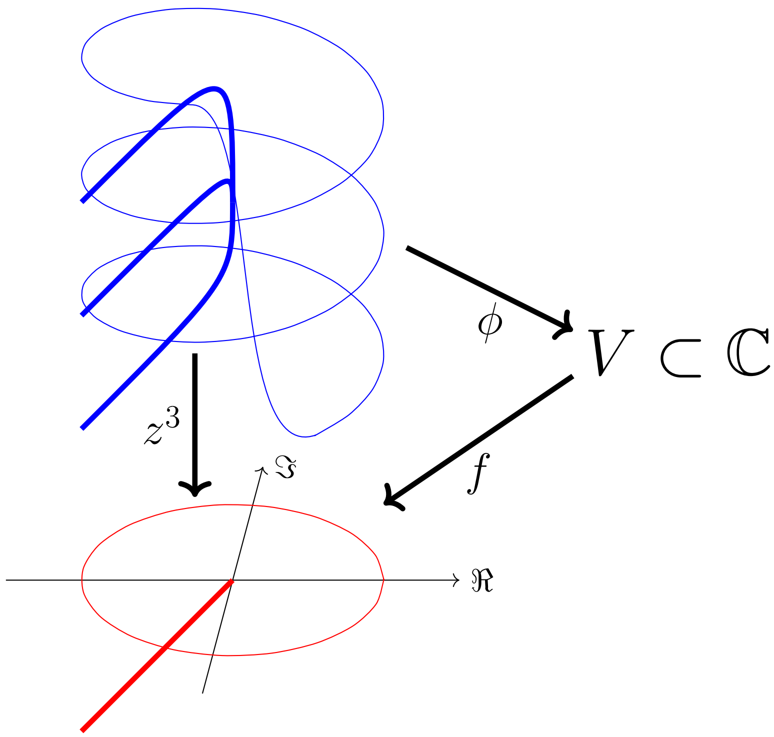


A holomorphic function near a point of multiplicity 3

Corollary 11.14 (Leaves near a point of higher order) *If $U \subset \mathbb{C}$ is open, $f \in \mathcal{O}(U)$ and $p \in U$ with $\operatorname{ord}(f,p) = k < \infty$. Then there is a neighbourhood $V \subset U$ of p such that*

$$\#(f^{-1}(x) \cap V) = \begin{cases} 1 & \text{if } x = p \\ k & \text{if } x \in V \setminus \{p\} \end{cases} .$$

Proof: By translating the problem (i.e. replace $f(z)$ by $f(p+z) - f(p)$), we can assume that $p = f(p) = 0$. By Theorem 11.13 we may further assume that $f(z) = z^k$. •

Problem 11.15 *Recall that a map $f: X \to Y$ between topological spaces is open if $f(U) \subset Y$ is open whenever $U \subset X$ is open. Show that for all $k \in \mathbb{N}$, the kth power map $\mathbb{C} \xrightarrow{z \mapsto z^k} \mathbb{C}$ is open.*

Corollary 11.16 (Open Mapping Theorem) *If $U \subset \mathbb{C}$ is a domain and $f \in \mathcal{O}(U)$ is nonconstant, then $f(U)$ is also a domain. In particular f is open.*

Proof: By continuity, $f(U)$ is connected. By the Theorem 11.13 on the local form of a holomorphic function, every point $p \in U$ has a neighbourhood $V \subset U$ with a biholomorphic map $\phi\colon \mathrm{B}_r(0) \xrightarrow{\cong} V$, so that $f(\phi(z)) = f(p) + z^{\operatorname{ord}(f,p)}$, hence $f(y) = f(p) + (\phi^{-1}(y))^{\operatorname{ord}(f,p)}$ for all $y \in V$. In particular

$$f(V) = f(p) + \left(\phi^{-1}(y)\right)^{\operatorname{ord}(f,p)} = f(p) + \mathrm{B}_r(0)^{\operatorname{ord}(f,p)} = \mathrm{B}_{r^{\operatorname{ord}(f,p)}}(f(p))$$

is open. •

Corollary 11.17 (Maximum Principle) *The absolute value of a nonconstant holomophic function on a domain does not assume its maximum on the domain.*

Proof: If $f \in \mathcal{O}(U)$ is not constant, then $f: U \to \mathbb{C}$ is open. Since the modulus $|\cdot| : \mathbb{C} \to \mathbb{R}_0^+$ is an open function, so is the composition $|f| : U \to \mathbb{R}_0^+$. Therefore $|f(U)| \subset \mathbb{R}_0^+$ is open, hence $\sup |f(U)| \notin |f(U)|$. •

This can be rephrased

Corollary 11.18 (Maximum Principle) *The absolute value of a nonconstant holomophic function on a connected compact subset of* $\mathbb{C}$ *assumes its maximum on the boundary of the subset.*

Since $\Re, \Im \colon \mathbb{C} \to \mathbb{R}$ are all open, we have statements similar to 11.17, 11.18.

Vanishing of the complex line integral characterizes holomorphic functions:

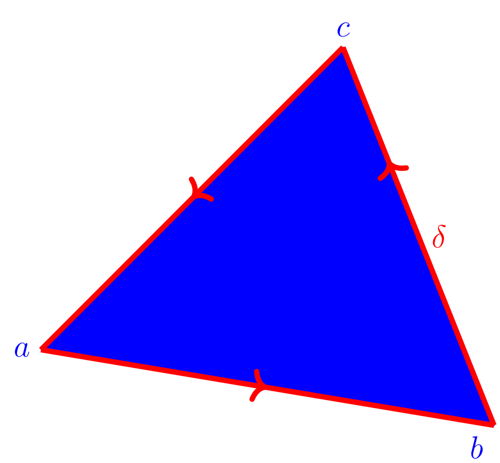
Theorem 11.19 (Morera's Theorem) *Let* $U \subset \mathbb{C}$ *be open,* $f: U \to \mathbb{C}$ *continuous. Assume that for every three points* (a, b, c), $a, b, c \in \mathbb{C}$, *whose convex hull* $\Delta(a, b, c)$ *lies in* U *we have that* $\oint_\delta f(z)\, dz = 0$, *where* δ *is the boundary curve of the triangle* $\Delta(a, b, c)$. *Then* $f \in \mathcal{O}(U)$.

A triangle is the convex hull of three points. Thus if $a, b, c \in \mathbb{C}$, then the triangle spanned by a, b, c is

$$\Delta(a, b, c) = \left\{ \alpha a + \beta b + \gamma c \,\middle|\, \alpha, \beta, \gamma \in \mathbb{R}_0^+, \alpha + \beta + \gamma = 1 \right\} .$$

The boundary curve of $\Delta = \Delta(a, b, c)$ is the concatenation of the edges of the triangle,

$$\delta(a, b, c)(t) = \begin{cases} a + t(b - a) & \text{if } 0 \le t \le 1 \\ b + (t-1)(c - b) & \text{if } 1 \le t \le 2 \\ c + (t-2)(a - c) & \text{if } 2 \le t \le 3 \end{cases} . \tag{11.20}$$



Problem 11.21 *There is some ambiguity here. First the triangle might be degenerate, for instance b might be on the segment $[a,c]$. Even if the triangle is not degenerate, one might reverse the orientation. Check that none of these matter here. Find a formula $X(a,b,c)$ determining the orientation of the curve* (11.20) *from the coordinates of a, b, c, if the points do not lie on a common line.*

Proof: Since holomorphicity is local, we may assume that $U = \mathrm{B}_r(p)$ for some $p \in \mathbb{C}$, $r > 0$, and we may also assume $p = 0$, $r = 1$, i.e. $U = \mathrm{B} = \mathrm{B}_1(0)$. We will define an antiderivative F of f. Then the theorem follows from Goursat's Theorem.

Up to a constant, an antiderivative of F must be given by line integrals over segments emanating in 0,

$$F(q) := \int_{\ell_q} f(z)dz = \int_0^1 f(tq)q\ dt$$

with $\ell_q(t) = tq$ for $t \in [0,1]$. We need to show that $F' = f$. To this end, for h so small that $p + h \in \mathrm{B}$,

$$\begin{aligned}\frac{F(p+h) - F(p)}{h} &= \frac{1}{h}\left(\int_{\ell_q} f(z)dz - \int_{\ell_{q+h}} f(z)dz\right)\\ &= \frac{1}{h}\int_{q \rightsquigarrow q+h} f(z)\ dz \quad \text{since} \quad \ell_q * (q \rightsquigarrow q+h) * \ell_{q+h}^{-1} \quad \text{bounds a triangle in} \quad \mathrm{B}\\ &= \frac{1}{h}\int_0^1 f(q+th)h\ dt\\ &= \int_0^1 f(q+th)\ dt \xrightarrow{h\to 0} f(q)\end{aligned}$$

because f is continuous. •

<u>**Notation:**</u> $\mathcal{H} = \{z \in \mathbb{C} \mid \Im z > 0\}$, the upper half plane.

Theorem 11.22 (Schwarz Reflection Principle) *Let $U \subset \mathbb{C}$ be open and invariant under complex conjugation, i.e. $z \in U \implies \overline{z} \in U$. Let f be a continuous function on U which is holomorphic on $U \cap \mathcal{H} = \{z \in U \mid \Im z > 0\}$ and $f(U \cap \mathbb{R}) \subset \mathbb{R}$. Then the function F on U defined by*

$$F(z) := \begin{cases} \overline{f(\overline{z})} \text{ if } \Im z \leq 0 \\ f(z) \text{ if } \Im z \geq 0 \end{cases} \tag{11.23}$$

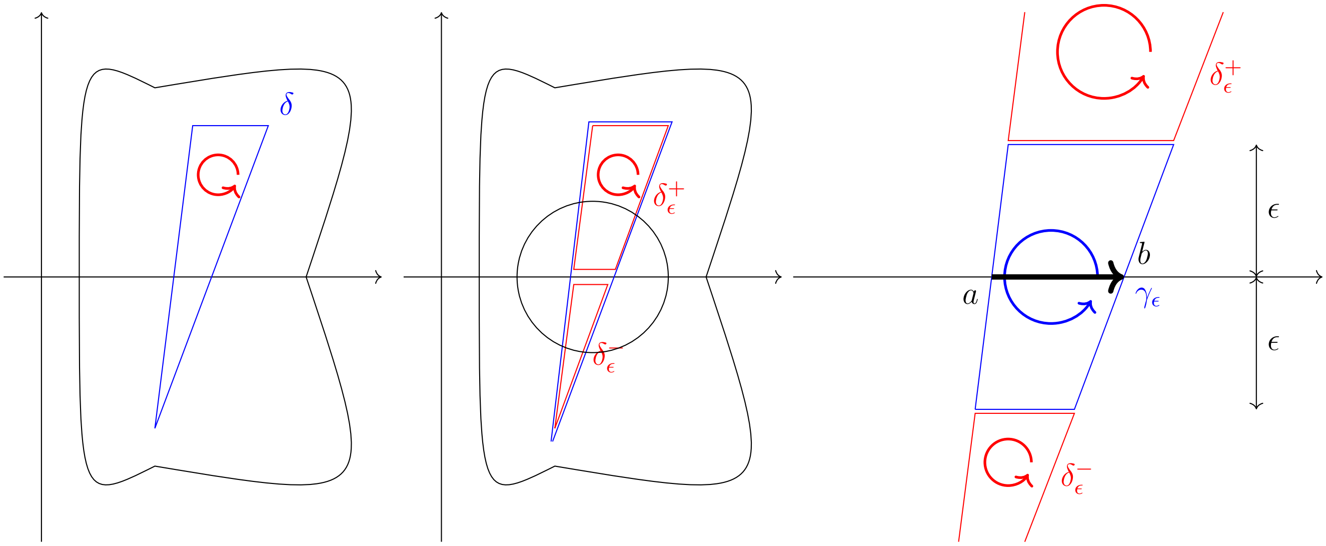
is holomorphic on all of U.

Problem 11.24 *Let $U \subset \mathbb{C}$ be open, $f\colon U \to \mathbb{C}$ be a holomorphic function, $cU = \{\overline{u} \mid u \in U\}$ the complex conjugate of the set U and $cfc\colon cU \to \mathbb{C}$ be the conjugate of f with complex conjugation, i.e.*

$$cfc(x) = \overline{f(\overline{x})}\ .$$

Prove that cU is open and that $cfc \in \mathcal{O}(cU)$.

Proof: You have just shown that the restriction $F|_{U\cap c\mathcal{H}}$ is holomorphic. Since the two definitions in (11.23) coincide on $\mathbb{R}\cap U$, F is continuous. In order to apply Morera's Theorem we need to prove that complex line integrals of F over triangles in U vanish. For triangles lying entirely in the upper or in the lower half plane this is follows from Cauchy's Integral Theorem, because F is holomorphic there. If the triangle does not lie in the upper nor in the lower half plane, we split it in a triangle and a quadrilateral,



Then

$$\oint_\delta f(z)\ dz = \oint_{\delta_\epsilon^+} f(z)\ dz + \oint_{\delta_\epsilon^-} f(z)\ dz + \oint_{\gamma_\epsilon} f(z)\ dz \xrightarrow{\epsilon\to 0} 0$$

The integrals over δ_ϵ^+ and δ_ϵ^- vanish for all positive ϵ by the Cauchy Integral Theorem. The limit of the integral over γ_ϵ converges,

$$\lim_{\epsilon\to 0}\oint_{\gamma_\epsilon} f(z)\ dz = \int_a^b f(x)\ dx + \int_b^a f(x)dx = 0\ .$$

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12 Isolated Singularities

If $p \in U \overset{\text{open}}{\subset} \mathbb{C}$ and $f \in \mathcal{O}(U\setminus p)$, then one says "$p$ is an **isolated singularity**" of f. The isolated singularity p is **removable** if f has a holomorphic extension $\tilde{f}$ to U, i.e $\tilde{f}\in\mathcal{O}(U)$, $\tilde{f}|_{U\setminus\{p\}} = f$. p is a **pole** if p is not removable and there is some $k\in\mathbb{N}$ so that p is a removable singularity of the function $\tilde{f}$ with $\tilde{f}(z) = (z-p)^k f(z)$. The smallest such k is the **order of the pole**. An **essential singularity** is an isolated singularity that is neither removable nor a pole.

Example 12.1 *The point $p = 0$ is a removable singularity of $f(z) = \frac{\sin(z)}{z}$. In fact, the holomorphic extension is given by the power series*

$$\tilde{f}(z) = \sum_{k=0}^{\infty}\frac{(-1)^k z^{2k}}{(2k+1)!}\ .$$

Similarly, the point 0 is a removable singularity for functions like

$$\frac{e^z - 1 - z - \frac{1}{2}z^2}{z^3} \quad , \quad \frac{1 - \cos(z)}{z^2} \ ,$$

and the power series of the holomorphic extension is easily derived. The value of the holomorphic extension at the removable singularity of a function f is denoted by

$$f(z)|_{z=p} \ .$$

For example

$$\left.\frac{\sin(z)}{z}\right|_{z=0} = 1 \quad , \quad \left.\frac{e^z - 1 - z - \frac{1}{2}z^2}{z^3}\right|_{z=0} = \frac{1}{6} \quad , \quad \left.\frac{1 - \cos(z)}{z^2}\right|_{z=0} = \frac{1}{2} \ .$$

Usually we will not distinguish between the function and its extension over a removable singularity.

Example 12.2 *The prototypical example of a pole of of order $k \in \mathbb{N}$ at p is the function*

$$f(z) = \frac{1}{(z-p)^k} \ .$$

Example 12.3 *The points $p = k\pi$, $k \in \mathbb{Z}$, are poles of order 1 of the function $f \in \mathcal{O}(\mathbb{C} \setminus 2\pi\mathbb{Z})$ with*

$$f(z) = \frac{1}{\sin(z)} \ .$$

These singularities are not removable since for any $k \in \mathbb{Z}$, the limit $\lim_{z \to 2\pi k} f(z)$ does not exist. On the other hand, since $\sin(\pi k + z) = (-1)^k \sin(z)$, we have

$$(z - \pi k)f(z) = (-1)^k \frac{z - \pi k}{\sin z - \pi k}$$

whose singularity at πk is removable,

$$(-1)^k \left.\frac{z - \pi k}{\sin z - \pi k}\right|_{z=\pi k} = (-1)^k \ .$$

Example 12.4 *The function $f(z) = e^{\frac{1}{z}}$ has an essential singularity at $p = 0$, because there is no $k \in \mathbb{N}$ so that*

$$z^k e^{\frac{1}{z}}$$

is bounded near $z = 0$. Similarly, $z = 0$ is an essential singularity of

$$\sin(1/z) \quad \textit{and} \quad \cos(1/z) \ .$$

12.1 Laurent Series

For $r, R \in \mathbb{R}_0^+$, $r \leq R$, the concentric annulus is

$$A_{r,R}(p) = \{z \in \mathbb{C} \mid r < |z-p| < R\} \ .$$

The following is analogous to 9.8 and 11.5.

Theorem 12.5 *Let* $A_{r,R}(p) \subset U \overset{open}{\subset} \mathbb{C}$ *and* $f \in \mathcal{O}(U)$*. Then there is a unique sequence* $(c_n)_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ *so that*

$$f(z) = \sum_{n\in\mathbb{Z}} c_n (z-p)^n \ . \tag{12.6}$$

This **Laurent Series** *converges absolutely and uniformly on every compact subset of* $A_{r,R}(p)$*. The derivative of* f *is*

$$f'(z) = \sum_{n\in\mathbb{Z}} n c_n (z-p)^{n-1} \ .$$

The Laurent coefficients c_n *are given by the formula*

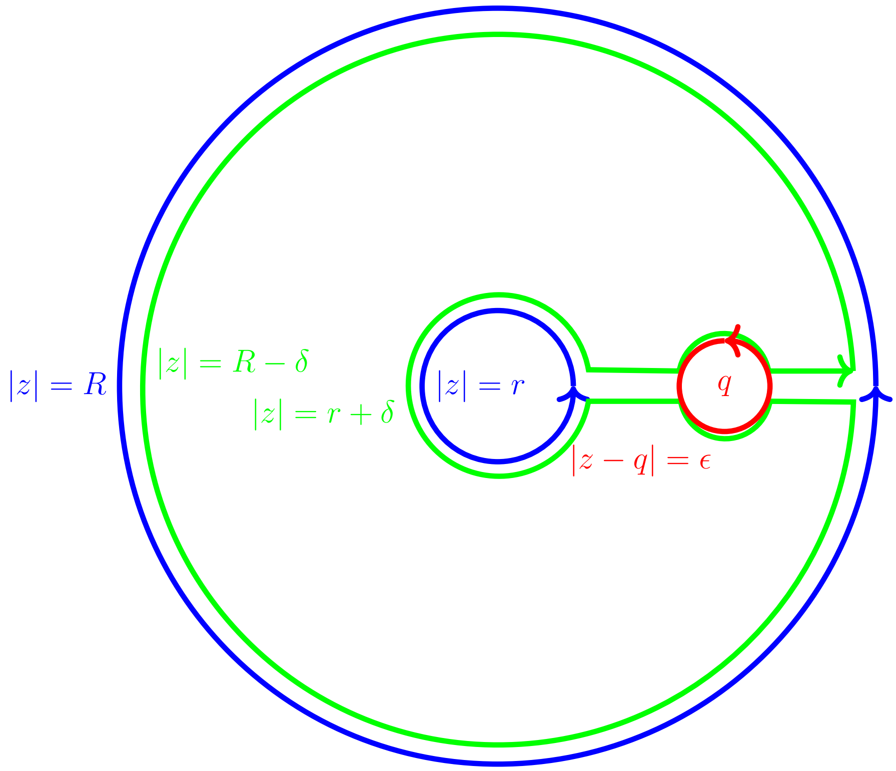
$$c_n = \frac{1}{2\pi i} \oint_{|z-p|=\rho} \frac{f(z)}{(z-p)^n} \, dz \quad \textit{for any} \quad \rho \in (r, R) \ . \tag{12.7}$$

Proof: We may assume $p = 0$. Let $q \in A_{r,R}(0)$ and choose δ so that

$$r < r + \delta < |q| < R - \delta < R \ .$$

By continuity of f, for $\epsilon > 0$ so that $\mathrm{B}_\epsilon \subset A_{r,R}(0)$ we have

$$f(q) = \lim_{\rho\to 0} \frac{1}{2\pi i} \oint_{|z-q|=\rho} \frac{f(z)}{z-q} \, dz = \oint_{|z-q|=\epsilon} \frac{f(z)}{z-q} \, dz \ .$$



By additivity of the complex line integral and the Cauchy intergral theorem for rectangular domains,

$$0 = \oint_{\gamma} \frac{f(z)}{z-q}\, dz = \oint_{|z|=R-\delta} \frac{f(z)}{z-q}\, dz - \oint_{|z|=r+\delta} \frac{f(z)}{z-q}\, dz - \oint_{|z-q|=\epsilon} \frac{f(z)}{z-q}\, dz$$

Thus

$$2\pi i f(q) = \oint_{|z|=R-\delta} \frac{f(z)}{z-q}\, dz - \oint_{|z|=r+\delta} \frac{f(z)}{z-q}\, dz\ . \tag{12.8}$$

As in the proof of Theorem 11.5, we now use the Taylor series for $\frac{1}{1-z}$, this time at 0 and ∞,

$$\frac{1}{1-z} = \begin{cases} \displaystyle\sum_{k=0}^{\infty} z^k & \text{if } |z| < 1 \\ \displaystyle -\sum_{k=1}^{\infty} z^{-k} & \text{if } |z| > 1 \end{cases}\ .$$

Since $r+\delta<|q|<R-\delta$ we can expand the integrals in (12.8),

$$\begin{aligned}
2\pi i f(q) &= \oint_{|z|=R-\delta} \frac{f(z)}{z-q}\,dz - \oint_{|z|=r+\delta} \frac{f(z)}{z-q}\,dz \\
&= \oint_{|z|=R-\delta} \frac{f(z)}{z}\frac{1}{1-\frac{q}{z}}\,dz - \oint_{|z|=r+\delta} \frac{f(z)}{z}\frac{1}{1-\frac{q}{z}}\,dz \\
&= \oint_{|z|=R-\delta} \frac{f(z)}{z}\sum_{k=0}^{\infty}\left(\frac{q}{z}\right)^k\,dz + \oint_{|z|=r+\delta} \frac{f(z)}{z}\sum_{k=1}^{\infty}\left(\frac{q}{z}\right)^{-k}\,dz \\
&= \sum_{k=0}^{+\infty}\left(\oint_{|z|=R-\delta}\frac{f(z)}{z^{k+1}}\,dz\right)q^k + \sum_{k=1}^{\infty}\left(\oint_{|z|=r+\delta}\frac{f(z)}{z^{k+1}}\,dz\right)q^{-k} \\
&= \sum_{k=-\infty}^{+\infty}\underbrace{\left(\oint_{|z|=\rho}\frac{f(z)}{z^{k+1}}\,dz\right)}_{2\pi i c_k}q^k
\end{aligned}$$

for all ρ, $r<\rho<R$, since the integrands are holomorphic on $A_{r,R}(0)$ which shows (12.7). •

Corollary 12.9 (Estimate of the Laurent Coefficients) *(in the notation of Theorem 12.5) For all $\rho\in(r,R)$,*

$$|c_k| \leq \frac{\max\{f(z) \mid |z-p|=\rho\}}{\rho^k}$$

Corollary 12.10 (Principal Part) *If $r,R\in[0,\infty]$, $r<R$, $p\in\mathbb{C}$ and $f\in\mathcal{O}(A_{r,R}(p))$, then*

$$f(z)=N(z-p)+H\left(\frac{1}{z-p}\right) \quad \textit{with} \quad N\in\mathcal{O}(\mathrm{B}_R(0)) \quad \textit{and} \quad H\in\mathcal{O}\left(\mathrm{B}_{1/r}(0)\right) \quad \textit{and} \quad H(0)=0\,.$$

H is called the **principal part of f at** *p.*

In the case $r=0$, the point p is an isolated singularity. The type of this singularity is determined by the principal part H, because N is holomorphic there.

Corollary 12.11 *Let $f\in\mathcal{O}(A_{0,R}(p))$. Let H be the principal part of f at p. Then*

1. *p is a removable singularity if $H=0$.*
2. *p is a pole of order k if H is a polynomial of degree k.*
3. *p is essential if H is not a polynomial.*

Since the first case in Corollary 12.11 is the only one where the function is bounded in a vicinity of the singularity, this immediately gives

Theorem 12.12 (Riemann Removable Singularities Theorem) *If $\Omega \subset \mathbb{C}$ is open, $p \in \Omega$, and $f \in \mathcal{O}(\Omega) \setminus \{p\}$ is bounded, then p is removable and thus f extends to a holomorphic functon on all of Ω.*

Theorem 12.13 (Casorati-Weierstrass) *Let $U \subset \mathbb{C}$ be open, $p \in U$ and $f \in \mathcal{O}(U \setminus \{p\})$. Then p is an essential singularity if and only if $f(U \setminus \{p\})$ is dense in $\mathbb{C}$.*

By **Picard's Theorem**, $\mathbb{C} \setminus f(U \setminus \{p\})$ contains at most one point.

Proof: If $f(U \setminus \{p\})$ is not dense in $\mathbb{C}$, then there is $q \in \mathbb{C}$ and $r > 0$ so that $f(U \setminus \{p\}) \subset \mathbb{C} \setminus \mathrm{B}_r(q)$. Thus $\left|\frac{1}{f(z)-q}\right| < \frac{1}{r}$, hence $\frac{1}{f(z)-q}$ has a removable singularity at p. •

Thus for an isolated singularity $p \in U \overset{\text{open}}{\subset} \mathbb{C}$ of a function $f \in \mathcal{O}(\mathrm{B}_r(p) \setminus \{p\})$ there are three possibilities:

1. <u>*removable:*</u> f is bounded near p,
   f extends holomorphically to p,
   $\lim_{z\to p} f(z)$ exists.
2. <u>*pole:*</u> for some $k \in \mathbb{N}$,
   p is a removable singularity of $(z-p)^k f(z)$,
   $\lim_{z\to p} f(z) = \infty$,
   $\frac{1}{f(z)}$ is bounded near p.
3. <u>*essential:*</u> Neither $f(z)$ nor $\frac{1}{f(z)}$ is bounded near p,
   Neither $\lim_{z\to p} f(z)$ nor $\lim_{z\to p} \frac{1}{f(z)}$ exist
   $f(\mathrm{B}_\rho(p) \setminus \{p\}) \subset \mathbb{C}$ is dense for all $0 < \rho \leq r$.

12.2 The Riemann Sphere, Meromorphic functions

Definition 12.14 *A* **meromorphic function** *on an open subset $U \subset \mathbb{C}$ is a function $f \in \mathcal{O}(U \setminus S)$ where $S \subset U$ is a discrete subset and all $s \in S$ are poles (of finite positive order). We denote the set of meromorphic functions on U by $\mathcal{M}(U)$.*

Thus meromorphic functions are functions without essential singularities. If $f \in \mathcal{M}(U)$ then for each $p \in U$, f is holomorphic in a neighbourhood of p or p is a removable singularity of $\frac{1}{f}$.

Let $\mathcal{S} = \mathbb{C} \cup \infty$ be a one point compactification of $\mathbb{C}$. Thus $\infty \notin \mathbb{C}$ and the topology of S is

$$\{U \mid U \subset \mathbb{C} \text{ open}\} \cup \{\mathcal{S} \setminus K \mid K \subset \mathbb{C} \text{ compact}\} .$$

This is the topology induced by the maps

$$z\colon \mathbb{C} \xrightarrow{z\mapsto z} \mathcal{S} \quad \text{and} \quad \frac{1}{z}\colon \mathbb{C} \xrightarrow{z\mapsto \frac{1}{z}, 0\mapsto\infty} \mathcal{S} .$$

Definition 12.15 *A map $f\colon U \to \mathcal{S}$, $U \subset \mathcal{S}$, is holomorphic if its compositions*

$$z \circ f \circ z \ , \ \frac{1}{z} \circ f \circ z \ , \ z \circ f \circ \frac{1}{z} \ , \ \frac{1}{z} \circ f \circ \frac{1}{z}$$

are holomorphic where defined. We write

$$\mathcal{O}(U, \mathcal{S}) = \{ f\colon U \to \mathcal{S} \,|\, f \ \text{a holomorphic map} \} \ .$$

We have a natural injection

$$\mathcal{M}(U) \hookrightarrow \mathcal{O}(U, \mathcal{S})$$

missing only those holomorphic maps $f\colon U \to \mathcal{S}$ which are constant ∞ on a component of U.

Example 12.16 *If $f \in \mathcal{O}(U \setminus \{p\})$, $p \in U \subset \mathbb{C}$, and p is a pole of order k, then $\frac{1}{f(z)}$ has a zero of order k at p.*

A polynomial of degree d has (a pole of) order d at ∞. Since $p(\infty) = \infty$, we need to look at $1/p(1/z)$,

$$\operatorname{ord}(p(z); z = \infty) = \operatorname{ord}\left(\frac{1}{p\left(\frac{1}{z}\right)}; z = 0 \right) = d$$

because

$$\frac{1}{a_d \left(\frac{1}{z}\right)^d + a_{d-1}\left(\frac{1}{z}\right)^{d-1} + \ldots a_1 \left(\frac{1}{z}\right) + a_0} = \frac{z^d}{a_d + a_{d-1}z + \ldots a_1 z^{d-1} + a_0 z^d} \quad , \quad a_d \neq 0 \ .$$

A holomorphic map $f\colon S \to S$ is a rational function, i.e. of the form $f(z) = \frac{p(z)}{q(z)}$ with polynomials p and q.

Meromorphic functions $U \to \mathbb{C}$ are holomoprhic maps $U \to S$.

13 More Problems

1. Compute

$$\int_0^{2\pi} \frac{1}{1 + \sin(t)^2} \, dt \ .$$

Solution: We can rewrite this as a complex line integral over a closed curve and apply the Residue Theorem,

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{1+\sin(t)^2}\, dt &= \int_0^{2\pi} \frac{1}{1+\left(\frac{e^{it}-e^{-it}}{2i}\right)^2} \frac{1}{ie^{it}} ie^{it}\, dt \\
&= \oint_{|z|=1} \frac{1}{1+\left(\frac{z-\frac{1}{z}}{2i}\right)^2} \frac{1}{iz}\, dz \\
&= \oint_{|z|=1} \frac{4iz}{-4z^2+z^4-2z^2+1}\, dz \\
&= \oint_{|z|=1} \frac{4iz}{z^4-6z^2+1}\, dz \\
&= \oint_{|z|=1} \underbrace{\frac{4iz}{(z^2-(3-2\sqrt{2}))(z^2-(3+2\sqrt{2}))}}_{J(z)}\, dz \\
&= 2\pi i \left(\mathbf{Res}\left(J(z); z=\sqrt{3-2\sqrt{2}}\right) + \mathbf{Res}\left(J(z); z=-\sqrt{3-2\sqrt{2}}\right)\right) \\
&= 2\pi i \left(\frac{4i\sqrt{3-2\sqrt{2}}}{2\sqrt{3-2\sqrt{2}}\,((3-2\sqrt{2})-(3+2\sqrt{2}))} + \frac{-4i\sqrt{3-2\sqrt{2}}}{-2\sqrt{3-2\sqrt{2}}\,((3-2\sqrt{2})-(3+2\sqrt{2}))}\right) \\
&= 2\pi i \left(\frac{4i}{2\,(-4\sqrt{2})} + \frac{-4i}{-2\,(-4\sqrt{2})}\right) = \pi\sqrt{2}\ .
\end{aligned}$$

2. Compute the integral
$$\int_{-\infty}^{\infty} \frac{\sin(t)}{t}\, dt\ .$$

Hint: Close the path of integration as in the picture below. Choose the relation between h and r so that the contribution of the integrals over β_*, ω_* and μ_* can be neglected in the limit $r \to \infty$. Compare the integral over α_r with that over the full circle.

Solution: This is the imaginary part of
$$\lim_{r\to\infty}\left(\int_{-r}^{-1/r} \frac{e^{it}}{t}\, dt + \int_{1/r}^{r} \frac{e^{it}}{t}\, dt\right) = \lim_{r\to\infty}\left(\int_{\gamma_r} \frac{e^{iz}}{z}\, dz + \int_{\delta_r} \frac{e^{iz}}{z}\, dz\right)$$

Here γ_r, δ_r, α_r, $\beta_{r,h}$ are the curves

$$\begin{aligned}
\gamma_r &: [-r,-1/r] \to \mathbb{C} \quad , \quad \gamma_r(t) = t\ , \\
\delta_r &: [1/r, r] \to \mathbb{C} \quad , \quad \delta_r(t) = t\ , \\
\alpha_r &: [0,\pi] \to \mathbb{C} \quad , \quad \alpha(t) = \frac{1}{r}e^{it} \\
\beta_{r,h} &: [-r,r] \to \mathbb{C} \quad , \quad \beta(t) = ih + t \\
\omega_{r,h} &: [0,h] \to \mathbb{C} \quad , \quad \omega(t) = -r + it \\
\mu_{r,h} &: [0,h] \to \mathbb{C} \quad , \quad \mu(t) = r + it
\end{aligned}$$

as in the picture below. Thus $\gamma_r * \alpha_r^{-1} * \delta_r * \mu_{r,h} * \beta_{r,h}^{-1} * \omega_{r,h}$ is closed. We estimate the integrals over the auxilliary curves:
$$\left|\int_{\omega_{r,h}} \frac{e^{iz}}{z}\, dz\right|, \left|\int_{\mu_{r,h}} \frac{e^{iz}}{z}\, dz\right| \leq \frac{h}{r}$$

$$\left| \int_{\beta_{r,h}} \frac{e^{iz}}{z} \, dz \right| \leq 2re^{-h}$$

If we choose $r = h^2$, both of these converge to 0 for $h \to \infty$.

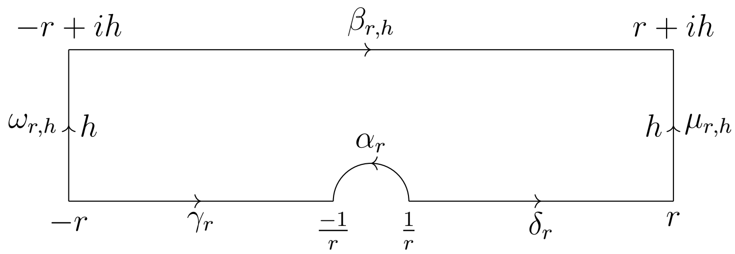
$$\lim_{r \to \infty} \int_{\alpha_r} \frac{e^{iz}}{z} \, dz = \frac{1}{2} \lim_{\epsilon \to 0} \int_{|z|=\epsilon} \frac{e^{iz}}{z} \, dz = \pi i$$

Since

$$\oint_{\gamma_r * \alpha_r^{-1} * \delta_r * \mu_{r,h} * \beta_{r,h}^{-1} * \omega_{r,h}} \frac{e^{iz}}{z} \, dz = 0$$

this yields

$$\int_{-\infty}^{\infty} \frac{\sin(t)}{t} \, dt = \pi \ .$$



3. Throughout this problem, zeros are counted with multiplicity. How many zeros has the polynomial $p(z) = z^5 + 9z^2 + 3$ in

   (a) $\mathrm{B}_2(0)$?

   (b) $A_{2,3}(0) = \{z \in \mathbb{C} \mid 2 < |z| < 3\}$?

   Solution: If $|z| = 2$ then
   $$\left|z^5 + 3\right| \leq 2^5 + 3 = 35 < 9\left|z^2\right| = 36 \ .$$
   By Rouché's Theorem 2.26, p has as many zeros in $\mathrm{B}_2(0)$ as z^2, i.e. 2.

   If $|z| = 3$ then
   $$\left|9z^2 + 3\right| \leq 81 + 3 = 84 < \left|z^5\right| = 3^5 = 243 \ .$$
   By Rouché's Theorem, p has as many zeros in $\mathrm{B}_3(0)$ as z^5, i.e. 5. Since none of these are on $S_2(0)$, we have 3 zeros in the annulus $A_{2,3}(0)$.

4. Show that the function $f \in \mathcal{O}(\mathbb{C})$ with $f(z) = e^z + z^5 + 1$ has exactly three zeros with negative real part.

   Hint: Compare $|e^z| = e^{\Re(z)}$ and $|z^5 + 1|$ for z imaginary and for $\Re(z) \leq 0$, $|z| = 4056$. The estimate is in your favour in order to apply Rouché's Theorem 2.26, except at $z = 0$. Use the zero counting winding number directly.

   Solution: Let γ be the boundary curve of $\Omega = \{z \in \mathbb{C} \mid \Im(z) \leq 0 \ , |z| < 7000\}$. For z on γ we have $|z^5 + 1| > |e^z|$ except at $z = 0$, so we can not use Rouché's Theorem directly. But as for the proof of that theorem, we look at the zero's counting winding number $w(f \circ \gamma, 0)$. Consider the homotopy
   $$H(s,t) = se^{\gamma(t)} + \gamma(t)^5 + 1 \ .$$

For $s \in [0,1]$, if $\gamma(t) \neq 0$ then $H(s,t) \neq 0$ because of the above estimates. If $\gamma(t) = 0$, then $H(s,t) = s+1$, by direct computation. Thus, in all cases $H\colon [0,1] \times [0,1] \to \mathbb{C} \setminus \{0\}$ is a homotopy relative endpoint of $f \circ \gamma$ with $g \circ \gamma$, $g(z) = z^5 + 1$ avoiding 0. Thus

$$w(f \circ \gamma, 0) = w(g \circ \gamma, 0) = 3$$

because $z^5 + 1$ has three simple zeros in the left half plane.

5. Consider the polynomial

$$p(z) = z^9 + 10z^6 + 15z^3 + 1 \ .$$

Determine the number of zeros of p in each of

$$\mathrm{B}_1(0) \quad , \quad \mathrm{B}_2(0) \quad , \quad \mathrm{B}_3(0) \quad \text{and} \quad \mathbb{C} \setminus \mathrm{B}_3(0) \ .$$

Hint: First count with multiplicities. Why does p not have higher order zeros?

Solution: We look at the boundaries of these discs:

(a) $|z = 1|$: Then $|15z^3| = 15 > 1 + 10 + 1 \geq |z^9 + 10z^9 + 1|$, hence $p(z)$ has as many zeros with $|z| < 1$ as z^3, i.e. 3.

(b) $|z| = 2$: Then $|10z^6| = 10 \cdot 2^6 > 8 \cdot 2^6 + 15 \cdot 2^3 + 1 \geq |z^9 + 15z^3 + 1|$, hence $p(z)$ has as many zeros with $|z| < 2$ as z^6, i.e. 6.

(c) $|z| = 3$: Then $|z^9| = 27 \cdot 3^6 > 10 \cdot 3^6 + 15 \cdot 3^3 + 1 \geq |10z^6 + 15z^3 + 1|$, hence $p(z)$ has as many zeros with $|z| < 3$ as z^9, i.e. 9.

(d) $|z > 3|$: Since $p(z)$ has degree 9 and all 9 zeros are in $\mathrm{B}_3(0)$, there are no zeros in $\mathbb{C} \setminus \mathrm{B}_3(0)$.

We thus have three zeros in each of the three annuli

$$A_{0,1}, A_{1,2}, A_{2,3} \ .$$

Since $p(z) = q(z^3)$ for some degree 3-polynomial q, if z is a zero the so are ϵz and $\epsilon^2 z$. Since 0 is not a zero, These are pairwise different, and have the same modulus, hence lie in the same annulus above.

6. Find a biholomorphic map

$$\Omega := \{z \in \mathbb{C} \,|\, 0 < \Re(z) \ , \ 0 < \Im(z) < \pi\} \xrightarrow{\phi} \Psi := \{z \in \mathbb{C} \,|\, 0 < \Im(z) < 2\pi\} \ .$$

Hint: power functions, exponential/log, Möbius transformations.

Solution:

$$\Omega \xrightarrow{\exp} \{|z| > 1, \Im(z) > 0\} \xrightarrow{1/z} \mathbb{E} \cap \{\Im(z) < 0\} \xrightarrow{\text{Cayley}} \{\Im(z) > 0, \Re(z) > 0\} \xrightarrow{z \mapsto z^4} \mathbb{C} \setminus \mathbb{R}_0^+ = \exp(\Psi) \ .$$

7. Classify the following domains up to biholomorphic equivalence. Briefly justify your claims.

(a) $\Omega_a = \mathbb{C} \setminus \mathbb{R}_0^+$

(b) $\Omega_b = \mathbb{E} \setminus \{0\}$

(c) $\Omega_c = \mathbb{C} \setminus (\{1,2\} \cup [3,\infty))$

(d) $\Omega_d = \{z \in \mathbb{C} \,|\, \Re(z) + 3\Im(z) < 2\}$

(e) $\Omega_e = \{z \in \mathbb{C} \,|\, \Re(z) > \Im(z)^4\}$

(f) $\Omega_f = \mathbb{E} \setminus \{0, \frac{1}{2}\}$

(g) $\Omega_g = \{z \in \mathbb{C} \,|\, 2 < |z| < 3\}$

(h) $\Omega_h = \mathbb{C}$

(i) $\Omega_i = \mathbb{C} \setminus [-1, 1] = \{z \in \mathbb{C} \,|\, |\Re(z)| > 1 \text{ or } \Im(z) \neq 0\}$

Solution:

(a) Ω_{7a}, Ω_{7d}, Ω_{7e} are simply connected and different from $\mathbb{C}$, hence mutually equivalent by the Riemann Mapping Theorem.

(b) $\Omega_{7b} \cong \Omega_{7i}$ because Inversion at -1, $z \mapsto 1/(z+1)$ takes Ω_{7i} biholomorphically to $\mathbb{C} \setminus (\{0\} \cup [1/2, \infty))$. This is biholomorphically equivalent to $\mathbb{E} \setminus \{0\}$.

(c) *None of $\Omega_{7c}, \Omega_{7f}, \Omega_{7g}, \Omega_{7i}$ is equivalent to any other in the list:*

$\mathbb{C}$ is not equivalent to any other domain. Otherwise, if $\mathbb{C} \cong G \underset{\subset}{\neq} \mathbb{C}$, then G would be biholomorphically equivalant to $\mathbb{E}$ by the Riemann Mapping Theorem. However, C is not equivalent to $\mathbb{E}$, for instance by Liouville's Theorem.

The annulus $\Omega_{7g} = A_{2,3}$ is not equivalent to $\Omega \setminus S$ for any open set Ω and a discrete subset $S \subset \Omega$. This is because a holomorphic map $\Omega \setminus S \to A_{2,3}$ is bounded and therefore all its isolated singularities S are removable, i.e. the map extends to all of Ω. By the open mapping theorem, this extension must map all the singularities to points in $A_{2,3}$.

Ω_{7c} is equivalent to $\mathbb{C} \setminus (\{-2, -1\} \cup [0, \infty)$. The square root takes this to $H \setminus \{i, i\sqrt{2}\}$ and the Cayley transform maps this to $\mathbb{E} \setminus \left\{0, \frac{\sqrt{2}-1}{\sqrt{2}+1}\right\}$.

8. Consider the polynomial

$$p(z) = 11z^9 + 15z^3 + 1 \ .$$

Counting roots with multiplicities determine the number of zeros of p in each of

$$\mathrm{B}_1(0) \quad , \quad \mathrm{B}_2(0) \quad , \quad \mathrm{B}_3(0) \quad \text{and} \quad \mathbb{C} \setminus \mathrm{B}_3(0) \ .$$

How many roots does the polynomial have disregarding multiplicities?

Solution: If $|z| = 1/3$, then

$$\left|11z^9 + 15z^3\right| \leq \frac{11}{3^9} + \frac{15}{3^3} < 1 \ ,$$

hence we have no zero in $\overline{\mathrm{B}_{1/3}(0)}$

If $|z| = 1$, then

$$\left|15z^3\right| = 15 > 11 + 1 \geq \left|11z^9 + 1\right|$$

and by Rouché's Theorem, the polynomial has as many zeros in $\mathrm{B}_1(0)$ as $15z^3$, i.e. 3.

If $|z| = 2$, then

$$\left|11z^9\right| = 11 \times 512 > 15 \times 8 + 1 \geq \left|15z^3 + 1\right|$$

and by Rouché's Theorem, the polynomial has 9 zeros in $\mathrm{B}_2(0)$.

Since this is the degree of the polynomial there are no more zeros, i.e. the number of zeros in $\mathrm{B}_3(0)$ is also 9, and there are no zeros in $\mathbb{C} \setminus \mathrm{B}_3(0)$.

The polynomial $p(z)$ "is a polynomial in z^3", i.e. there is a polynomial $q(u) \in \mathbb{C}[u]$, namely

$$q(u) = 11u^3 + 15u + 1 \ ,$$

so that

$$p(z) = q(u)|_{u=z^3} \ .$$

A consequence of this is that if $p(a) = 0$, then $p(\epsilon a) = p(\epsilon^2 a) = 0$ for $\epsilon^3 = 1 \neq \epsilon = \frac{1}{2} + \frac{\sqrt{3}}{2}$. Thus the non-zero roots of p come in triples $a, \epsilon a, \epsilon^2 a$. Since $p(0) \neq 0$, if a were a zero of p with multiplicity 2, then by the first part, the zeros of p would have to be $b, \epsilon b, \epsilon^2 b$ simple zeros for some $b \in \mathrm{B}_1(0)$, and $a, \epsilon a, \epsilon^2 a$ all double zeros and $1 < |a| < 2$. The zeros of $q(u)$ must then be a^3 (simple) and b^3 (double). To see that q has no double zeros, we compute the greatest common divisor of q and $q'(u) = 33u^2 + 15$,

$$11u^3 + 15u + 1 = (33u^2 + 15)\frac{u}{3} + 10u + 1$$

but $10u + 1$ irreductible. Hence q and q' are coprime. Thus q has three different zeros and p has 9 different zeros.

9. It is known that two annuli $A_{r,R} = \mathrm{B}_R(0) \setminus \overline{\mathrm{B}_r(0)}$, $A_{r',R,}$ are biholomorphically equivalent if and only if $R/r = R'/r'$. Prove the weaker statement: "There is $r > 0$ so that for all ρ, $0 < \rho < r$, the annulus $A_{\rho,1}$ is not biholomorhically equivalent to $A_{1/2,1}$."

   Hint: Assuming the contrary, i.e. a sequence of biholomorphic maps

   $$A_{r_n,1} \xrightarrow{\phi_n} A_{1/2,1} \quad , \quad r_n \xrightarrow{n\to\infty} 0 \ ,$$

   show that there would have to exist a biholomorhic map

   $$\mathbb{E} \setminus \{0\} \xrightarrow{\phi} A_{1/2,1}$$

   which is impossible because of the nature of its singularity at 0.

10. Find a biholomorhic map $\mathbb{E} \to \mathbb{E} \setminus \mathbb{R}_0^+$.

    Solution: $z \mapsto c\left(\sqrt{c^{-1}(z)}\right)^2$.

11. Compute the integral $\displaystyle\int_0^\infty \frac{\sqrt{t}}{1+t^4}\, dt$.

    Solution: The integral is equal to

    $$\lim_{r\to 0, R\to\infty} \int_r^R \frac{\sqrt{t}}{1+t^4}\, dt = \lim_{r\to 0, R\to\infty} \int_\gamma \frac{\sqrt{z}}{1+z^4}\, dz$$

    where $\gamma_{r,R}(t) \colon [r, R] \to \mathbb{C}$, $\gamma_{r,R}(t) = t$. We close γ with the paths

    $$\alpha, \beta \colon [0, \pi/2] \to \mathbb{C}, \omega \colon [r, R] \to \mathbb{C}$$

    $$\alpha_r(t) = re^{2\pi it}$$
    $$\beta_R(t) = Re^{2\pi it}$$
    $$\omega_{r,R}(t) = it$$

The loop $\gamma_{r,R} * \beta_R * \omega_{r,R}^{-1} * \alpha_r^{-1}$ is a boundary curve for the domain $\Omega_{r,R} = \{\rho e^{it} \,|\, r < \rho < R,\ 0 < t < \pi/2\}$. We estimate or compute the integrals over the auxillary paths:

$$\left|\int_{\alpha_r} \frac{\sqrt{z}}{1+z^4}\ dz\right| \le \frac{\pi r}{2} \frac{\sqrt{r}}{(1-r^4)^2} \xrightarrow{r\to 0} 0$$

$$\left|\int_{\beta_R} \frac{\sqrt{z}}{1+z^4}\ dz\right| \le \frac{\pi R}{2} \frac{\sqrt{R}}{(R^4-1)^2} \xrightarrow{R\to \infty} 0$$

$$\int_{\omega_{r,R}} \frac{\sqrt{z}}{1+z^4}\ dz = \int_r^R \frac{\sqrt{it}}{1+(it)^4}\ i\ dt = i^{3/2}\int_r^R \frac{\sqrt{t}}{1+t^4}\ dt = i^{3/2}\int_{\gamma_{r,R}} \frac{\sqrt{z}}{1+z^4}\ dz$$

If $r < 1 < R$ then there is one singularity of the integrand in $\Omega_{r,R}$ at $i^{1/2} = e^{\pi i/4} = \frac{1+i}{\sqrt{2}}$ The winding number of $\gamma_{r,R} * \beta_R * \omega_{r,R}^{-1} * \alpha_r^{-1}$ around this singularity is 1. The residue of the integrand is

$$\mathbf{Res}\left(\frac{\sqrt{z}}{1+z^4}; z = i^{1/2}\right) = \frac{i^{1/4}}{(i^{1/2}-i^{3/2})(i^{1/2}-i^{5/2})(i^{1/2}-i^{7/2})} \ .$$

By the residue theorem the loop integral is

$$\frac{2\pi i i^{1/4}}{(i^{1/2}-i^{3/2})(i^{1/2}-i^{5/2})(i^{1/2}-i^{7/2})} \ .$$

For the integral over γ we get

$$\frac{1}{1-i^{3/2}} \frac{2\pi i^{5/4}}{(i^{1/2}-i^{3/2})(i^{1/2}-i^{5/2})(i^{1/2}-i^{7/2})}$$

$$= \frac{1}{1-i^{3/2}} \frac{2\pi i^{5/4}}{\sqrt{2}\ 2\ i^{1/2}\sqrt{2}\ i} = \frac{1}{i^{1/4}-i^{7/4}} \frac{\pi}{2} = \frac{\pi}{4\cos(\pi/8)} = \frac{\pi}{4\sqrt{\frac{1}{2\sqrt{2}}+\frac{1}{2}}}$$

12. Let $\gamma\colon [0,2] \to \mathbb{C}$ be the curve with $\gamma(t) = (1-t) + it^6(2-t)^8$. Compute

$$\int_\gamma \frac{1}{z}\ dz \ .$$

Solution: The integrand is holomorphic on the closed upper half-plane minus 0, $X = \{z \in \mathbb{C} \,|\, \Im(z) \ge 0 \text{ and } z \ne 0\}$. The curve γ is homotopic relative endpoints to the curve $\tilde{\gamma}(t) = 1 - t + i\sqrt{1-(1-t)^2}$, $t \in [0,2]$, which is a reparametrization of the upper semicircle $\hat{\gamma}\colon [0,\pi] \to \mathbb{C}$, $\hat{\gamma}(t) = e^{it}$. The integral is

$$\int_\gamma \frac{1}{z}\ dz = \int_{\hat{\gamma}} \frac{1}{z}\ dz = \int_0^\pi \frac{1}{e^{it}} ie^{it}\ dt = i\pi \ .$$

Alternatively the function $f(z) = \frac{1}{z}$ has an antiderivative on X, $\frac{1}{z} = \left.\frac{d}{du}\right|_{u=z} \ln(u)$ where $\ln(re^{i\phi}) = \ln(r) + i\phi$ for $r \in \mathbb{R}^+$, $\phi \in [0,\pi]$, is the branch of the logarithm defined in the upper half-plane. Thus

$$\int_\gamma \frac{1}{z}\ dz = [\ln(u)]_{u=\gamma(-1)}^{u=\gamma(1)} = \ln(1) - \ln(-1) = i\pi \ .$$

13. Compute the integral

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{1+x^4} dx \ .$$

Solution:

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{1+x^4} dx = \Re \int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^4} dx \ .$$

We close the path of integration by a semi circle at ∞ in the upper half-plane. The singularities of the integrand in the upper half-plane are simple poles at ϵ, ϵ^3, where $\epsilon = e^{i\pi/4} = \frac{1+i}{\sqrt{2}}$. The residues of the integrand at ϵ and ϵ^3 are

$$\mathbf{Res}\left(\frac{e^{ix}}{1+x^4}; x = \epsilon\right) = \frac{e^{i\epsilon}}{(\epsilon-\epsilon^3)(\epsilon-\epsilon^5)(\epsilon-\epsilon^7)}$$

$$\mathbf{Res}\left(\frac{e^{ix}}{1+x^4}; x = \epsilon^3\right) = \frac{e^{i\epsilon^3}}{(\epsilon^3-\epsilon)(\epsilon^3-\epsilon^5)(\epsilon^3-\epsilon^7)}$$

By the Residue Theorem

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^4} dx &= 2\pi i \left(\frac{e^{i\epsilon}}{(\epsilon-\epsilon^3)(\epsilon-\epsilon^5)(\epsilon-\epsilon^7)} + \frac{e^{i\epsilon^3}}{(\epsilon^3-\epsilon)(\epsilon^3-\epsilon^5)(\epsilon^3-\epsilon^7)}\right) \\
&= 2\pi i \frac{e^{i\epsilon}}{\sqrt{2}(1+i)\sqrt{2}i\sqrt{2}} + \frac{e^{i\epsilon^3}}{-\sqrt{2}i\sqrt{2}(-1+i)\sqrt{2}} \\
&= \frac{2\pi i}{i\sqrt{8}} \left(\frac{e^{i\epsilon}}{1+i} + \frac{e^{i\epsilon^3}}{-(-1+i)}\right) \\
&= \frac{\pi}{\sqrt{2}} \left(\frac{e^{i\epsilon}}{1+i} + \frac{e^{i\epsilon^3}}{1-i}\right) \\
&= \frac{\pi}{\sqrt{2}} \left(\frac{e^{i\frac{1+i}{\sqrt{2}}}}{1+i} + \frac{e^{i\frac{-1+i}{\sqrt{2}}}}{1-i}\right) \\
&= \frac{\pi e^{-1/\sqrt{2}}}{\sqrt{2}} \left(\frac{e^{\frac{i}{\sqrt{2}}}}{1+i} + \frac{e^{\frac{-i}{\sqrt{2}}}}{1-i}\right) \\
&= \frac{\pi e^{-1/\sqrt{2}}}{2\sqrt{2}} \left((1-i)e^{\frac{i}{\sqrt{2}}} + (1+i)e^{\frac{-i}{\sqrt{2}}}\right)
\end{aligned}$$

The real part of this is

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\cos(x)}{1+x^4} dx &= \frac{\pi e^{-1/\sqrt{2}}}{2\sqrt{2}} \left((1-i)e^{\frac{i}{\sqrt{2}}} + (1+i)e^{\frac{-i}{\sqrt{2}}}\right) \\
&= \frac{\pi e^{-1/\sqrt{2}}}{\sqrt{2}} \left(\cos\left(\frac{1}{\sqrt{2}}\right) + \sin\left(\frac{1}{\sqrt{2}}\right)\right)
\end{aligned}$$

14. Consider the function $f\colon \mathbb{C} \setminus \{1,2\} \to \mathbb{C}$ with

$$f(z) = \frac{1}{z^2-3z+2} \ .$$

Find the Laurent series of f convergent in the annuli

$$\{z \in \mathbb{C} \,|\, |z| < 1\} \quad , \quad \{z \in \mathbb{C} \,|\, 1 < |z| < 2\} \quad \text{and} \quad \{z \in \mathbb{C} \,|\, 2 < |z|\}$$

Solution: Partial fraction decomposition gives

$$f(z) = \frac{1}{z^2 - 3z + 2} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$= \frac{1}{1-z} - \frac{1/2}{1-z/2} = \frac{-1/z}{1-1/z} - \frac{1/2}{1-z/2} = \frac{-1/z}{1-1/z} + \frac{1/z}{1-2/z}$$

which gives convergent Laurent series

$$\sum_{k=0}^{\infty} z^k - \frac{z^k}{2^{k+1}} = \sum_{k=0}^{\infty} \left(1 - \frac{1}{2^{k+1}}\right) z^k \quad \text{for} \quad |z| < 1 \ ,$$

$$\sum_{k=1}^{\infty} -z^{-k} - \sum_{k=0}^{\infty} \frac{z^k}{2^{k+1}} \quad \text{for} \quad 1 < |z| < 2 \ ,$$

$$\sum_{k=1}^{\infty} -z^{-k} + 2^{k-1} z^{-k} = \sum_{k=1}^{\infty} (2^{k-1} - 1) z^{-k} \quad \text{for} \quad 2 < |z| \ .$$

15. Let $\gamma \colon [0,1] \to \mathbb{C}$ be the curve with

$$\gamma(t) = t + it + \cos(\sin(\pi t) + it(t-1)) \ .$$

Compute the complex line integral

$$\int_\gamma z e^z \, dz \ .$$

Solution: The integrand is an entire function in z. An antiderivative for this is

$$F(z) = z e^z - e^z \ .$$

Thus the integral is

$$\int_\gamma z e^z \, dz = F(\gamma(1)) - F(\gamma(0)) = F(1+i) - F(0) = (1+i)e^{1+i} - e^{1+i} + 1 = 1 + ie^i + ie$$

16. Compute the integral

$$\int_0^{2\pi} \frac{\sin(t)}{1 + \sin(t)\cos(t)} \, dt \ .$$

Solution: We rewrite this as a complex line integral, substituting

$$z = e^{it} \quad , \quad \cos(t) = \frac{z + 1/z}{2} \quad , \quad \sin(t) = \frac{z - 1/z}{2i}$$

and get

$$\int_0^{2\pi} \frac{\sin(t)}{1 + \sin(t)\cos(t)} \, dt = \oint_{|z|=1} \frac{\frac{z-1/z}{2i}}{1 + \frac{z+1/z}{2} \frac{z-1/z}{2i}} \, \frac{1}{iz} \, dz$$

$$= \oint_{|z|=1} \frac{2z^2-2}{-4z^2+iz^4-i}\, dz = -2i \oint_{|z|=1} \frac{z^2-1}{z^4+4iz^2-1}\, dz \;.$$

The zeros of the denominator of the integrand are

$$\pm\sqrt{-2i \pm \sqrt{-4+1}} = \pm\sqrt{(-2\pm\sqrt{3})i} \;.$$

Inside the unit disc are

$$\pm\sqrt{(-2+\sqrt{3})i} \;,$$

and these are simple poles of the integrand. We need the residues

$$\mathbf{Res}\left(\frac{z^2-1}{z^4+4iz^2-1}; z = \pm\sqrt{(-2+\sqrt{3})i}\right)$$

$$= \mathbf{Res}\left(\frac{z^2-1}{(z^2+i(2+\sqrt{3})}\;\frac{1}{z\mp\sqrt{(-2+\sqrt{3})i}}; z = \pm\sqrt{(-2+\sqrt{3})i}\right)$$

$$= \frac{(-2+\sqrt{3})i-1}{(-2+\sqrt{3})i+i(2+\sqrt{3})}$$

By the residue theorem, the integral is

$$2\pi i(-2i)\left(\mathbf{Res}\left(\frac{z^2-1}{z^4+4iz^2-1}; z = \sqrt{(-2+\sqrt{3})i}\right) + \mathbf{Res}\left(\frac{z^2-1}{z^4+4iz^2-1}; z = -\sqrt{(-2+\sqrt{3})i}\right)\right)$$

17. Compute the integral

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{(1+x^2)(4+x^2)} dx \;.$$

Solution:

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{(1+x^2)(4+x^2)} dx = \Re \int_{-\infty}^{\infty} \frac{e^{ix}}{(1+x^2)(4+x^2)} dx \;.$$

We close the path of integration by a semi circle at ∞ in the upper half-plane. The singularities of the integrand in the upper half-plane are simple poles at i and $2i$. The residues of the integrand at i and $2i$ are

$$\mathbf{Res}\left(\frac{e^{ix}}{(1+x^2)(4+x^2)}; x = i\right) = \frac{e^{i^2}}{(2i)(4+i^2)} = \frac{e^{-1}}{6i}$$

$$\mathbf{Res}\left(\frac{e^{ix}}{(1+x^2)(4+x^2)}; x = 2i\right) = \frac{e^{2i^2}}{(1+(2i)^2)4i} = \frac{e^{-2}}{-12i}$$

By the Residue Theorem

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(1+x^2)(4+x^2)} dx = 2\pi i\left(\frac{e^{-1}}{6i} - \frac{e^{-2}}{12i}\right) \;,$$

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{(1+x^2)(4+x^2)} dx = 2\pi\Re\left(\frac{e^{-1}}{6} - \frac{e^{-2}}{12}\right)$$

$$= \frac{\pi}{3e} - \frac{\pi}{6e^2}$$

18. Let $(w_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ be so that $\sum_{n=1}^{\infty} \frac{1}{|w_n|} < \infty$. Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{z - w_n} \tag{13.1}$$

converges locally uniformly (compactly) on $\mathbb{C} \setminus \{w_n \mid n \in \mathbb{N}\}$.

Solution: For any $R \in \mathbb{R}^+$, we have

$$N(R) := \#\{n \in \mathbb{N} \mid |w_n| < R\} = \#\left\{n \in \mathbb{N} \,\middle|\, \frac{1}{|w_n|} > \frac{1}{R}\right\} \leq \sum_{n=1}^{\infty} \frac{R}{|w_n|} < \infty\ .$$

If $|z| < R < 2R < |w|$ then $|z - w| > |w/2|$. For $|z| < R$ we estimate the tail of (13.1),

$$\left|\sum_{n>N(2R)} \frac{1}{z - w_n}\right| \leq \sum_{n>N(2R)} \left|\frac{1}{z - w_n}\right| \leq \sum_{n>N(2R)} \frac{2}{|w_n|} \xrightarrow{R\to\infty} 0$$

independent of z (as long as $|z| < R$). This shows that the sum (13.1) converges uniformly on all sets

$$V_R := \mathrm{B}_R(0) \cap \mathbb{C} \setminus \{w_n \mid n \in \mathbb{N}\}\ .$$

Since $\mathbb{C} \setminus \{w_n \mid n \in \mathbb{N}\} = \bigcup_R V_R$ the series converges locally uniformly.

19. Show that the isolated singularities of a biholomorhic map are removable with at most one pole of order 1. Thus, let $\Omega \subset \mathbb{C}$ be open, $p, q \in \Omega$, $p \neq q$, and $f\colon \Omega \setminus \{p, q\} \to V \subset \mathbb{C}$ be biholomorphic. Then p, q are removable or, if p is not removable, then it is a pole of order 1 and q is removable.

This proves the following

Theorem 13.2 *If $\Omega \subset S = \mathbb{C} \cup \{\infty\}$ is open, $P \subset \Omega$ finite and $f\colon \Omega \setminus P \to V \subset S$ biholomorphic, then f extends to a biholomorphic map $\hat{f}\colon \Omega \to \hat{V} \subset S$ where $V \subset \hat{V} \overset{open}{\subset} S$ and $\hat{V} \setminus V = \hat{f}(P)$.*

Solution: An isolated singularity p can not be essential. To see this, let $x \in \Omega \setminus \{p\}$. By the Open Mapping Theorem, for all $r, s > 0$ sufficiently small, $f(\mathrm{B}_r(p) \setminus \{p\})$ and $f(\mathrm{B}_s(x))$ are open. By the Casorati-Weierstrass Theorem, $f(\mathrm{B}_r(p) \setminus \{p\})$ is dense in $\mathbb{C}$, i.e. intersects every non-empty open subset of $\mathbb{C}$. Thus for all $r, s > 0$, there is $y \in f(\mathrm{B}_r(p) \setminus \{p\}) \cap f(\mathrm{B}_s(x))$. If $r + s < |x - p|$, this conflicts with injectivity of f.

Thus p is a pole (or removable) and f extends to a holomorphic map $\tilde{f}\colon \Omega \setminus \{q\} \to S = \mathbb{C} \cup \{\infty\}$. By Theorem 11.13 the extension $\tilde{f}$ can not be injective near p if p were a pole of order ≥ 2.

Finally if p and q are both poles of first order, then $f(\mathrm{B}_r(p) \setminus \{p\}) \supset \mathbb{C} \setminus \overline{\mathrm{B}_R(0)}$, $f(\mathrm{B}_s(q) \setminus \{q\}) \supset \mathbb{C} \setminus \overline{\mathrm{B}_T(0)}$ for r, s sufficiently small and some $R, T > 0$. But $\mathbb{C} \setminus \overline{\mathrm{B}_R(0)} \cap \mathbb{C} \setminus \overline{\mathrm{B}_T(0)} \neq \emptyset$ for all $T, R > 0$. As before this conflicts with injectivity of f if $r + s < |p - q|$.

20. Compute the complex line integral

$$\oint_{|z|=1} z + \overline{z} + |z|\ dz\ .$$

Solution: This is straightforward from the definition:

$$\oint_{|z|=1} z+\overline{z}+|z| \; dz = \int_0^{2\pi} \left(e^{it}+e^{-it}+1\right) ie^{it} \; dt$$
$$= i\int_0^{2\pi} e^{2it}+1+e^{it} \; dt = 2\pi i$$

One might also have remembered from class or elsewhere that $\oint_{|z|=1} 1/z \; dz = 2\pi i$. Since for all z on the curve in question we have $|z|=1$ and $\overline{z}=1/z$ and since the "function z" is holomorphic everywhere, we can compute this as

$$\oint_{|z|=1} z+\overline{z}+|z| \; dz = \oint_{|z|=1} z+\frac{1}{z}+1 \; dz$$
$$= \oint_{|z|=1} \frac{1}{z} \; dz = 2\pi i \; .$$

21. Let f, g be entire functions, $g(0)=0$, $g(z)\neq 0$ for all $z\in\mathbb{C}\setminus\{0\}$ and $g'(0)=1$. Prove that

$$\oint_{|z|=r} \frac{f(z)}{g(z)} \; dz = 2\pi i f(0)$$

for all $r>0$.

Solution: By the Cauchy Integral Formula,

$$2\pi i f(0) = \oint_{|z|=r} \frac{f(z)}{z} \; dz \; .$$

The difference of the integrals

$$\oint_{|z|=r} \frac{f(z)}{z} \; dz - \oint_{|z|=r} \frac{f(z)}{g(z)} \; dz = \oint_{|z|=r} \frac{f(z)}{z} - \frac{f(z)}{g(z)} \; dz$$
$$= \oint_{|z|=r} \frac{f(z)g(z)-zf(z)}{zg(z)} \; dz$$

does not depend on $r>0$ since the integrand is holomorphic on $\mathbb{C}\setminus\{0\}$. Since $g(0)=0$, $g'(0)=1$ there is $h\in\mathcal{O}(\mathbb{C})$ with

$$g(z) = z(1+zh(z)) \; .$$

The difference of the integrals becomes

$$\oint_{|z|=r} \frac{f(z)z(1+zh(z))-zf(z)}{z^2(1+h(z))} \; dz = \oint_{|z|=r} \frac{f(z)z^2h(z)}{z(1+zh(z))} \; dz = \oint_{|z|=r} \frac{zf(z)h(z)}{(1+zh(z))} \; dz = 0$$

because the integrand is holomorphic on all of $\mathbb{C}$.

22. Let f be an entire function. Show that

$$\max\left\{(\Re f(a+ib))^3 \;\middle|\; a^4+b^4\leq 1\right\} = \max\left\{(\Re f(a+ib))^3 \;\middle|\; a^4+b^4=1\right\} \; .$$

Do we also have

$$\max\left\{\sin(\Re f(a+ib)) \;\middle|\; a^4+b^4\leq 1\right\} = \max\left\{\sin(\Re f(a+ib)) \;\middle|\; a^4+b^4=1\right\} \; ?$$

Solution: Since the f is entire it is open. Also the maps $\mathbb{C} \to \mathbb{C}, z \mapsto z^3$ and $\Re\colon \mathbb{C} \to \mathbb{R}$ are open. The composition of open maps is open, hence $g\colon z \mapsto (\Re f(z))^3$ is an open map and hence $\{(\Re f(a+ib))^3 \,|\, a^4+b^4<1\} = g(\{a+ib \,|\, a^4+b^4<1\})$ is an open subset of $\mathbb{R}$. On the other hand, g is continuous, hence assumes extrema on every compact set, and the set $\mathrm{B}^4 = \{a+ib \,|\, a^4+b^4<1\}$ is compact. It follows that g assumes its maximum only on the boundary $\{a+ib \,|\, a^4+b^4=1\}$.

The second statement is also true. Since f is continuous and $\overline{\mathrm{B}}^4$ compact, $f(\overline{\mathrm{B}}^4) = \overline{f(\overline{\mathrm{B}}^4)}$ is also compact. Since f is open, $f(\mathrm{B}^4)$ is open, hence $f(\mathrm{B}^4) \subset \operatorname{interior} f(\overline{\mathrm{B}}^4)$, hence $f(\partial \mathrm{B}^4) \supset \partial f(\overline{\mathrm{B}}^4)$. Now for every compact subset Y of $\mathbb{C}$, we have $\Re Y = \Re\partial Y$. Therefore

$$\Re(f(\overline{\mathrm{B}^4})) = \Re(\partial f(\overline{\mathrm{B}^4})) \subset \Re(f(\partial \mathrm{B}^4)) \ .$$

23. Determine all entire functions f with

$$|f(z)| \le |z| \ln(1+|z|) \tag{13.3}$$

Solution: The Cauchy estimate for the Taylor coefficients c_n of such a function gives

$$|c_n| \le \frac{r\ln(1+r)}{r^n} \begin{cases} \xrightarrow{r\to\infty} 0 \text{ if } n>1 \\ \xrightarrow{r\to 0} 0 \text{ if } n=0,1 \end{cases} \ .$$

Hence all Taylor coefficients of a function f satisfying (13.3) are 0.

24. The function $f\colon \mathbb{C}\setminus\mathbb{Z} \to \mathbb{C}$ with

$$f(z) = \frac{z}{\sin(\pi z)}$$

is holomorphic. Let $\sum\limits_{n=0}^{\infty} c_n(z-i)^n = f(z)$ be the Taylor series of f at i. What is the radius of convergence of this power series?

Solution: Since $\sin(0)=0$, $\sin'(0)=1$ there is an entire function h with $h(0)=1\neq 0$ so that $\sin(z) = zh(z)$. Hence

$$f(z) = \frac{z}{\sin(\pi z)} = \frac{z}{zh(z)} = \frac{1}{h(z)} \ .$$

Therefore f extends to a function $1/h$ which is holomorphic at 0. Clearly, f and $1/h$ have the same Taylor series at i. The largest r so that $1/h$ is holomorhic on $\mathrm{B}_r(i)$ is $\sqrt{2}$, this is therefore the radius of convergence of the Taylor series of f at i.

25. Find a domain $U \subset \mathbb{C}$, a sequence $(a_n)_{n\in\mathbb{N}} \in U^{\mathbb{N}}$ so that $\lim_{n\to\infty} a_n$ exists (in $\mathbb{C}$), and two functions $f, g \in \mathcal{O}(U)$ that coincide on this sequence (i.e. $f(a_n)) = g(a_n))$ for all $n\in\mathbb{N}$) but so that $f\neq g$.

Solution: Let $U = \mathbb{R}^+ + i\mathbb{R} = \{z\in\mathbb{C} \,|\, \Re(z)>0\}$, $(a_n)_{n\in\mathbb{N}} = \left(\frac{1}{\pi n}\right)_{n\in\mathbb{N}} \in U^{\mathbb{N}}$, $g=0$ and $f(z) = \sin(1/z)$.

26. Let $U\subset\mathbb{C}$ be open and $f\in\mathcal{O}(U)$ be injective. Prove that $f'(u)\neq 0$ for all $u\in U$.

Solution: If $f'(u)=0$ for some $u\in U$, then $\operatorname{ord}(f,u)=k\ge 2$. By the Corollary on the number of leaves of a holomorphic map near a point of order k, there is a neighbourhood V of u in U so that f is k to 1 on $V\setminus\{u\}$, in particular f is not injective there.

27. By the Riemann Mapping Theorem there is a biholomorphic map $\mathbb{C}\setminus\mathbb{R}_- \to \mathbb{E} = \mathrm{B}_1(0)$. Find a formula!

Solution: Let $c\colon \mathcal{H} \xrightarrow{\cong} \mathbb{E}$ be the Cayley transform, $s\colon z\to -z^2$. Since $s\colon \mathcal{H}\to\mathbb{C}\setminus\mathbb{R}_0^-$ is biholomorphic, the composition $s\circ c^{-1}$ maps $\mathbb{E}$ biholomorphically to $\mathbb{C}\setminus\mathbb{R}_-$.

28. Compute the Laurent series of

$$f(z) = \frac{1}{z(z^2-4)}$$

around 0.

Hint: Partial fractions, but only partially.

Solution: The partial partial fraction decomposition gives

$$\begin{aligned}
f(z) = \frac{1}{z(z^2-4)} &= \frac{-1}{4z} + \frac{z}{4} \times \frac{1}{z^2-4} \\
&= \frac{-1}{4z} + \begin{cases} \frac{1}{4z}\frac{1}{1-\frac{4}{z^2}} & \text{if } 2 < |z| \\ \frac{-z}{16}\frac{1}{1-\frac{z^2}{4}} & \text{if } 0 < |z| < 2 \end{cases} \\
&= \frac{-1}{4z} + \begin{cases} \frac{1}{4z}\sum_{k=0}^{\infty}\left(\frac{4}{z^2}\right)^k & \text{if } 2 < |z| \\ \frac{-z}{16}\sum_{k=0}^{\infty}\left(\frac{z^2}{4}\right)^k & \text{if } 0 < |z| < 2 \end{cases}
\end{aligned}$$

29. Is there a biholomorphic map $A_{1,2}(0) \leftrightarrow A_{1,\infty}(0)$?

Hint: Look at this in the right direction. The principal part is principal. Or invert $(z \mapsto 1/z)$ to pull ∞ to 0. Some or all of Liouville, Laurent, Schwarz.

Solution: If $\phi\colon A_{1,\infty}(0) \to A_{1,2}(0)$ is holomorphic, then ϕ is given by its Laurent series,

$$\phi(z) = \sum_{k\in\mathbb{Z}} c_k z^k \ .$$

The coefficients c_k satisfy the estimate

$$|c_k| \le \frac{\max\{|\phi(z)| \mid |z| = r\}}{r^k} \le \frac{2}{r^k} \quad \text{for all} \quad r > 1 \ .$$

Thus $c_k = 0$ for all $k \ge 0$ and ϕ is equal to its principal part,

$$\phi(z) = H(1/z),\ \ H\colon \mathrm{B}_1(0) \to A_{1,2}(0) \text{ biholomorphic, } H(0) = 0$$

impossible.

30. We defined $\arctan\colon \mathbb{R} \to \mathbb{R}$ by

$$\arctan(x) := \int_0^x \frac{1}{1+u^2}\, du \ .$$

(a) Does this extend to an entire function? Either compute the power series of this function or all of its Laurent series at 0.

Hint: You need not calculate derivatives. $\frac{1}{1-q}$ =...

Solution: If $A \in \mathcal{O}(\mathbb{C})$ were such an entire extension, then A' and $h(z) = \frac{1}{1+z^2}$ would be holomorphic on $\mathbb{C} \setminus \{-i, +i\}$ which coincide on $\mathbb{R}$. By the identity theorem, we would have $A' = h$ on all of $\mathbb{C} \setminus \{-i, +i\}$ and therefore A' would have a pole at i (and one at -i), impossible for the derivative of an entire function.

The Laurent series of $f(z) = \frac{1}{1+z^2}$ around 0 are

$$\frac{1}{1+z^2} = \sum_{k=0}^{\infty} (-z^2)^k = \sum_{k=0}^{\infty} (-1)^k z^{2k} \quad \text{for} \quad |z| < 1$$

and

$$\frac{1}{1+z^2} = \frac{1}{z^2} \frac{1}{\frac{1}{z^2}+1} = \frac{1}{z^2} \sum_{k=0}^{\infty} (-z^{-2})^k = -\sum_{k=1}^{\infty} (-1)^k z^{-2k} \quad \text{for} \quad |z| > 1 \ .$$

(b) Let $U = \mathbb{C} \setminus \{(1-t)i \,|\, t \in \mathbb{R}_0^+\}$ and $F \in \mathcal{O}(U)$ be so that

$$F(1) = \arctan(1) = \frac{\pi}{4} \quad \text{and} \quad F'(z) = \frac{1}{1+z^2} \quad \text{for all} \quad z \in U \ .$$

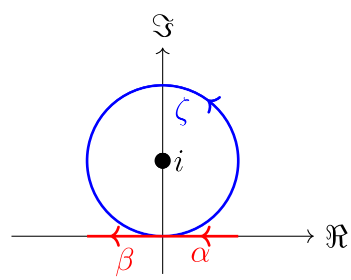
Compute $F(-1)$.

Solution: The set U is star shaped, center $2i$, because if $u \in U$, then the segment $[u, 2i] \subset U$. Thus U is a simply connected domain and $h(z) = \frac{1}{1+z^2}$ is holomorphic on U. We therefore have $F(z) - F(1) = \int_\gamma \frac{1}{1+z^2}\, dz$ for all curves $\gamma\colon [0, L] \to U$ with $\gamma(0) = 1$ and $\gamma(L) = -1$. We use the concatenation $\gamma = \alpha * \zeta * \beta$ where

$$\begin{aligned} &\alpha(t) = 1 - t \quad \text{for} \quad t \in [0,1] \\ &\zeta(t) = i + e^{it} \quad \text{for} \quad -\pi/2 \le t \le 3\pi/2 \\ &\beta(t) = -t \quad \text{for} \quad t \in [0,1] \end{aligned}$$

This gives

$$\begin{aligned} F(-1) - F(1) &= \int_\gamma \frac{1}{1+z^2}\, dz = \int_\alpha + \int_\zeta + \int_\beta = \left(\int_\alpha + \int_\beta \right) + \int_\zeta \\ &= \int_1^{-1} \frac{1}{1+x^2}\, dx + \oint_{|z-i|=1} \frac{1}{1+z^2}\, dz \\ &= -\frac{\pi}{2} + \oint_{|z-i|=1} \frac{1}{z+i} \frac{1}{z-i}\, dz \\ &= -\frac{\pi}{2} + \left. \frac{1}{z+i} \right|_{z=i} \times \oint_{|z-i|=1} \frac{1}{z-i}\, dz \\ &= -\frac{\pi}{2} + \frac{1}{2i} \times 2\pi i = \frac{\pi}{2} \ . \end{aligned}$$



The curves α, β, γ.

31. Is there a closed continuous curve $\gamma\colon [0,1] \to \mathbb{C}$ with infinitely many winding numbers, i.e. so that

$$\{w(\gamma, a) \,|\, a \in \mathbb{C} \setminus \gamma([0,1])\} \quad \text{is infinite?}$$

Solution:

$$\gamma(t) = e^{4\pi i t}, \frac{1}{2} + \frac{1}{2}e^{8\pi i(t-1/2)}, \frac{3}{4} + \frac{1}{4}e^{16\pi i(t-3/4)}$$

$$\gamma(t) = \begin{cases} \left(1 - \frac{1}{2^n}\right) + \frac{1}{2^n} e^{i\pi 2^{n+2}\left(t-1+\frac{1}{2^n}\right)} & \text{if } 1 - \frac{1}{2^n} \le t \le 1 - \frac{1}{2^{n+1}} \\ 1 & \text{if } t = 1 \end{cases}$$

32. Show that for every $p, q \in \mathbb{C}$, $R > r > 0$, so that $R > r + |p - q|$ there is $\rho \in [0, 1]$ so that the domain $G = \mathrm{B}_R(p) \setminus \overline{\mathrm{B}_r(q)}$ is biholomorphically equivalent to $\{z \in \mathbb{C} \mid \rho < |z| < 1\}$.

    Hint: Möbius transformations map circles to circles.

    Solution: The condition on the radii means that $\overline{\mathrm{B}_r(q)} \subset \mathrm{B}_R(p)$. The map

    $$G \to \mathbb{E} \setminus \mathrm{B}_{r/R}((q-p)/R) \quad , \quad z \mapsto (z-p)/R$$

    is biholomorphic. We can therefore assume that $q = 0$, $\mathrm{B}_R(p) = \mathbb{E} \supset \overline{\mathrm{B}_r(q)}$. If $q \neq 0$, then for $t \in [-1/|q|, 1/|q|]$, consider the the Möbiustransform w_{tq} with

    $$w_{tq}(z) = \frac{z - tq}{-t\overline{q}z + 1}$$

    as in Lemma 4.4. The center of $w_{tq}(\overline{\mathrm{B}_q(r)})$ depends continuously on t, always lies on $\mathbb{R}q$, and is equal to q for $t = 0$ and -1 for $t = -1/|q|$. By the intermediate value theorem, it is 0 for some $t \in [0, 1/|q|]$.

33. Compute $\displaystyle\int_0^\infty \frac{1}{(1+x^2)\sqrt{x}}\, dx$.

    Solution: We can extend the square root to the closed upper half plane without 0. The integrand $f(z)$ then satisfies $f(-z) = -if(z)$. By definition

    $$\int_0^\infty f(x)\, dx = \lim_{T\to\infty} \int_{1/T}^T f(x)\, dx \ .$$

    To close the path of integration we use the curves

    $$\begin{aligned} \gamma_T(t) &= t \in [1/T, T] \\ \alpha_T(t) &= Te^{it} \quad , \quad t \in [0, \pi] \\ \omega_T(t) &= -t \quad , \quad t \in [1/T, T] \\ \beta_T(t) &= \frac{1}{T}e^{it} \quad , \quad t \in [0, \pi] \end{aligned}$$

    Then the concatentation

    $$\mu_T = \gamma_T * \alpha_T * \omega_T^{-1} * \beta_T^{-1}$$

    is closed. For $T > 1$ the path μ_T surrounds the only singularity of the integrand $f(z)$ with winding number

    $$w(\mu_T, i) = 1 \ .$$

    The residue of f there is

    $$\mathbf{Res}\left(\frac{1}{(1+z^2)\sqrt{z}}; z = i\right) = \mathbf{Res}\left(\frac{1}{z+i}\frac{1}{\sqrt{z}}\frac{1}{z-i}; z = i\right) = \frac{1}{2i\sqrt{i}} \ .$$

Thus construction gives a

$$2\pi i\mathbf{Res}\,(f;i) = \frac{\pi}{\sqrt{i}} = \lim_{T\to\infty}\oint_{\mu_T} f(z)\ dz$$
$$= \lim_{T\to\infty}\int_{\gamma_T} f(z)\ dz + \underbrace{\lim_{T\to\infty}\int_{\alpha_T} f(z)\ dz}_{=0} - \underbrace{\lim_{T\to\infty}\int_{\omega_T} f(z)\ dz}_{=i\int_{\gamma_T} f(z)\ dz} - \underbrace{\lim_{T\to\infty}\int_{\beta_T} f(z)\ dz}_{=0}\ .$$

Solving this for the integral over γ gives

$$\int_0^\infty \frac{1}{(1+x^2)\sqrt{x}}\ dx = \lim_{T\to\infty}\int_{\gamma_T} f(z)\ dz = \frac{1}{1-i}\frac{\pi}{\sqrt{i}} = \frac{\pi}{\sqrt{2}}$$

34. Find a biholomorphic map $\mathbb{C}\setminus\{i+3t \,|\, t\in[0,1]\} \to \mathbb{E}\setminus\{0\}$.

Hint: Of course the specific nature of the segment removed from $\mathbb{C}$ is irrelevant here.

Solution: We do this in a number of steps:

$$\mathbb{C}\setminus\{i+3t \,|\, t\in[0,1]\} \xrightarrow{z\mapsto\frac{z-i}{3}} \mathbb{C}\setminus\{t \,|\, t\in[0,1]\} \xrightarrow{z\mapsto\frac{1}{z}-1} \mathbb{C}\setminus(\mathbb{R}_0^+\cup\{-1\})$$
$$\xrightarrow{z\mapsto\sqrt{z},\ \sqrt{-1}=i} \mathcal{H}\setminus\{i\} \xrightarrow{z\mapsto\frac{z-i}{z+i}} \mathbb{E}\setminus 0\ .$$

35. Let $U\subset\mathbb{C}$ be a domain and $f\in\mathcal{O}(U)$ be non-constant. Let $K\subset U$ be compact with non-empty interior

$$\overset{\circ}{K} = \bigcup_{V\subset K, V \text{ open}} V\ .$$

Assume that $|f|$ is constant on the boundary $\partial K = K\setminus\overset{\circ}{K}$. Show that then f must have a zero in $\overset{\circ}{K}$.

Solution: The function $|f|$ is continuous and therefore has extrema on the compact set K, say

$$|f(k_0)| \le |f(k)| \le |f(k_1)| \quad \text{for all} \quad k\in K\ .$$

By the open mapping theorem, the function $f\colon K\to\mathbb{C}$ is open. The modulus function $|\cdot|:\mathbb{C}\to\mathbb{R}_0^+$ is open on $\mathbb{C}\setminus\{0\}$, hence the composition $|f|$ is open on $K\setminus f^{-1}(0)$.

If f has no zero on K, then $|f|(\text{interior}\,K)$ must be an open subset of $[|f(k_0)|, |f(k_1)|]$, hence can not contain $|f(k_0)|$, $|f(k_1)|$. Thus $k_0,k_1\in\partial K$. But $|f|$ is constant on ∂K, hence

$$|f(k)| = |f(k_0)| = |f(k_1)| =: c \quad \text{for all} \quad k\in K$$

In particular $f(K)\subset cS^1$. By assumption interior $K\neq\emptyset$. Again by the open mapping theorem, $f(\text{interior}\,K)$ is a non-empty open subset of $\mathbb{C}$, but as such can not lie in cS^1, the interior of this set being emtpy.

36. Let $f\in\mathcal{O}(\mathbb{C})$ be non-constant and $c\in\mathbb{R}^+$. Show that

$$\overline{\{z\in\mathbb{C} \,|\, |f(z)|<c\}} = \{z\in\mathbb{C} \,|\, |f(z)|\le c\}\ . \tag{13.4}$$

Solution: Clearly $\{z\in\mathbb{C} \,|\, |f(z)|<c\}$ is contained in both sets. The closure of a set A is contained in every closed set containing the set A. Since the right hand set in (13.4) is closed, this proves one inclusion.

By the open mapping theorem, $|f|:\mathbb{C}\to\mathbb{R}_0^+$ is open. Thus if $|f(z_0)|=c>0$ for some $z_0\in\mathbb{C}$, then every neighbourhood U of z_0 is mapped to an neighbourhood $|f(U)|$ of c in $\mathbb{R}_0^+$, i.e. every neighbourhood U of z_0 intersects $\{z\in\mathbb{C}\,|\,|f(z)|<c\}$.

If X is any topological space and $A\subset X$ then $x\in\overline{A}$ if and only if every neighbourhood of x intersects A.

37. Let $p\in\mathbb{C}[z]$ be a polynomial and $c\in\mathbb{R}^+$. Show that

$$\#\pi_0(\{z\in\mathbb{C}\,|\,|p(z)|<c\})\le \operatorname{degree} p\ .$$

Hint: For a topological space X. $\pi_0(X)$ denotes the set of path components of X. A path component of X is a maximal path connected subset of X.

Solution: Let p be a non-constant polynomial. Let $Y\in\pi_0\{z\in\mathbb{C}\,|\,|p(z)|<c\}$. Then Y is open, and, since $\lim_{z\to\infty}p(z)=\infty$, Y is bounded. It follows that $\overline{Y}$ is compact. By problem 36, $|p(z)|=c$ for all $z\in\partial Y=\overline{Y}\setminus Y$. By problem 35, p has a zero in Y. We thus have shown that p has a zero in each component of Y. Consequently the number of components is less than the number of zeros of p which is less than the degree of p.

38. Find a biholomorphic map

$$G=\{x+iy\,|\,x\in\mathbb{R},y\in(0,2\pi)\}\to\mathbb{E}=\mathrm{B}_1(0)\ .$$

Solution: The exponential map takes G to $\mathbb{C}\setminus\mathbb{R}_0^+$. The square root function with $\sqrt{i}=\frac{1+i}{2}$ maps this domain to the upper half plane, which is biholomorphically equivalent to the disc by the Cayley transform. Thus a biholomorphic map $\phi\colon G\to\mathbb{E}$ as required is given by

$$\phi(z)=\frac{e^{z/2}-i}{e^{z/2}+i}\ .$$

39. Show that a holomorphic map $f\colon\mathcal{H}\to\mathcal{H}=\{z\in\mathbb{C}\,|\,\Im(z)>0\}$ with two fixed points is the identity.

Solution: Conjugating such a map with the Cayley transform gives a map $g=c\circ f\circ c^{-1}\colon\mathbb{E}\to\mathbb{E}$ with two fixed points. By conjugating again with a Möbius transform if needed we can assume that one of the fixed points of g is 0. By the Schwarz Lemma g must then be the identity.

40. Compute

$$\int_0^\infty\frac{\sqrt{t}}{1+t^2}\,dt.$$

Hint: The picture for this problem is below. The square root on the curves γ and δ is not the same! The integrand extends meromorphically to the region surrounded by $\gamma_r*\beta_r*\delta_r^{-1}*\alpha_r^{-1}$. Show that the contributions of the integrals over α_r and β_r become negligible as $r\to\infty$. What is the relation of the integrand on γ to that on δ?

Solution: We use the curves

$$\begin{aligned}\gamma_r\colon[1/r,r]\to\mathbb{C}\quad&,\quad\gamma_r(t)=t\ ,\\ \delta_r\colon[1/r,r]\to\mathbb{C}\quad&,\quad\delta_r(t)=t\ ,\\ \alpha_r\colon[0,2\pi]\to\mathbb{C}\quad&,\quad\alpha(t)=\frac{1}{r}e^{it}\\ \beta_r\colon[0,2\pi]\to\mathbb{C}\quad&,\quad\beta(t)=re^{it}\end{aligned}$$

Since the integrand is bounded near zero, $\lim_{r\to\infty}\int_{\alpha_r}\frac{\sqrt{z}}{1+z^2}\,dz = 0$. Since the integrand satisfies

$$\left|\frac{\sqrt{z}}{1+z^2}\right| \le |z|^{-3/2} \ ,$$

$$\left|\int_{\beta_r}\frac{\sqrt{z}}{1+z^2}\,dz\right| \le 2\pi r \times r^{-3/2} \xrightarrow{r\to\infty} 0 \ .$$

On δ_r we need to use the extension of the square root. Thus

$$\int_{\delta_r}\frac{\sqrt{z}}{1+z^2}\,dz = -\int_{\gamma_r}\frac{\sqrt{z}}{1+z^2}\,dz \ .$$

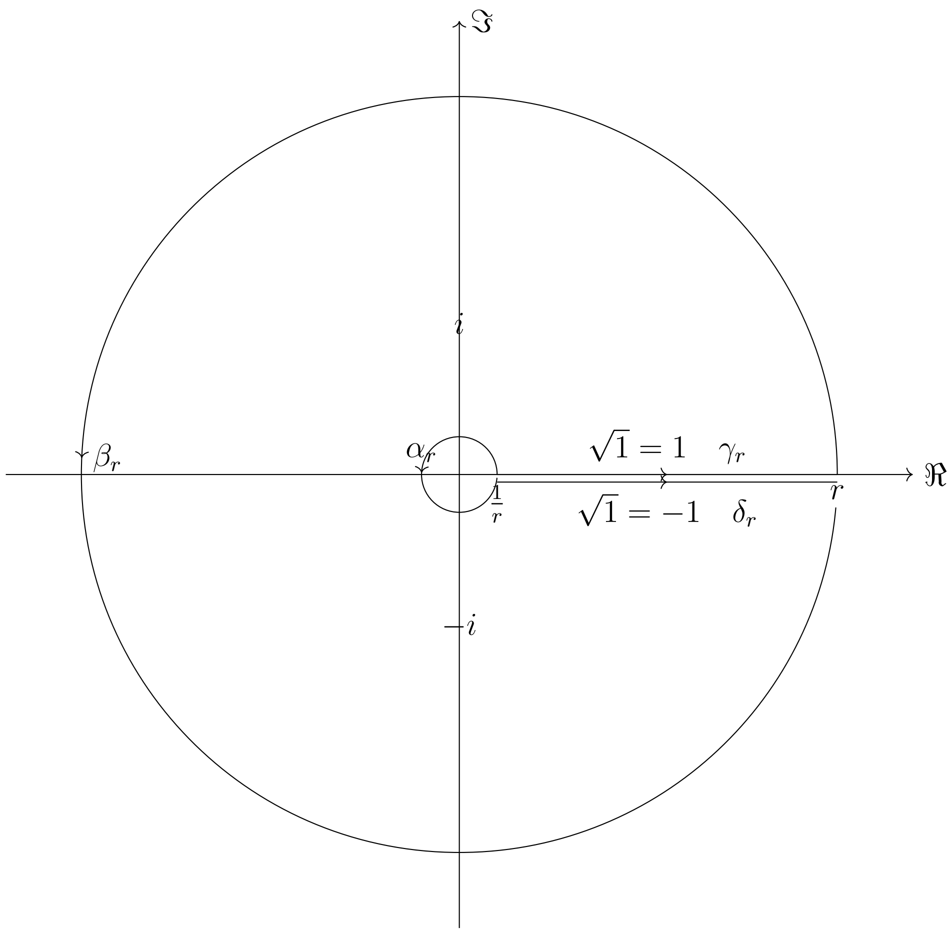
By the residue theorem, for $r > 1$,

$$\oint_{\gamma_r*\beta_r*\delta_r^{-1}*\alpha_r^{-1}}\frac{\sqrt{z}}{1+z^2}\,dz = 2\pi i\left(\mathbf{Res}\left(\frac{\sqrt{z}}{1+z^2}; z=i\right) + \mathbf{Res}\left(\frac{\sqrt{z}}{1+z^2}; z=-i\right)\right)$$

$$= 2\pi i\left(\frac{1+i}{2i\sqrt{2}} + \frac{-1+i}{-2i\sqrt{2}}\right) = \pi\sqrt{2} \ .$$

The integral in question is half of this,

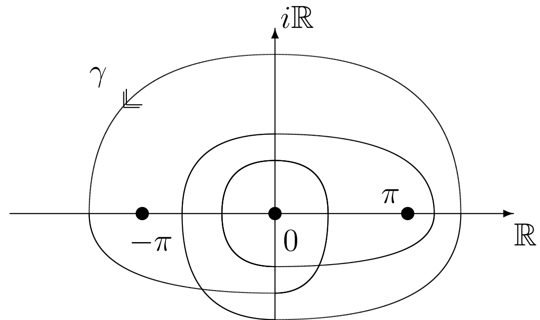
$$\int_0^\infty \frac{\sqrt{t}}{1+t^2}\,dt = \frac{\pi}{\sqrt{2}} \ .$$



41. Let γ be the simple curve as shown in the picture. Compute the integrals

(a)
$$\oint_\gamma \frac{\cos(z/2)}{\sin(z)}\, dz\ ,$$

(b)
$$\oint_\gamma \frac{z^2(z-\pi)}{(\sin(z))^2}\, dz\ .$$



Solution: The singularities, winding numbers and residues are

Singularity	Pole order	winding number	Residue(41a)	Residue(41b)
$-\pi$	2	1	0	$5\pi^2$
0	0	3	1	0
$+\pi$	1	2	0	π^2

where

$$\textbf{Res}\left(\frac{z^2(z-\pi)}{(\sin(z))^2}, z=-\pi\right) = \frac{d}{dz}\bigg|_{z=-\pi} -\pi \frac{z^2(z-\pi)}{(\sin(z))^2}(z+\pi)^2 = \left(\frac{d}{dz}\bigg|_{z=-\pi} -\pi(z^2(z-\pi))\right)\times \frac{(z+\pi)^2}{(\sin(z))^2}\bigg|_{z=-\pi}.$$

Hence

$$\textbf{Res}\left(\frac{z^2(z-\pi)}{(\sin(z))^2}, z=-\pi\right) = (3\pi^2+2\pi^2)\times 1 = 5\pi^2\ .$$

By the Residue Theorem the integrals are

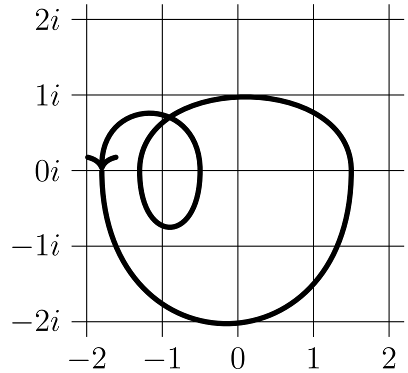
(a) $\displaystyle\oint_\gamma \frac{\cos(z/2)}{\sin(z)}\, dz = 2\pi i \times (1\times 3) = 6\pi i$

(b) $\displaystyle\oint_\gamma \frac{z^2(z-\pi)}{(\sin(z))^2}\, dz = 2\pi i \times (1\times 5\pi^2 + 2\times \pi^2) = 14\pi^3 i,$

42. Compute the integral

$$\oint_\gamma \frac{z}{\sin(\pi z)\cos(i\pi z)}\, dz$$

where γ is as shown in the picture.



Hint: If f is odd, i.e. $f(z) = -f(-z)$, or even, i.e. $f(z) = f(-z)$, what is the relation between the residues $\mathbf{Res}\,(f, p)$ and $\mathbf{Res}\,(f, -p)$? Check your answer to this against $f(z) = \frac{1}{z}$!

What is the relation between $\mathbf{Res}\,(f(\alpha z), z = 0)$ and $\mathbf{Res}\,(f(z), z = 0)$?

You do not need to work out all the singularities in this problem!

Solution: Since

$$2i\sin(z) = e^{iz} - e^{-iz} = 0$$

if and only if $e^{iz} = \pm 1$, $z \in \pi\mathbb{Z}$ and

$$2\cos(z) = e^{iz} + e^{-iz} = 0$$

if and only if $e^{iz} = \pm i$, $z \in \frac{\pi}{2} + \pi\mathbb{Z}$, the integrand has simple poles at $\mathbb{Z} \cup i\left(\frac{1}{2} + \mathbb{Z}\right)$. The singularities surrounded by γ are

$$-1, 0, 1, \frac{i}{2}, -\frac{i}{2}, -\frac{3i}{2}$$

The singularity at 0 is removable. The integrand is even and therefore has opposite residues at opposite points. All singularities have winding number 1 with exception of -1 where it is 2. Thus we only need the residues

$$\begin{aligned}
\mathbf{Res}\left(\frac{z}{\sin(\pi z)\cos(i\pi z)}; z = -1\right) &= \frac{-1}{\cos(-i\pi)}\mathbf{Res}\left(\frac{1}{\sin(\pi z)}; z = -1\right) \\
&= \frac{-1}{\cos(i\pi)}\frac{-1}{\pi} \\
&= \frac{1}{\pi\cos(i\pi)} = \frac{2}{\pi(e^{-\pi} + e^{\pi})}
\end{aligned}$$

$$\begin{aligned}
\mathbf{Res}\left(\frac{z}{\sin(\pi z)\cos(i\pi z)}; z = -\frac{3i}{2}\right) &= \frac{-\frac{3i}{2}}{\sin(-\frac{3\pi i}{2})}\mathbf{Res}\left(\frac{1}{\cos(i\pi z)}; z = -\frac{3i}{2}\right) \\
&= \frac{\frac{3i}{2}}{\sin(\frac{3\pi i}{2})}\frac{-i}{\pi} \\
&= \frac{\frac{3i}{2}}{\sin(\frac{3\pi i}{2})}\frac{-i}{\pi} = \frac{3i}{\pi(e^{-3\pi/2} - e^{3\pi/2})}
\end{aligned}$$

and the integral is $2\pi i$ times the sum of these.

43. Let $G, F \subset \mathbb{C}$, $G, F \neq \mathbb{C}$ be simply connected domains and $h, \hat{h}\colon G \to F$ be biholomorphic. Assume that there is $x \in G$ so that

    (a) $\hat{h}(x) = h(x)$ and

(b) $\hat{h}'(x)/h'(x) \in \mathbb{R}^+$.

Prove that $h = \hat{h}$.

Hint: State the Riemann Mapping Theorem and the Schwarz Lemma.

Solution: By the Riemann Mapping Theorem, there is a biholomorhic map $\phi\colon F \to \mathbb{E}$. Since the group of biholomorphic self maps of $\mathbb{E}$ acts transitively on $\mathbb{E}$, we can assume $\phi(h(x)) = 0$.

$$\Psi := \phi \circ \hat{h} \circ h^{-1} \circ \phi^{-1} \colon \mathbb{E} \to \mathbb{E}$$

is biholomorphic and $\Psi(0) = 0$. Then by the Schwarz Lemma, Ψ is a rotation, i.e. $\Psi(z) = \Psi'(0)z$ for all $z \in \mathbb{E}$ and $|\Psi'(0)| = 1$. By the chain rule,

$$\Psi'(0) = \phi'(h(x)) \times \hat{h}'(x) \times \left(h^{-1}\right)'(h(x)) \times \left(\phi^{-1}\right)'(0)$$

$$= \phi'(\phi^{-1}(0)) \times \hat{h}'(x) \times \left(h^{-1}\right)'(h(x)) \times \left(\phi^{-1}\right)'(0) = \hat{h}'(x)/h'(x) \in \mathbb{R}^+$$

by assumption. It follows that $\Psi'(0) = 1$, $\Psi = \mathrm{id}_{\mathbb{E}}$, hence $\hat{h} \circ h^{-1} = \mathrm{id}_F$, $\hat{h} = h$.