Euclidean Geometry MT351A

Recommended Reading: Patrick D. Barry, *Geometry with Trigonometry*

0 Homework

1. Assume that a plane has n points, $n \in \mathbb{N}$, $n \geq 3$, and that for some $k \in \mathbb{N}$, $k \geq 2$, there are lines f and g so that

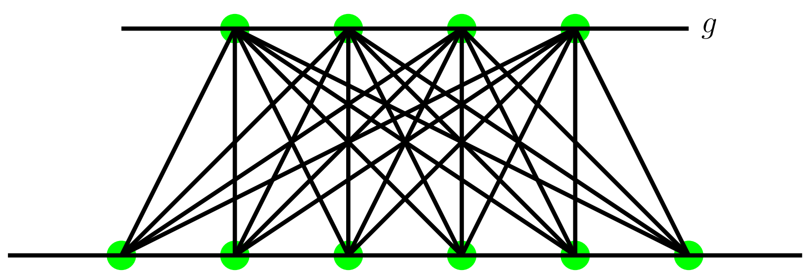
   (i) f and g are different and parallel, i.e. there is no point lying on f and on g

   (ii) There are $n - k$ points lying on f

   (iii) There are k points lying on g.

   (a) Draw such a plane in the case $n = 10$ and $k = 4$.
   Solution:

   The 6-4-plane

   (b) Show that any line ℓ in this plane different from f and g must be of the form

   $$\ell = (FG)$$

   where F lies on f and G lies on g. Furthermore F and G are the only points on (FG).
   Solution: All points in this plane lie on f or on g. If ℓ is any line different from f and g, then by axiom 4, ℓ can intersect f or g in at most one point. If ℓ did not intersect f, then, since all points not on f lie on g, all points on ℓ would have to lie in g, hence ℓ and g would have to be equal, since they have at least two points in common. Thus ℓ must intersect f in one point F say. Similarly one shows that ℓ must intersect g in one point G, say. Thus $\ell = FG$. There is no third point on ℓ because all points of this plane lie on f or g.

2. Is there a plane having a point Z lying in all lines of the plane? Prove your answer.

   Solution: No. To see this, assume Z lies on all lines of the plane. By axiom 1, there is a line l and by assumption Z lies on l. By Axiom 2 there is second point $L \neq Z$ on l. By axiom 3 there is point K not on l, in particular $K \neq Z$, $K \neq L$. By axiom 4 there is a unique line $k = (KZ)$ through K and Z. By axiom 4, there also is a unique line (KL) through K and L. We claim that Z does not lie on (KL): If Z were on (KL), then Z and L would be on (K, L)

and by the uniqueness of axiom 4, we would have $l = (ZL) = (KL)$. In particular K would lie on l, contradicting our choice of K. Thus Z does not lie on (KL).

3. How many total orders (see Definition 2.2) are there on a set with five elements, like $\{1, 2, 3, 4, 5\}$?

   Solution: This is the same as the number of permutations, i.e. 5!.

4. Define a relation $\ll$ on $\mathbb{R} \times \mathbb{R}$ so that for any $x, y, a, b \in \mathbb{R}$ we have

$$(x, y) \ll (a, b) \overset{\text{def}}{\iff} (x \leq a, x \neq a) \text{ or } (x = a, y \leq b) \ .$$

   Prove that $\ll$ so defined is a total order on $\mathbb{R}^2$.

   Hint: Use that $\leq$ is a total order on $\mathbb{R}$.

   Solution: For $x \in \mathbb{R}$ we clearly have $(x, x) \ll (x, x)$.
   Now assume $(x, y) \ll (a, b)$ and $(a, b) \ll (x, y)$. There are two cases:

   (a) $x = a$. Then we have $y \leq b$ and $b \leq y$, hence $y = b$ because $\leq$ is a total order on $\mathbb{R}$. Thus $(x, y) = (a, b)$.

   (b) $x < a$ and $a < x$, but this is impossible.

   Finally assume that

$$(x, y) \ll (a, b) \quad \text{and} \quad (a, b) \ll (r, s) \ . \tag{4-1}$$

   Here we need to distinguish a number of cases.

   (a) $x = a = r$. Then we have $y \leq b$ and $b \leq s$, hence $x = r$ and $y \leq s$ and therefore $(x, y) \ll (r, s)$.

   (b) $x < a$ and $a = r$. Then we have $x < r$ and therefore $(x, y) \ll (r, s)$.

   (c) $x = a$ and $a < r$. Then we have $x < r$ and therefore $(x, y) \ll (r, s)$.

   (d) $x < a$ and $a < r$. Then we also have $x < r$ and therefore $(x, y) \ll (r, s)$.

   In all cases we have shown that (4-1) implies $(x, y) \ll (r, s)$

5. Recall the standard plane defined in 1.9.3, and the notation $\ell_{a,b,c}$ for lines in this plane. Consider the points $A = (2, 3)$ and $B = (4, 4)$ in the standard plane. Find $a, b, c \in \mathbb{R}$ so that A and B lie on the line $\ell_{a,b,c}$.

   Hint: There are several solutions to this.

   Solution: We need to solve the system of linear equations corresponding to $A \in \ell_{a,b,c}$ and $B \in \ell_{a,b,c}$, i.e.

$$2a + 3b = c$$
$$4a + 4b = c$$

   Subtracting the two equations gives

$$2a + b = 0 \quad , \quad b = -2a \quad , \quad c = 2a - 6a = -4a \ .$$

   The points A, B lie on the line $\ell_{a,-2a,-4a}$ for all $a \in \mathbb{R}$, $a \neq 0$.

6. Let A, B, C be points in an absolute geometry, so that B lies between A and C. Show that

$$|A,B|\,|B,C| \leq \frac{|A,C|^2}{4} \ .$$

Hint: Translate this into a problem about real numbers. Look at Proposition 3.2.

Solution: Let $f\colon AC \to \mathbb{R}$ be the isometric bijection of Proposition 3.2, so that $f(A) = 0$ and $f(C) = c \geq 0$. Let $b := f(B)$. We only need to show that

$$b(c-b) \leq \frac{c^2}{4} \ .$$

But this holds for all real numbers b, c because

$$\frac{c^2}{4} - b(c-b) = \left(\frac{c}{2} - b\right)^2 \geq 0 \ .$$

7. Let $A, B, C \in l$ be points on a common line in an absolute geometry. Let M be the midpoint of $[A, B]$, i.e. so that $|M, A| = |M, B| = |A, B|/2$. Prove that if C is not between A, B then

$$|C,A| + |C,B| = 2\,|C,M| \ .$$

Hint: This is a simple statement about real numbers.

Solution: If $A = B$ then $M = A = B$ and the statement is trivial. We may therefore assume that $A \neq B$. We prove this in two ways:
1. Proof using the isometrie between lines and $\mathbb{R}$ of Proposition 3.2: Let $<<$ be the natural order on the line AB which has $A << B$. Let $a, b, c, m \in \mathbb{R}$ be the real numbers corresponding to the points A, B, C, M respectively, under an order preserving isometry $\lambda_{AB,A,<<}$ as in Proposition 3.2. We then have $a < b$, $m = (a+b)/2$ and $c \notin (a, b)$, hence $c \leq a$ or $c \geq b$. We need to show that

$$|c-a| + |c-b| = 2\,|c-m| \ . \tag{7-2}$$

We have the two cases

(a) $c \leq a$: Then $c \leq m$, $c \leq b$ and we can eliminate the absolute values in (7-2),

$$|c-a| + |c-b| = a - c + b - c = a + b - 2c = 2(m-c) = 2\,|c-m|$$

(b) $c \geq b$: Then $c \geq m$, $c \geq a$. Again we can eliminate the absolute values in (7-2),

$$|c-a| + |c-b| = c - a + c - b = 2c - (a+b) = 2(c-m) = 2\,|c-m|$$

2. Direct proof: We have the two cases

(a) $C << A$: Then $C << A << M << B$ and by additivity of the distance,

$$|C,A| + |A,M| = |C,M| \quad \text{and} \quad |C,M| + |M,B| = |C,B| \ .$$

Since M is the midpoint of $[A, B]$, hence $|A, M| = |M, B|$, we get

$$|C,A| = |C,M| - |A,M|$$

$$|C,B| = |C,M| + |M,B| = |C,M| + |A,M| \ .$$

Adding the two equations proves the claim.

(b) $B << C$: Then $A << M << B << C$ and by additivity of the distance,

$$|C,A| = |C,M| + |A,M| \quad \text{and} \quad |C,M| = |C,B| + |M,B| \ .$$

Since M is the midpoint of $[A,B]$, hence $|A,M| = |M,B|$, we get

$$|C,A| = |C,M| + |A,M|$$

$$|C,B| = |C,M| - |M,B| = |C,M| - |A,M| \ .$$

As before, adding the two equations proves the claim.

8. In this problem you may only use the axioms for absolute geometry, in particular Axiom 3.12. Let A, B, C, D and A', B', C, D' be quadruples of pairwise different points in an absolute geometry. Assume also that D lies in the interior region of $\angle ABC$ and that D' lies in the interior region of $\angle A'B'C'$. Assume that

$$|A,B| = |A',B'| \quad , \quad |B,C| = |B',C'| \quad , \quad |C,D| = |C',D'| \quad , \tag{8-3}$$

$$|\angle ABC| = |\angle A'B'C'| \quad , \quad |\angle BCD| = |\angle B'C'D'| \ . \tag{8-4}$$

Show that then

$$|A,D| = |A',D'| \ .$$

Hint: You will need to use the congruence axiom 3.12 several times.

Solution: By the congruence axiom SAS, Axiom 3.12, we have $(B,C,D) \cong (B'C'D')$. Consequently,

$$|\angle CBD| = |\angle C'B'D'| \quad \text{and} \quad |B,D| = |B',D'| \ . \tag{8-5}$$

By assumption, $D \in \mathrm{IR}\,(\angle ABC)$, $D' \in \mathrm{IR}\,(\angle A'B'C')$. By the additivity of the angle measure,

$$|\angle ABC| = |\angle ABD| + |\angle DBC|$$
$$|\angle A'B'C'| = |\angle A'B'D'| + |\angle D'B'C'|$$

Subtracting the two equations and using the assumption (8-4) together with (8-5) gives

$$|\angle ABD| = |\angle A'B'D'| \ .$$

From (8-3) and (8-5) we also have

$$|A,B| = |A',B'| \quad \text{and} \quad |B,D| = |B',D'|$$

By SAS we have $(ABD) \cong (A'B'C')$, in particular

$$|A,D| = |A',D'| \ .$$

9. Let (A, B, C) be a triangle in an absolute geometry, i.e. a triple of non-collinear points. Assume that the sum of the interior angles of the triangle (A, B, C) is different from 180. Show that then there are at least two parallels to AB through C.

Hint: This geometry must be non-Euclidean, the parallel-postulate can not hold.

Solution: Let A', B' be points in this plane, so that A and A' are in different half-planes of BC and B and B' are in different half-planes of AC. By the construction axiom for the angle measure, we can choose these points so that

$$|\angle BAC| = |\angle B'CA| \quad \text{and} \quad |\angle ABC| = |\angle A'CB| \ .$$

As in the proof of the existence of parallels in absolute geometry, Theorem 3.35, the Alternate Angles Theorem implies that

$$B'C||AB||CA' \ .$$

Since the sum of the interior angles in the triangle (A, B, C) is not 180,

$$|\angle B'CA| + |\angle ACB| + |\angle A'CB| = |\angle BAC| + |\angle ACB| + |\angle ABC| \neq 180 \ .$$

Thus $\angle B'CA'$ is not a straight angle and therefore the lines $B'C$ and CA' are different parallels to AB.

10. Let A, B, C, D be four pairwise different points in a Euclidean plane. Let Z be a fifth point so that

$$\begin{aligned} D, Z &\in \mathrm{IR}\,(\angle ABC)\,, \\ A, Z &\in \mathrm{IR}\,(\angle BCD)\,, \\ B, Z &\in \mathrm{IR}\,(\angle CDA)\,, \\ C, Z &\in \mathrm{IR}\,(\angle DAB)\,, \\ |A, Z| = |B, Z| &= |C, Z| = |D, Z| \end{aligned}$$

Show that

$$|\angle ABC| + |\angle ADC| = 180 \ .$$

Hint: "Angle chase": The triangles (AZB), (BZC), etc. are isosceles. The geometry is assumed Euclidean. Thus the angle sum in a triangle is 180.

Solution: Since Z lies in the interior regions as above, additivity of the angle measure gives

$$|\angle ABC| + |\angle ADC| = |\angle ABZ| + |\angle ZBC| + |\angle ADZ| + |\angle ZDC|$$

$$|\angle BCD| + |\angle DAB| = |\angle BCZ| + |\angle ZCD| + |\angle DAZ| + |\angle ZAB|$$

Since these are angles in isosceles triangles,

$$|\angle ABZ| = |\angle ZAB| \quad , \quad |\angle ZBC| = |\angle BCZ| \quad \text{etc}$$

Thus

$$|\angle ABC| + |\angle ADC| = |\angle BCD| + |\angle DAB|$$

Since $D \in \mathrm{IR}\,(\angle ABC)$ additivity of the angle measure gives

$$|\angle ABC| = |\angle ABD| + |\angle DBC| \ .$$

Similarly,

$$|\angle ADC| = |\angle ADB| + |\angle BDC| \ .$$

Since in a Euclidean plane, the angles of a triangle add up to 180, we now have

$$\begin{aligned} &|\angle ABC| + |\angle ADC| + |\angle BCD| + |\angle DAB| \\ &= |\angle ABD| + |\angle DBC| + |\angle ADB| + |\angle BDC| + |\angle BCD| + |\angle DAB| \\ &= (|\angle ABD| + |\angle ADB| + |\angle DAB|) + (|\angle DBC| + |\angle BDC| + |\angle BCD|) = 2 \times 180 \end{aligned}$$

11. Recall the various representations of a complex number z as a pair of real numbers (**Cartesian coordinates**), a formal sum, a (2×2)-matrix, or in terms of modulus and argument (**polar coordinates**):

$$\begin{aligned} z &= (a, b) \\ &= a + ib = \Re z + i \Im z \\ &= \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \\ &= |z|\, e^{i \arg z} \text{ if } z \neq 0 \end{aligned}$$

where

$$\begin{aligned} \Re(z) &= a \quad \textbf{real part} \\ \Im(z) &= b \quad \textbf{imaginary part} \\ |z| &= \sqrt{\det \begin{pmatrix} a & -b \\ b & a \end{pmatrix}} = \sqrt{a^2 + b^2} = \sqrt{z\overline{z}} \quad \textbf{modulus} \\ \overline{z} &= (a, -b) = a - ib = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}^t = |z|\, e^{-i \arg z} \quad \textbf{complex conjugate} \end{aligned}$$

If $z \neq 0$, then $\arg z \in \mathbb{R}/2\pi\mathbb{Z}$ is a (equivalence class of) real number(s) so that

$$\frac{z}{|z|} = e^{i \arg z} = \cos(\arg z) + i \sin(\arg(z)) \; .$$

$\arg z$ is only determined up to integer multiples of 2π!

For each of the following complex numbers compute the real part $\Re(z)$, the imaginary part $\Im(z)$, the modulus $|z|$ and an argument $\arg z$.

(a) $1 + i$

Solution: $\Re = 1$, $\Im = 1$, $|1 + i| = \sqrt{2}$, $\arg(1 + i) = \frac{\pi}{4} + 2\pi\mathbb{Z}$

(b) $1 + i\sqrt{3}$

Solution: $\Re = 1$, $\Im = \sqrt{3}$, $\left|1 + i\sqrt{3}\right| = 2$, $\arg(1 + i) = \frac{\pi}{3} + 2\pi\mathbb{Z}$

(c) $\dfrac{1}{3 + 2i}$

Hint: Try to make the denominator real.

Solution: Modulus and argument are immediately calculated:

$$\left| \frac{1}{3 + 2i} \right| = \frac{1}{|3 + 2i|} = \frac{1}{\sqrt{13}}$$

$$\arg\left(\frac{1}{3 + 2i} \right) = -\arg(3 + 2i) = -\arctan\frac{2}{3} + 2\pi\mathbb{Z}$$

In order to compute real and imaginary part, we multiply with the complex conjugate of the denominator, so that it becomes real:

$$\frac{1}{3+2i} = \frac{3-2i}{(3+2i)(3-2i)} = \frac{3-2i}{13}$$

hence $\Re = \frac{3}{13}$, $\Im = \frac{-2}{13}$.

(d) $\dfrac{1+2i}{1+3i}$

Solution: As before, modulus and argument are immediately read off:

$$\left|\frac{1+2i}{1+3i}\right| = \frac{|1+2i|}{|1+3i|} = \frac{\sqrt{5}}{\sqrt{10}} = \frac{1}{\sqrt{2}} \ .$$

$$\arg\left(\frac{1+2i}{1+3i}\right) = \arg(1+2i) - \arg(1+3i) = \arctan(2) - \arctan(3) = \arctan(2) - \arctan(3) \ ,$$

all modulo $2\pi\mathbb{Z}$.

For real and imaginary part we compute

$$\frac{1+2i}{1+3i} = \frac{(1+2i)(1-3i)}{10} = \frac{7-i}{10}$$

hence

$$\Re\frac{1+2i}{1+3i} = \frac{7}{10} \quad \text{and} \quad \Im\frac{1+2i}{1+3i} = -\frac{1}{10} \ .$$

12. Let M be a Möbius transformation of the upper half-plane with

$$M\left(\frac{i}{2}\right) = 2i \quad \text{and} \quad M(2i) = \frac{i}{2} \ .$$

Find $M(0)$, $M(\infty)$, $M(1+i)$.

Hint: How does M act on the imaginary axis $i\mathbb{R}^+$?

Solution: M preserves the imaginary axis $i\mathbb{R}^+$ since it maps hyperbolic lines to hyperbolic lines and interchanges two points on $i\mathbb{R}^+$. Hence $M(z) = \frac{-1}{z}$. In particular, $M(0) = \infty$, $M(\infty) = 0$,

$$M(1+i) = \frac{-1}{1+i} = \frac{i-1}{2}$$

13. In the hyperbolic upper half-plane find the point A so that

$$|i, A| = \frac{1}{5}\,|i, 2i| = \frac{1}{6}\,|A, 2i| \ . \qquad (13\text{-}6)$$

Hint: By the sharp triangle inequality in Theorem 3.27, the points $A, i, 2i$ lie on the same line. See also problem 17.

Solution: We have

$$|A, 2i| = |A, i| + |i, 2i| \qquad (13\text{-}7)$$

hence, by the sharp triangle inequality in Theorem 3.27, i lies between A and $2i$ and $A = \alpha i$ for some $\alpha \in (0,1]$. Since the distance on the hyperbolic line $C(0) = iR^+$ is

$$|\lambda i, \mu i| = \left|\ln \frac{\lambda}{\mu}\right|$$

we get from (13-6) that

$$\ln \frac{1}{\alpha} = \frac{1}{5} \ln 2$$

hence

$$\alpha = \frac{1}{\sqrt[5]{2}} .$$

14. Determine all Möbius transformations M of the upper half-plane mapping the line $C(-1,1) = \{e^{it} \mid 0 < t < \pi\}$ to $C(0) = i\mathbb{R}^+$, i.e. so that

$$\forall t \in (0,\pi) \exists \lambda \in \mathbb{R}^+ : M(i\lambda) = e^{it} .$$

Hint: Look at ideal points. Problem 18 might help.

Solution: Such a Möbius transformation M needs to map $\{-1,1\}$ to $\{0,\infty\}$. Thus

$$M(-1) = 0 \quad , \quad M(1) = \infty$$

or

$$M(1) = 0 \quad , \quad M(-1) = \infty .$$

If M is of the first kind, then

$$\frac{-1}{M(z)}$$

is of the second. Möbius transformations of the first kine are

$$M(z) = a\frac{z+1}{z-1}$$

where we need to choose a so that M preservers the upper half-plane. This amounts to

$$\det \begin{pmatrix} a & a \\ 1 & -1 \end{pmatrix} = -2a > 0$$

hence $a < 0$. Thus the Möbius transformations we look for are

$$M(z) = \frac{az+a}{z-1} \quad \text{or} \quad \frac{-1}{\frac{az+a}{z-1}} = -\frac{z-1}{az+a} \quad \text{with} \quad a < 0 ,$$

or

$$M(z) = \frac{az+a}{-z+1} \quad \text{or} \quad \frac{z-1}{az+a} \quad \text{with} \quad a > 0 .$$

15. Let k be a natural number. Assume that there is a triangulation of the **surface of genus 2** with v vertices, e edges and t triangles. Also assume that exactly k edges meet at every vertex. Show that $k \geq 7$. Assuming that $k = 7$ is possible, what is the corresponding f-vector (v,e,t)?

Hint: This is similar to problem 19.

Solution: The Euler characteristic of F_2 is -2, hence we must have

$$v - e + t = -2$$

Since every edge bounds two triangles we also have

$$2e = 3t$$

Now also k edges meet in each vertex and every edge hits two different vertices. Thus

$$kv = 2e = 3t$$

Eliminating v and e from these equations gives

$$\frac{3}{k}t - \frac{3}{2}t + t = \left(\frac{3}{k} - \frac{3}{2} + 1\right) t = \left(\frac{3}{k} - \frac{1}{2}\right) t = -2 \ .$$

Since $t > 0$ we must have $k > 6$, i.e. $k \geq 7$, and

$$t_k = \frac{4k}{k-6}$$

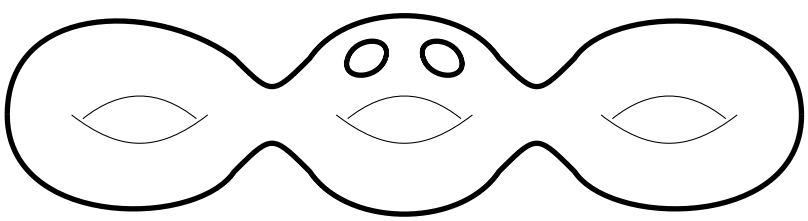
must be a positive integer. Since $t_7 = 28 \in \mathbb{Z}$ the smallest possible value is $k = 7$, which gives the f-vector

$$(12, 42, 28)$$

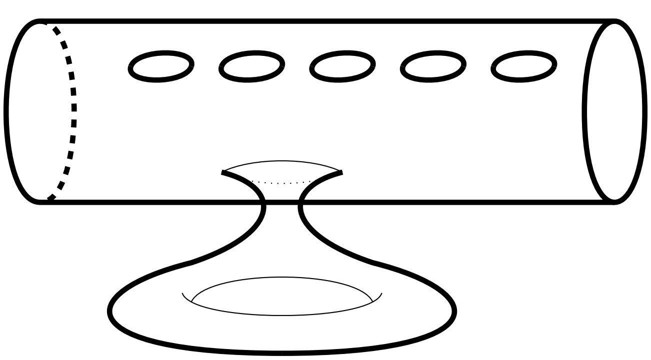
16. Compute the Euler-characteristic of each of the surfaces shown below.

    Hint: You need not consider a whole triangulation of these surfaces for this problem. Assume a triangulation of a surface and modify this to give a triangulation of this surface with a disc missing. Look at problem 20.

    (a) surface of genus 3 with 2 discs removed.

    (b) cylinder with a handle and 5 discs removed.

Solution: Removing a disc amounts to removing one triangle in a sufficiently large triangulation. The surface of genus 3 has Euler characteristic

$$\chi(F_2) = -4 = t - e + f \ .$$

Thus

$$\chi(F_2 \setminus \text{two discs}) = \chi(F_2) - 2 = -6 \ .$$

The cylinder can be obtained by removing two discs from a sphere. Since the torus has Euler-characteristic 0, we compute

$$\chi(\text{cylinder+handle} \setminus 5 \text{ discs}) = \chi(\text{torus} \setminus 7 \text{ discs}) = -7 \ .$$

17. In the hyperbolic upper half-plane find points P, Q so that

$$|i, P| = |P, Q| = |Q, 2i| = \frac{|i, 2i|}{3} \ .$$

Hint: By the sharp triangle inequality in Theorem 3.27, the points P, Q must lie on the segment $[i, 2i]$.

Solution: For $\lambda \in \mathbb{R}^+$ we hae $|i, \lambda i| = |\ln \lambda|$. We must have $P = i\pi$, $Q = i\psi$ for some $\pi, \psi \in \mathbb{R}^+$. The given equality translates to

$$\ln \pi = \ln \frac{\psi}{\pi} = \frac{\ln 2}{\psi} = \frac{\ln 2}{3} = \ln(2^{1/3})$$

for π, ψ, hence

$$\pi = 2^{1/3} \quad , \quad \psi = 2^{2/3} \ .$$

18. Determine all Möbius transformations M of the upper interchanging $i+1$ and $i-1$, i.e. so that

$$M(i+1) = i-1 \quad , \quad M(i-1) = i+1 \ .$$

Solution: Such a Möbius transformation M must fix the midpoint of the segment $[i-1, i+1]$ and therefore map the unique line intersecting this segment in this midpoint to itself. This line is $C(0) = i\mathbb{R}^+$ whose ideal points are 0 and ∞. If M fixes these ideal points, then $M(z) = az$, $a \in \mathbb{R}^+$, but this can not interchange $i \pm 1$. Thus M interchanges $0, \infty$ and must therefore be of the form $M(z) = \frac{a}{z}$. To determine a we solve

$$M(i-1) = \frac{a}{i-1} \stackrel{!}{=} i+1 \ ,$$

hence $a = (i-1)(i+1) = -2$,

$$M(z) = \frac{-2}{z} \ .$$

19. Let k be a natural number. Assume that there is a triangulation of the sphere with v vertices, e edges and t triangles. Also assume that exactly k edges meet at every vertex. Which are the possible values for k? For each of these, which are the possible f-vectors (v, e, t)?

Solution: The Euler characteristic of the sphere sis 2, hence we must have

$$v - e + t = 2$$

Since every edge bounds two triangles we also have

$$2e = 3t$$

Now also k edges meed in each vertex and every edge hits two different vertices. Thus

$$kv = 2e$$

Eliminating e and t from these equations gives

$$v - \frac{k}{2}v + \frac{k}{3}v = 2$$

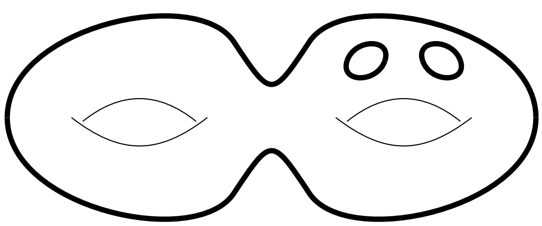
$$v(6-k) = 12 \ .$$

Thus $k < 6$ and $6 - k$ divides 12, hence $k = 5, 4, 3, 2$. The possible f-vectors are

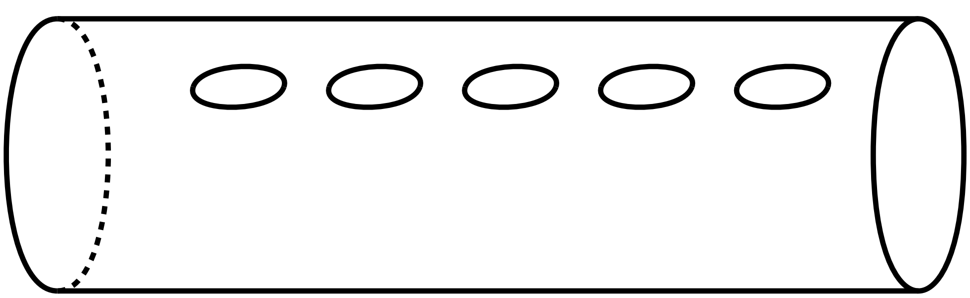
$$\begin{array}{ll} k=5 & (12,30,20) \\ k=4 & (6,24,12) \\ k=3 & (4,6,4) \\ k=2 & (3,3,2) \end{array}$$

20. Compute the Euler-characteristic of each of the surfaces shown below

(a) surface of genus 2 with 2 discs removed.



(b) cylinder with 5 discs removed.



Solution: Removing a disc amounts to removing one triangle in a sufficiently large triangulation. The surface of genus 2 has Euler characteristic

$$\chi(F_2) = -2 = t - e + f \ .$$

Thus

$$\chi(F_2 \setminus \text{two discs}) = \chi(F_2) - 2 = -4 \ .$$

The cylinder can be obtained by removing two discs from a sphere. Since the sphere has Euler-characteristic 2, we compute

$$\chi(\text{cylinder} \setminus 5 \text{ discs}) = \chi(S^2 \setminus 7 \text{ discs}) = \chi(S^2) - 7 = -5 \ .$$

21. For each of the following complex numbers compute the real part $\Re(z)$, the imaginary part $\Im(z)$, the modulus $|z|$ and an argument $\arg z$.

   (a) $e^{2+i\pi/3}$
   Solution: $\Re = e^2/2$, $\Im = e^2\sqrt{3}/2$, $|.| = e^2$, $\arg() = \frac{\pi}{3} + 2\pi\mathbb{Z}$

   (b) $3 + 3i$

   (c) $-\sqrt{3} + i$
   Solution: $\Re = -\sqrt{3}$, $\Im = 1$, $|.| = 2$, $\arg() = \frac{5\pi}{6} + 2\pi\mathbb{Z}$

   (d) $\dfrac{1}{4-i}$
   Hint: Try to make the denominator real.
   Solution: $\Re = \frac{4}{17}$, $\Im = \frac{1}{17}$, $|| = 1/\sqrt{17}$, $\arg() = \arctan(1/4) + 2\pi\mathbb{Z}$.

   (e) $\dfrac{(1+i)^3}{1+3i}$
   Solution:

22. Let M be a Möbius transformation of the upper half-plane with

$$M(1+i) = i \quad \text{and} \quad M(i) = \lambda i$$

for some $\lambda \in \mathbb{R}^+$. Find $a, b, c, d \in \mathbb{R}$ so that $M(z) = \frac{az+b}{bz+d}$. Find the hyperbolic distance $|i, i+1|$.

Solution: M needs to map the hyperbolic line $(1+i, i)$ through $1+i$ and i to the hyperbolic line $(i, 2i)$ through i and λi. The ideal points of these lines are

$$(1+i, i) = C\left(\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right)$$
$$(i, \lambda i) = C(0, \infty)$$

and up to a possible inversion, the Möbius transformations that map the lines this way are of the form

$$M(z) = \beta \frac{z - \frac{1-\sqrt{5}}{2}}{z - \frac{1+\sqrt{5}}{2}} = \beta \frac{2z - 1 + \sqrt{5}}{2z - 1 - \sqrt{5}}$$

for some α, β. We compute β from the condition $M(1+i) = i$,

$$\beta \frac{2+2i-1+\sqrt{5}}{2+2i-1-\sqrt{5}} = \beta \frac{2i+1+\sqrt{5}}{2i+1-\sqrt{5}} \stackrel{!}{=} i \ ,$$

which leads to

$$\beta = \frac{-2+i-i\sqrt{5}}{2i+1+\sqrt{5}} = \frac{-2}{1+\sqrt{5}} = \frac{1-\sqrt{5}}{2} \ .$$

The possible Möbius transformations with the required mappping properties are therefore $M(z)$ or its inversion $\frac{-1}{M(z)}$, i.e.

$$M_1(z) = \frac{1-\sqrt{5}}{2} \frac{2z-1+\sqrt{5}}{2z-1-\sqrt{5}}$$
$$M_2(z) = \frac{-1}{M_1(z)}$$

These map i to

$$M_1(i) = \frac{1-\sqrt{5}}{2} \frac{2i-1+\sqrt{5}}{2i-1-\sqrt{5}} = \frac{1-\sqrt{5}}{2} \frac{-2}{1+\sqrt{5}} i = \frac{\sqrt{5}-1}{\sqrt{5}+1} i$$
$$M_2(i) = \frac{\sqrt{5}+1}{\sqrt{5}-1} i \ .$$

In particular, the hyperbolic distance is

$$|i, 1+i| = \ln \frac{\sqrt{5}+1}{\sqrt{5}-1} = \ln \frac{3+\sqrt{5}}{2} \ .$$

1 Introduction - Axioms, Theorems, Definitions - Diagrams

For simplicity in this introduction we will look at the first, simplest group of axioms for Euclidean geometry, the **Incidence axioms**. These give the relations between points and lines that will be assumed in the sequel.

Euclid's definitions, "A point is that which has no parts, or which has no magnitude" or "a line is length without breadth" would not be considered defintions nowadays, because they express new notions in terms of others (have no parts, magnitude, length, breadth) that have not yet been defined. Fortunately, already Euclid's development of geometry never refers to these definitions. The fact that a point has no parts is never used in the proof of any theorem.

We will therefore start simply by naming the objects the theory is to deal with. This is a mere convention, giving names to the objects involved in specifying the structure of a plane, the **undefined terms**. These are

1.1 Undefined terms

Customarily, the undefined terms for a plane are

points, lines, lies on

This is to say that in a plane there may be points, lines and some points may lie on some lines.

As a concession to style, we will be a bit sloppy about this and also say "a line contains a point", "a line goes through a point" and the like for "a point lies on a line".

1.2 Axioms (postulates)

Axioms state some facts we agree to accept about the undefined terms. In the case of a plane,

Axiom 1.1 (Incidence Axioms for a Plane)

1. *There is at least one line.*
2. *For each line ℓ there are at least two different points P, Q so that P lies on ℓ and Q lies on ℓ.*

3. *For each line ℓ there is a point P so that P does not lie on ℓ.*

4. *For every two points P, Q, P different from Q, there is a unique line ℓ so that P lies on ℓ and Q lies on ℓ.*

1.3 Theorems

Theorems are consequences of these axioms. In this respect, the words "Theorem", "Corollary", "Proposition", "Lemma" are synonyms. These words are used to give a hint about the importance of a theorem and its relation to others.

A Corollary is a (more or less immediate) consequence.
A Proposition is a Theorem of lesser importance.
A Lemma, or "auxilliary theorem" is a often technical statement needed in the proof of some other theorem, but not considered to be of interest in its own.

A theorem needs a **proof**, i.e. a sequence of statements each following from its predecessor by one or more of the axioms (or already established theorems). For example:

Theorem 1.2 *In a plane there are three points which are not collinear, (i.e. do not lie on the same line).*

Proof:

1. *Axiom 1:* there is a line ℓ.
2. *Axiom 2:* there are two points P, Q so that P lies on ℓ and Q lies on ℓ, and P and Q are different.
3. *Axiom 3:* there is a point R so that R does not lie on ℓ.
4. Therefore R is different from P and R is different from Q.
5. Thus the plane contains the three pairwise different points P, Q, R.
6. *Axiom 4, the uniqueness part:* If h were a line so that P, Q and R lie on h, then $h = \ell$. But then R would lie on ℓ, contradicting step 3.
7. We conclude that the points P, Q, R do not lie on the same line.

•

1.4 Definitions, Notation

These are not strictly necessary for the development of the theory. They can always be removed from a statement by expanding, "inlining". For instance,

Definition 1.3 *If h and g are lines in the plane and P lies on h and P lies on g, we say that h* **and** *g* **intersect in** *P. We say that two lines intersect if they intersect in some point of the plane.*

Definition 1.4 *Two lines in the plane are called* **parallel** *if they are equal or if they do not intersect.*

Some definitions depend on axioms (or even theorems).

Definition 1.5 *If P and Q are different points and ℓ is the (by axiom 4) unique line so that P lies on ℓ and Q lies on ℓ then we write*

$$\ell = PQ \ .$$

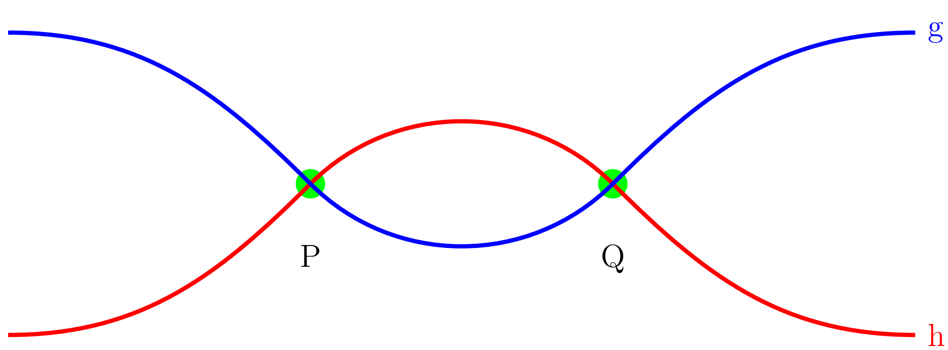
If we did not have uniqueness in axiom 4, that is if a line were not determined by two different points on it, we could have

$$h = PQ \quad \text{and} \quad g = PQ \quad \text{and} \quad h \neq g \ .$$

Definition 1.6 *A set M of points of a plane is called* **collinear** *if there is a line ℓ in the plane so that every point X in M lies on ℓ. If $M = \{A, B, C, \ldots\}$ is collinear, we say the points $A, B, C, \ldots$ are collinear.*

1.5 The role of diagrams

On might illustrate the uniqueness part of Axiom 4 and the potential problem with Definition 1.5 by a diagram,



By Axiom 4 this can not happen in a plane.

Diagrams can be very helpful to visualize the incidence structure of a plane.

By axioms 1,2,3 there are at least three points in a plane, and one can satisfy the plane axioms with three points A, B, C by taking as lines the sets

$$f = \{A, B\} \quad , \quad g = \{B, C\} \quad , \quad h = \{A, C\} \ .$$

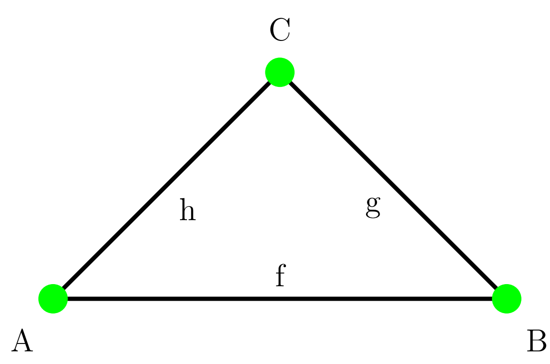
A point $X \in \{A, B, C\}$ lies in a line ℓ if $X \in \ell$.

We can say this without set-notation:
Points: A, B, C
Lines: f, g, h
lie on: A lies on f, A does not lie on g, A lies on h
B lies on f, B lies on g, B does not lie on h
C does not lie on f, C lies on g, C lies on h.

One might put this in a table,

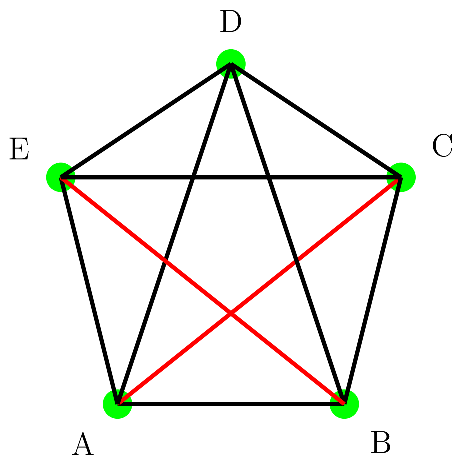
lies on	f	g	h
A	y	n	y
B	y	y	n
C	n	y	y

or a diagram



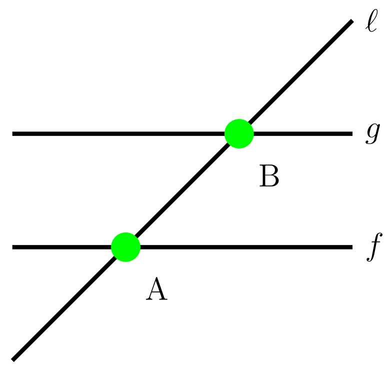
Diagrams can also mislead, for instance by showing intersections that do not really exist. For an example consider the "plane of 2-sets on 5 points". This is a plane with five points so that for every two points X, Y of these the line through these points does not contain any other points.

2-set plane on five points



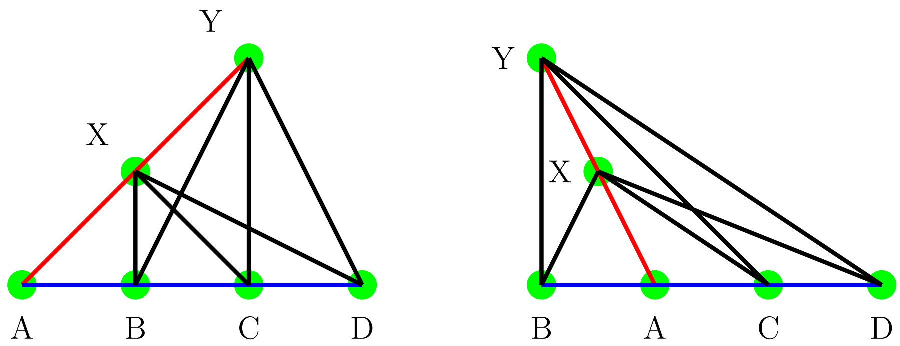
Only the green dots represent points in this plane. The lines AC and BE for instance (red) are parallel

Diagrams also tend to omit cases. One might prove the following: "If f and g are parallel lines in a plane and a line ℓ intersects f, then ℓ must also intersect g." One might "prove" this from the diagram

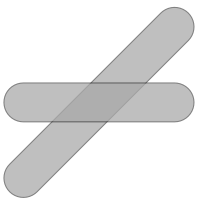


Of course the 2-set plane on five points we just discussed shows that this conclusion is wrong. Two parallel lines need not look as in this picture.

A diagram may insinuate too much structure. For instance the following diagrams show the same plane. This plane has one line with 4 points (blue), one line with 3 points (red) and 6 lines with 2 points (black).



A plane does not know anything about "betweenness".



Poincaré : "Geometry is the art of reasoning well from badly drawn figures"

1.6 Planes with four points

The following determines all planes with four points:

Theorem 1.7 *Assume that the plane contains exactly four points. Then there are two cases:*

1. *There is a line, say ℓ so that three points, say A, B, C, lie on ℓ. Then the fourth point X does not lie on ℓ and there are three lines a, b, c different from ℓ. The points lying on a are A and X, the ones on b are B and X and the ones on c are C and X. There are no lines other than ℓ, a, b, c.*

2. *On each line there are exactly two points. If A, B, C, D denote the points in the plane, then there are six lines*
$$\ell_{AB} \quad , \quad \ell_{AC} \quad , \quad \ell_{AD} \quad , \quad \ell_{BC} \quad , \quad \ell_{BD} \quad , \quad \ell_{CD} \tag{1.8}$$
and ℓ_{AB} contains the points A, B and not C, D, ℓ_{AC} contains the points A, C and not B, D, and so on.

Proof:

1. *Axiom 2:* there are at least two points on each line.

2. *Axiom 3:* for each line there is a point not on this line.

3. *1 and 2:* On each line there are two or three points. We thus have two cases:

   (a) There is a line, say ℓ with three points A, B, C.

      i. *Axiom 3:* There is a point X not on ℓ

      ii. *Axiom 4:* There are lines a, b, c so that A, X lie on a, B, X lie on b, C, X lie on c.

      iii. *the plane has only four points:* Every pair of points lies in one of the lines ℓ, a, b, c. Hence there is no other line. There are no points besides A, X on a, no points besides B, X on b, and no points besides C, X on c.

   or

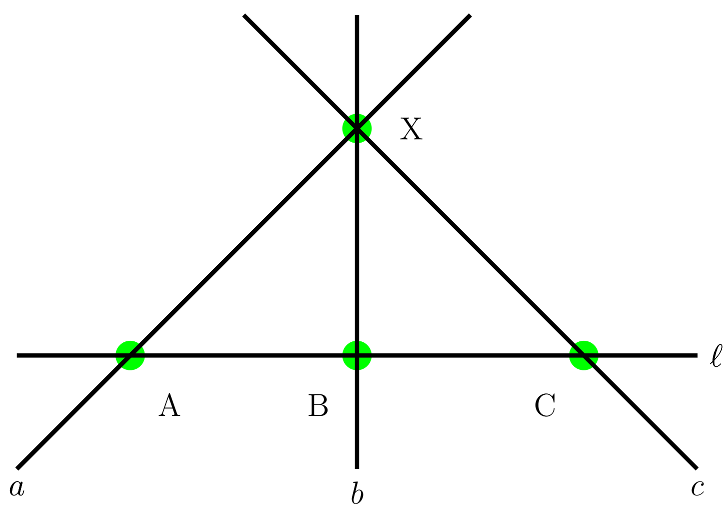
   (b) on each line there are exactly two points.

      i. *Axiom 4:* Pairs X, Y of points, X different from Y correspond biuniquely to lines.

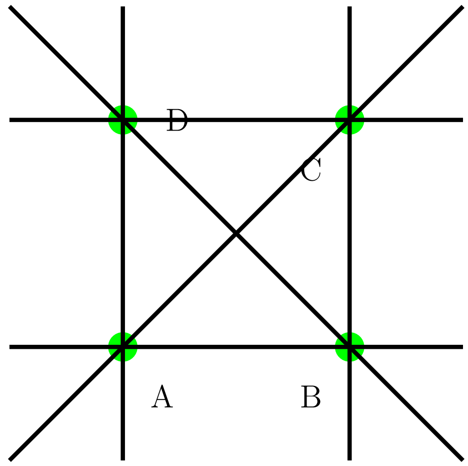
      ii. Since in a collection A, B, C, D of four points there exactly six such pairs, we have the lines as claimed in (1.8).

•

4-point plane, three points on one line



4-point plane, two points on each line



1.7 The Fano Plane

The Fano plane we describe in the following theorem is the smallest example of a **Projective Plane**. A projective plane is one where every two lines intersect; thus parallel lines must be equal. In a projective plane one can interchange the notions of points and lines, see section **??**.

Theorem 1.9 *Assume that in our plane there are exactly three points on each line, and that every two lines intersect. Then this plane has seven points and seven lines, and if the points are*

$$A, B, C, P, Q, R, Z \ ,$$

the triples lying on a common line are

$$ABC \quad , \quad APZ \quad , \quad BQZ \quad , \quad CRZ \quad , \quad PQC \quad , \quad PBR \quad , \quad AQR \ . \tag{1.10}$$

Proof: The theorem makes two assumptions,

1. *3pts:* On every line there are exactly three points
2. *isct:* Every two lines intersect

Now we can use these, the axioms, and theorem 1.2 in our proof:

1. *Axiom 1:* there is a line ℓ.
2. *3pts:* there are exactly three points on ℓ. We name them A, B, C.
3. *Axiom 3:* there is a point Z so that Z does not lie on ℓ.

4. *Axiom 4:* There are unique lines a, b, c, so that A, Z lie on a, B, Z lie on b and C, Z lie on c.

5. *3pts:* On each of the lines a, b, c there is a third point; P on a, Q on b, R on c.

6. *Axiom 4, uniqueness:* The lines a, b, c, ℓ are pairwise different.

7. *Axiom 4, uniqueness:* The point P does not lie on ℓ, b, c. Q does not lie on ℓ, a, c. R does not lie on ℓ, a, b.

8. *Axiom 4:* Let p be the line containing Q and R, q the line containing P, R and r be the line containing P, Q respectively.

9. *Axiom 4, uniqueness:* p does not contain B, C. q does not contain A, C, r does not contain A, B.

10. *isct:* The lines p, q, r intersect ℓ.

11. *9 and 10:* p intersects ℓ in the point A, q in B, r in C.

12. *summary:* We have points A, B, C, Z, P, Q, R and lines ℓ, a, b, c, p, q, r. The points lying on these lines are as in (1.10),

$$\underbrace{ABC}_{\ell} \quad , \quad \underbrace{APZ}_{a} \quad , \quad \underbrace{BQZ}_{b} \quad , \quad \underbrace{CRZ}_{c} \quad , \quad \underbrace{PQC}_{r} \quad , \quad \underbrace{PBR}_{q} \quad , \quad \underbrace{AQR}_{p} \; . \tag{1.11}$$

13. There are no more points: Assume X were a point different from A, B, C, Z, P, Q, R.
    (a) *Axiom 4:* There is a unique line g through A and X.
    (b) *3pts:* This line g is different from the lines ℓ, a, b, c, p, q, r because these lines have already 3 points among A, B, C, Z, P, Q, R.
    (c) *Axiom 4, uniqueness:* Z does not lie on g.
    (d) *isct:* g must intersect b in a point $\hat{B}$ and c in a point $\hat{C}$, say.
    (e) *Axiom 4, uniqueness:* $\hat{B}, \hat{C}$ are different. Otherwise they would both be equal to Z, the unique intersection point of b and c, which is not on g.
    (f) $\hat{B}$ is different from A because A is not on b. Similarly $\hat{C}$ is different from A because A is not on c.
    (g) *3pts:* There are 4 points $A, X, \hat{B}, \hat{C}$ on g, a contradiction.

14. *Axiom 4, uniqueness:* Every pair of points is contained in one of the lines. Hence there is no line different from ℓ, a, b, c, p, q, r.

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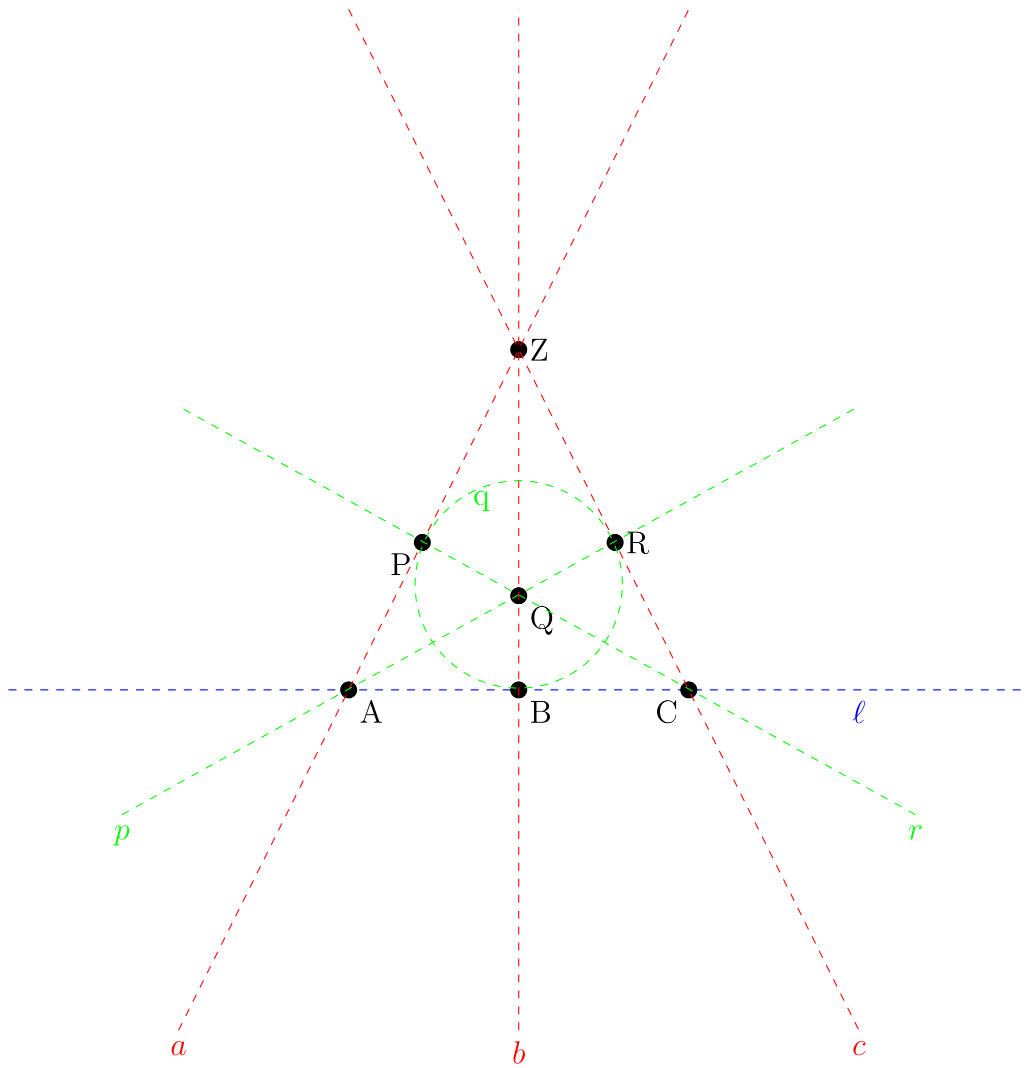


The Fano plane

1.8 Problems

1.8.1 Draw all planes with 5 points.

Hint: There are four such planes. One of them must be the 2-set plane on five points.

As in the proof of theorem 1.7 you might start considering lines with as many points as possible.

Thus, since we have only 5 points in total, and since not all can lie on one line by axiom 3, there can be at most 4 points on a line.

Since two lines can only have two points in common we can not have arbitrarily many lines with four points. In fact there can be at most one, why?

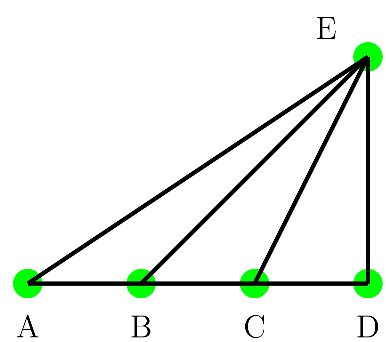
Now work your way down, assuming less and less points on the lines.

Solution: We will assume throughout that our plane has five points A, B, C, D, E.

1. *There is a line ℓ with 4 points:* Then all lines k different from ℓ can have at most 2 points. Otherwise k would intersect ℓ in at least three points, which would violate Axiom 4.

   Assume that the 4 points on ℓ are A, B, C, D, and that E is not in ℓ. The lines AE, BE, CE, DE intersect in the point E and meet the line ℓ in the points A, B, C, D respectively. Thus these lines can have only two points, otherwise they would intersect ℓ in 3 points which is forbidden by axiom 4.

   one line with 4 points

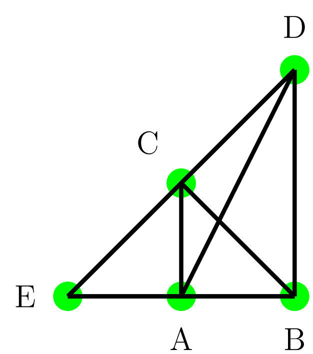
   

2. *The maximum number of points on a line is 3:* We can have at most two lines with three point in which case the two must intersect. Thus we have two subcases:

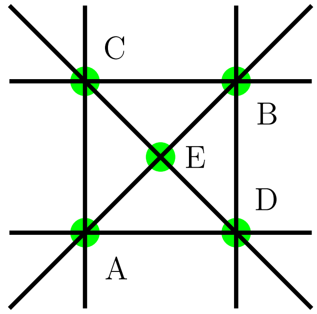
   (a) *There are two lines with 3 points:* We call these lines l and k. Since the plane has 5 points, there is exactly one point E on l and on k. We can name the other 4 points A, B, C, D and so that E, A, B are the points on l, and E, C, D are the points on k. The lines different from l and k must contain 2 points each among A, B, C, D and can not contain E. Thus the lines are

   $$k, l, AC, AD, BC, BD$$

   2 lines with 3 points

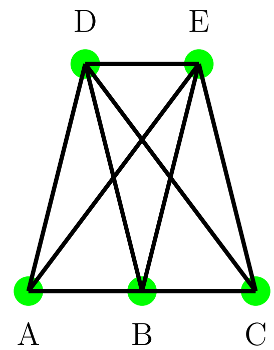
   

This diagram shows the same plane:



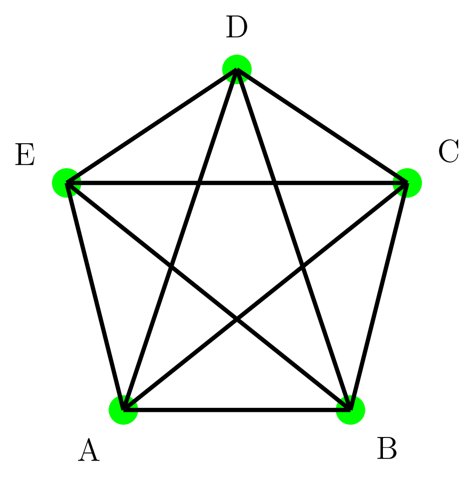
(b) *one line with 3 points*

one line with 3 points



3. *There are no lines with 3 or more points:* Then all lines have 2 points, and by axiom 4, for every 2 points there is a unique line through them. Thus this must be the 2-set plane.

2-set plane on five points



1.9 Some More Examples

1.9.1 A non-plane

We define points, lines and the relation "lie on" as follows:

Points in this candidate plane are natural numbers different from one. Thus the set of points is $\mathcal{P} = \mathbb{N} \setminus \{1\}$.
Lines are sets $\{a, b, c\}$ of three natural numbers so that $1 < a < b$ and $c = ab$. Thus the set of lines is $\mathcal{L} = \{\{a, b, ab\} \mid a, b \in \mathbb{N}, 1 < a < b\}$.
A point $P \in \mathcal{P}$ **lies on** a line $\ell \in \mathcal{L}$ if $P \in \ell$.

This is not a plane, because $\{2, 3, 6\}, \{2, 6, 12\} \in \mathcal{L}$ are different lines but the points 2 and 6 lie on both. Thus Axiom 4 is violated.

1.9.2 The Integer Plane

Points are pairs of integers. Thus set of points is $\mathcal{P} = \mathbb{Z} \times \mathbb{Z}$.
Lines are, as in the standard plane, sets of solutions of linear equations, i.e. sets of the form

$$\ell_{a,b,c} = \left\{(x, y) \in \mathbb{Z}^2 \mid ax + by = c\right\}$$

with $a, b, c \in \mathbb{Z}$, and a, b not both zero . Thus the set of lines is

$$\mathcal{L} = \{\ell_{a,b,c} \mid a, b, c \in \mathbb{Z}, (a, b) \neq (0, 0), \} \ .$$

A point $P \in \mathcal{P}$ **lies on** a line $\ell \in \mathcal{L}$ if $P \in \ell$.

This is not a plane because some lines do not have any points, which would violate Axiom 2: If any common divisor of a and b does not divide c then there are no integers x, y with $ax + by = c$.

To remedy this, we add a condition,

$$\gcd(a, b)|c \quad \text{which is to be read as} \quad a|c \quad , \quad b|c \quad \text{if} \quad b = 0 \quad \text{resp} \quad a = 0$$

in the definition of lines. Thus the integer plane is the plane where

Points are pairs of integers. Thus set of points is $\mathcal{P} = \mathbb{Z} \times \mathbb{Z}$.
Lines are, as in the standard plane, sets of solutions of linear equations, i.e. sets of the form

$$\ell_{a,b,c} = \left\{(x, y) \in \mathbb{Z}^2 \mid ax + by = c\right\}$$

with $a, b, c \in \mathbb{Z}$, and a, b not both zero and so that the greatest common divisor $\gcd(a, b)$ of a and b divides c, $\gcd(a, b)|c$. Thus the set of lines is

$$\mathcal{L} = \{\ell_{a,b,c} \,|\, a, b, c \in \mathbb{Z}, (a, b) \neq (0, 0), \gcd(a, b)|c\} \ .$$

A point $P \in \mathcal{P}$ **lies on** a line $\ell \in \mathcal{L}$ if $P \in \ell$.

We show that this defines a plane:

1. <u>*Axiom 1:*</u> $\ell_{0,1,0} = \mathbb{Z} \times \{0\}$ is a line.
2. <u>*Axiom 2:*</u> Assume $a, b, c \in \mathbb{Z}$ and $\gcd(a, b)|c$, say $c = k \gcd(a, b)$ for some $k \in \mathbb{Z}$. Then there are $x, y \in \mathbb{Z}$ so that $ax + by = \gcd(a, b)$, hence $a(kx) + b(ky) = c$ and $P = (kx, ky)$ lies on $\ell_{a,b,c}$. This shows that there is at least one point on each line. Since a, b are not both zero, the points $P = (kx, ky)$ and $Q = (kx + b, ky - a)$ are different. Since
$$a(kx - b) + b(kx - a) = akx + bky - ab + ba = akx + bky = c$$
the point Q also lies on $\ell_{a,b,c}$.
3. <u>*Axiom 3:*</u> If $a \neq 0$ and $P = (x, y)$ lies on the line $\ell_{a,b,c}$, then $ax + by = c$, hence $a(x + 1) + by = c + a \neq c$ and the point $R = (x + 1, y)$ does not lie on $\ell_{a.b.c}$. Similarly, if $b \neq 0$ and $P = (x, y)$ lies on the line $\ell_{a,b,c}$, then $ax + by = c$, hence $ax + b(y + 1) = c + b \neq c$ and the point $R = (x, y + 1)$ does not lie on $\ell_{a,b,c}$.
4. <u>*Axiom 4:*</u> Let $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ be two points, i.e. $p_1, p_2, q_1, q_2 \in \mathbb{Z}$, and so that $P \neq Q$, i.e. $p_1 \neq q_1$ or $p_2 \neq q_2$. We look for $a, b, c \in \mathbb{Z}$ so that both P and Q lie on $\ell_{a,b,c}$, i.e.
$$\begin{aligned} ap_1 + bp_2 &= c \\ aq_1 + bq_2 &= c \end{aligned}$$
Subtracting the second equation from the first leads to
$$a(p_1 - q_1) + b(p_2 - q_2) = 0 \ .$$
We can solve this by
$$a = q_2 - p_2 \quad , \quad b = p_1 - q_1$$
but also divide the greatest common divisor, i.e. put
$$a = \frac{q_2 - p_2}{\gcd(q_2 - p_2, p_1 - q_1)} \quad , \quad b = \frac{p_1 - q_1}{\gcd(q_2 - p_2, p_1 - q_1)} \ . \tag{1.12}$$
Note that now a, b are coprime, i.e. $\gcd(a, b) = 1$. Since P and Q are different, the denominators are not zero here.

   Inserting (1.12) in one of the two initial equations gives
$$c = p_1 \frac{q_2 - p_2}{\gcd(q_2 - p_2, p_1 - q_1)} + p_2 \frac{p_1 - q_1}{\gcd(q_2 - p_2, p_1 - q_1)} \ .$$
Clearly $\gcd(a, b)|c$. Thus there is a line $\ell_{a,b,c}$ and P, Q lie on $\ell_{a,b,c}$.

   It also follows from the above that any other triple a', b', c' so that P and Q lie on $\ell_{a',b',c'}$ must be an integer multiple of a, b, c and thus $\ell_{a',b',c'} = \ell_{a,b,c}$.

1.9.3 The Standard Plane $\mathbb{R}^2$

The **standard plane** is the plane whose points are pairs of real numbers and whose lines are the (affine) lines, i.e. the sets of solutions of nontrivial linear equations. Thus the lines are sets of the form

$$\ell_{a,b,c} = \{(x,y) \mid ax + by = c\} \quad \text{for} \quad a,b,c \in \mathbb{R},\ (a,b) \neq (0,0).$$

A point $P = (p,q) \in \mathbb{R}^2$ lies on the line $\ell_{a,b,c}$ if $P \in \ell_{a,b,c}$, i.e. if $ap + bq = c$.

2 Ordered Geometry, Planes with Segments, Natural Orders on Lines

2.1 Total Orders

Definition 2.1 *A relation R on a set M is a subset of the cartesian product, $R \subset M \times M$. Usually we use some symbol like $\sim$, $=$, $<$, $<<$, $\ll$ which we write between the related elements, i.e. instead of*

$$(a,b) \in \ll \qquad \text{we write} \quad a \ll b$$

Definition 2.2 *A* **total order** *on a set M is a relation $\ll$ such that for all $x,y,z \in M$ the following hold:*

1. $x \ll y$ *or* $y \ll x$ — *total*
2. $x \ll x$ — *reflexive*
3. *if* $x \ll y$ *and* $y \ll x$ *then* $x = y$ — *antisymmetric*
4. *if* $x \ll y$ *and* $y \ll z$ *then* $x \ll z$ — *transitive*

A **partial order** *on M is a relation that satsifies the last 3 of these conditions but not necessarily the first.*

If R is a relation, then its **reciprocal**, **reverse** or **opposite** is the relation $\overline{R}$ so that

$$a\ R\ b \quad \Longleftrightarrow \quad b\ \overline{R}\ a\ .$$

Proposition 2.3 *The reciprocal of a total order is again a total order.*

Proof: Let R be a total order on the set M, and let $\overline{R}$ be its reciprocal. We check the items in Definition 2.2 for $\overline{R}$. By definition of the reciprocal, for $a, b \in M$, aRb is equivalent to $b\overline{R}a$, hence, for $a, b, c \in M$,

1. "$a\overline{R}b$ or $b\overline{R}a$" is equivalent to "bRa or aRb", equivalently "aRb or bRa" which holds since R is a total order.

2. $a\overline{R}a$ is equivalent to aRa.

3. If $a\overline{R}b$ and $b\overline{R}a$ then, by definition of the reciprocal, bRa and aRb. Since R is a total order this implies that $a = b$.

4. If $a\overline{R}b$ and $b\overline{R}c$ then bRa and cRb, hence cRa. Again by the definition of the reciprocal this gives $a\overline{R}c$.

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Definition 2.4 *If $\ll$ is a total order on a set M and $A, B, C \in M$, then we say B* **lies between** A **and** C **wrt** $\ll$, *if $A \neq B \neq C$ and*

$$A \ll B \ll C \quad \text{or} \quad C \ll B \ll A\ .$$

A few immediate consequences from this definition:

1. If B lies between A and C, it also lies between C and A.

2. If B lies between A and C then $A \neq C$.

3. If B lies between A and C wrt the total order $\ll$, then it also lies between A and C wrt the reciprocal total order $\overline{\ll}$. This because of the equivalence

$$A \ll B \ll C \quad \Longleftrightarrow \quad C \overline{\ll} B \overline{\ll} A$$

Thus a total order gives the same betweenness relation as its reciprocal.

2.2 Equivalence Relations

Definition 2.5 *An equivalence relation $\sim$ on a set M is a reflexive, symmetric, transitive relation, i.e. such that for every $x, y, z \in M$ we have*

1. *$x \sim x$* *reflexive*
2. *$x \sim y$ if and only if $y \sim x$* *symmetric*
3. *if $x \sim y$ and $y \sim z$ then $x \sim z$* *transitive*

If $\sim$ is an equivalence relation on M and $x \in M$, then the **equivalence class of** *x is*

$$[x] = \{m \in M \mid m \sim x\}$$

We draw a few immediate consequences from this definition. Let $\sim$ denote an equivalence relation on M and let $[x]$ denote the equivalence class of $x \in M$.

1. For $x, y \in M$ we have $[x] = [y]$ or $[x] \cap [y] = \emptyset$,
2. We have $x \sim y$ if and only if $[x] = [y]$

2.3 Order Axioms - Axioms for Planes with Segments

We will use the undefined terms

points, **lines**, **lie on** and **natural order**.

The axioms for a plane with segments include the axioms 1-4 for planes, which concern the undefined terms points, lines, lie on. Additionally we postulate

Axiom 2.6 (Order Axioms I)

1. *A natural order is a total order on the points of a line.*
2. *There are two natural orders on each line ℓ and these are mutually reciprocal total orders on the points of ℓ.*

Notation 2.7 *For a line ℓ in a plane with segments, we denote the natural orders on ℓ by $\underset{\ell}{\leq}$, $\underset{\ell}{\geq}$.*

Thus for points A, B on a line ℓ we have $A \underset{\ell}{\leq} B$ if and only if $B \underset{\ell}{\geq} A$, and we always have $A \underset{\ell}{\leq} B$ or $A \underset{\ell}{\geq} B$.

Definition 2.8 (between) *If P, Q, R are points in a plane with segments, we say Q* **lies between** *P*,*R if*

1. *P is different from R,*
2. *Q lies on the line PR,*
3. *Q lies between P and R with respect to one (and hence any) of the natural orders of PQ*

Thus Q lies between P and R if and only if Q lies on the line PR, $P \neq Q \neq R$, and

$$P \underset{PR}{\leq} Q \underset{PR}{\leq} R \quad \text{or} \quad P \underset{PR}{\geq} Q \underset{PR}{\geq} R\,.$$

Definition 2.9 (Half-Planes) *If in a plane with segments, the points A, B do not lie on the line ℓ, then we say A* **and** *B* **lie in the same half-plane of** *ℓ, if there is no point on ℓ that lies between A and B. We write $A \underset{\ell}{\sim} B$ for this.*

If $A \underset{\ell}{\not\sim} B$ then we will say **the line** *ℓ* **separates** *A* **from** *B*

Axiom 2.10 (Order Axioms II) *"Lying in the same half-plane of a line ℓ", $\underset{\ell}{\sim}$, is an equivalence relation on the points not lying on this line. It has at most two equivalence classes. This means that:*

3. *If A is a point not on the line ℓ, then A lies in the same half-plane as A.*
4. *If a point A lies in the same half-plane of a line ℓ as a point B, then the point B lies in the same half-plane as A.*
5. *If a point A lies in the same half-plane of a line ℓ as a point B, and the point B lies in the same half-plane as the point C, then A lies in the same half-plane as the point C.*
6. *If A, B are points not lying in the same half-plane of a line ℓ, and C is any point not lying on ℓ, then A and C lie in the same half-plane or B and C lie in the same half-plane.*

2.4 Segments, Rays, Angles, The Interior Region

Definition 2.11 (Segments) *If A, B are points in a plane with segments, then the segment A, B is*

$$[A, B] := \{A, B\} \cup \{X \mid X \text{ a point between } A, B\}$$

We can rewrite this in terms of the natural order:

Proposition 2.12 *Let A, B be points in a plane with segments. Then*

1. *If $A \neq B$ and $A \underset{AB}{\leq} B$ then*
$$[A, B] = \left\{ X \,\middle|\, X \text{ lies on } AB, A \underset{AB}{\leq} X \underset{AB}{\leq} B \right\}$$
2. *If $A \neq B$ and $A \underset{AB}{\geq} B$ then*
$$[A, B] = \left\{ X \,\middle|\, X \text{ lies on } AB, A \underset{AB}{\geq} X \underset{AB}{\geq} B \right\}$$
3. *If $A = B$ then*
$$[A, B] = [A, A] = \{A\} \ .$$

Theorem 2.13 (Pasch's "Axiom") *Let A, B, C be non-collinear points and ℓ be a line in a plane with segments. Suppose that none of the points A, B, C lies on ℓ and that ℓ intersects the segment $[A, B]$. Then ℓ intersects either segments $[B, C]$ or $[A, C]$.*

Thus if ℓ separates A from B then it must separate A from C or B from C.

Proof: By definition, A and B do not lie in the same half-plane of ℓ. By Order Axiom 6, the point C lies in the same half-plane with A or with B. Thus there are two cases:

1. $\underline{C \underset{\ell}{\sim} A}$: If $C \underset{\ell}{\sim} B$, then by transitivity (Order Axiom 5) we would have $A \underset{\ell}{\sim} B$, contrary to the assumption. We thus have $C \underset{\ell}{\not\sim} B$, and the line ℓ intersects the segment $[B, C]$ and not the segment $[A, C]$.
2. $\underline{C \underset{\ell}{\sim} B}$: This is the same, with the roles of A and B interchanged. Thus, If $C \underset{\ell}{\sim} A$, then by transitivity (Order Axiom 5) we would have $B \underset{\ell}{\sim} A$, contrary to the assumption. We thus have $C \underset{\ell}{\not\sim} A$, and the line ℓ intersects the segment $[A, C]$ and not the segment $[B, C]$.

•

Definition 2.14 (Rays) *Let A, B be different points in a plane with segments. Interchanging the natural orders on AB if necessary, we may also assume that*

$$A \underset{AB}{\leq} B \ .$$

Then the ray $[A, B$ is the set

$$[A, B = \left\{ X \,\middle|\, X \text{ a point on } AB, A \underset{AB}{\leq} X \right\} \ .$$

There is an equivalent way of stating this, avoiding the ambiguity of the natural order:

$$[A, B = \{X \mid X \text{ a point on } AB, A \text{ not between } X, B\} \ . \tag{2.15}$$

Definition 2.16 *An* **angle** *is a set of two rays with the same vertex: For $A, X, B \in \mathcal{P}$, $A \neq X \neq B$,*

$$\angle AXB := \{[X, A \ , \ [X, B\}$$

A **zero angle** *is an angle $\angle AXB$ whose rays coincide, $[X, A \ = [X, B$. A* **straight angle** *is an angle $\angle AXB$ whose rays form a line, i.e. $X \in [A, B]$.*

Definition 2.17 *A set V of points is* **convex** *if*

$$\forall A, B \in V : [A, B] \subset V \ .$$

Definition 2.18 (Half-Plane) *If ℓ is a line and B a point not on ℓ, we denote by $H(\ell, B)$ the* **half plane of ℓ containing** *B, i.e. the equivalence class of B with respect to the relation $\underset{\ell}{\sim}$.*

$$H(\ell, B) := \left\{ X \,\middle|\, X \underset{\ell}{\sim} B \right\} \ .$$

From the definition of segments we can rewrite this as

$$H(\ell, B) = \{X \mid [X, B] \cap \ell = \emptyset\} \ .$$

Definition 2.19 (interior region) *The* **interior region of a non straight angle** *$\angle AXB$ is defined to be empty in case of the zero angle and*

$$\mathrm{IR}\,(\angle AXB) = H((AX), B) \cap H((BX), A)$$

if $[X, A \ \neq \ [X, B$.

2.5 The 2-set plane

There are exactly two total orders on a set with two elements and these are the opposite of each other. There is therefore no choice for the natural orders. Let $\mathcal{P}$ be a set with at least three elements, and consider the 2-set plane on $\mathcal{P}$. In the 2-set plane over $\mathcal{P}$,

Points are elements of $\mathcal{P}$,

Lines are subsets of $\mathcal{P}$ containing 2 elements, i.e. subsets of the form

$$\{A, B\} \quad , \quad A, B \in \mathcal{P} \quad , \quad A \neq B \ .$$

A point A **lies on** the line ℓ if $A \in \ell$

Natural Orders on the line $\ell = \{A, B\}$ are $\underset{\ell}{\leq}, \underset{\ell}{\geq}$ so that $A \underset{\ell}{\leq} B$, respectively $A \underset{\ell}{\geq} B$.

We have already seen that this satisfies the axioms 1-4 for a plane. The order axioms are also satisfied, because in this plane (with the above natural orders), a point never lies between two other points. Thus for a line ℓ and points A, B not on ℓ, we always have $A \underset{\ell}{\sim} B$. The relation "$\underset{\ell}{\sim}$" is an equivalence erlation on points not on ℓ with one equivalence class.

As an example consider the 2-set plane of $\{1, 2, 3, 4\}$, i.e. the one with points $1, 2, 3, 4$, lines

$$\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}$$

and the natural orders $\underset{\{A,B\}}{\leq}, \underset{\{A,B\}}{\geq}$ have

$$A \underset{\{A,B\}}{\leq} B \quad \text{or} \quad A \underset{\{A,B\}}{\geq} B \ .$$

Each line has only one half-plane. For instance the half-plane of $\{1, 2\}$ is $\{3, 4\}$.

2.6 The standard plane

Recall the standard plane from section 1.9.3. In this plane,

Points are pairs $A = (x, y) \in \mathbb{R}^2$ of real numbers

Lines are (sets of) solutions of non-trivial linear equations, i.e. sets of the form

$$\ell_{a,b,c} = \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, ax + by = c\right\} \quad \text{for} \quad a, b, c \in \mathbb{R}, (a, b) \neq (0, 0) \ .$$

A point (x, y) **lies on** the line $\ell_{a,b,c}$ if $(x, y) \in \ell_{a,b,c}$, i.e. if

$$ax + by = c \ .$$

The **natural orders** on a line $\ell_{a,b,c}$ are given by parametrizing this line. Thus, for every line $\ell_{a,b,c}$ there are $P, v \in \mathbb{R}^2$, $v \neq 0$, so that

$$\ell_{a,b,c} = \ell_{P,v} := \{P + tv \,|\, t \in \mathbb{R}\} .$$

We then define the natural orders $\underset{\ell_{P,v}}{\leq}$, $\underset{\ell_{P,v}}{\geq}$ so that

$$P + tv \underset{\ell_{P,v}}{\leq} P + sv \iff t \leq s$$

$$P + tv \underset{\ell_{P,v}}{\geq} P + sv \iff t \geq s$$

Clearly these two are reciprocal. Up to interchanging $\underset{\ell_{P,v}}{\leq}$, $\underset{\ell_{P,v}}{\geq}$, these total orders also do not depend on the choice of P and v to parametrize the line. Thus if $\ell_{P,v} = \ell_{Q,w}$ for some $P, Q, v, w \in \mathbb{R}^2$, $v, w \neq 0$, then

3 Absolute Geometry

In Absolute Geometry we indroduce notions of distance and angle measure. We will use the undefined terms

points, **lines**, **lie on**, **natural order**, **distance**, **angle measure**

In an absolute geometry we assume that points, lines, lie on and natural order satisfy the incidence axioms (for a plane), the order axioms (plane with segments). Additionally we assume axioms involving the new undefined terms distance and angle measure. We list these in three groups, axioms for distance, axioms for the angle measure and the congruence axiom.

Axiom 3.1 (Axioms for the Distance) *Below, A, B are points in an absolute geometry. Then*

0. *The* **distance between two points** *is a real number.*

   Notation: We denote the distance between points A, B by $|A, B|$.
1. *(Positivity) $|A, B| \geq 0$ and $|A, B| = 0$ if and only if $A = B$.*
2. *(Symmetry) $|A, B| = |B, A|$*
3. *(Additivity) If $Q \in [A, B]$ then $|A, B| = |A, Q| + |Q, B|$*

4. *(Construction) Let ℓ be a line, Q a point on ℓ and $\underset{\ell}{\leq}$ a natural order on ℓ. Then for every $\lambda \in \mathbb{R}$, $\lambda \geq 0$, there are unique points S, R on ℓ so that*

$$R \underset{\ell}{\leq} A \underset{\ell}{\leq} S \quad \textit{and}$$

$$|R, A| = |A, S| = \lambda \ .$$

Proposition 3.2 *Let Q be a point lying on a line ℓ in an absolute geometry. Let $<<$ be one of the natural orders on ℓ. Then*

$$\lambda_{\ell,Q,<<} \colon \ell \longrightarrow \mathbb{R}$$

$$\lambda_{\ell,Q,<<}(X) = \begin{cases} |Q, X| & \textit{if } Q << X \\ -|Q, X| & \textit{if } X << Q \end{cases}$$

is an isometric order preserving bijection.

This means that for every points A, B on the line ℓ we have

1. $|A, B| = |\lambda_{\ell,Q,<<}(A) - \lambda_{\ell,Q,<<}(B)|$, isometric
2. $A << B \iff \lambda_{\ell,Q,<<}(A) \leq \lambda_{\ell,Q,<<}(B)$. order preserving

Proof: To prove surjectivity, let $\lambda \in \mathbb{R}$. By the construction axiom there are points R, S so that $R << Q << S$ and $|RQ| = |\lambda| = |QS|$. If $\lambda \geq 0$, then $\lambda_{\ell,Q,<<}(S) = |Q, S| = |\lambda| = \lambda$. If $\lambda \leq 0$, then $\lambda_{\ell,Q,<<}(R) = -|Q, R| = -|\lambda| = \lambda$.

Injectivity follows from the uniqueness of the points R, S in the construction axiom for the distance. If $\lambda_{\ell,Q,<<}(X) = \lambda_{\ell,Q,<<}(Y) =: \lambda$ then $|Q, X| = |Q, Y|$. The only ambiguity this leaves would be that X, Y are on different sides of Q. But then the signs of $\lambda_{\ell,Q,<<}(X)$ and $\lambda_{\ell,Q,<<}(Y)$ would be different.

To see that $\lambda_{\ell,Q,<<}$ is isometric and order preserving, let A, B be points on ℓ. Wlog we may assume that $A << B$. We need to distinguish three cases:

1. $\underline{Q << A << B:}$ Then the first case in the definition of $\lambda_{\ell,Q,<<}$ applies, and by the additivity axiom for the distance, we can compute

   $$\lambda_{\ell,Q,<<}(B) = |Q, B| = |Q, A| + |A, B| = \lambda_{\ell,Q,<<}(A) + |A, B| \ . \tag{3.3}$$

   By positivity of the distance, we have $|A, B| \geq 0$. Now (3.3) gives $\lambda_{\ell,Q,<<}(B) \geq \lambda_{\ell,Q,<<}(A)$ as required to be order preserving, and

   $$|\lambda_{\ell,Q,<<} - \lambda_{\ell,Q,<<}(A)| = \lambda_{\ell,Q,<<} - \lambda_{\ell,Q,<<}(A) = |A, B|$$

   as needed for isometry.

2. <u>$A << Q << B$:</u> Then the first case in the definiton of $\lambda_{\ell,Q,<<}$ applies for B and the second for A. Thus, exploiting additivity and positivity of the distance as in the case before, we compute

$$\lambda_{\ell,Q,<<}(B) = |Q,B| = |A,B| - |A,Q| = |A,B| + \lambda_{\ell,Q,<<}(A) \ . \tag{3.4}$$

We have

$$\lambda_{\ell,Q,<<}(B) \geq 0 \geq \lambda_{\ell,Q,<<}(A) \ ,$$

which shows that $\lambda_{\ell,Q,<<}$ is order preserving, and (3.4) yields

$$|\lambda_{\ell,Q,<<}(B) - \lambda_{\ell,Q,<<}(A)| = \lambda_{\ell,Q,<<}(B) - \lambda_{\ell,Q,<<}(A) = |A,B|$$

for isometry.

3. <u>$A << B << Q$:</u> Now the second case in the definition of $\lambda_{\ell,Q,<<}$ applies for both A and B. Thus, since $|A,Q| = |A,B| + |B,Q|$,

$$\lambda_{\ell,Q,<<}(B) = -|Q,B| = -|Q,A| + |A,B| = \lambda_{\ell,Q,<<}(A) + |A,B| \ . \tag{3.5}$$

In particular

$$\lambda_{\ell,Q,<<}(B) \geq \lambda_{\ell,Q,<<}(A)$$

which shows that $\lambda_{\ell,Q,<<}$ is order preserving also in this case. Since both $\lambda_{\ell,Q,<<}(B), \lambda_{\ell,Q,<<}(A) \leq 0$, we also have

$$|\lambda_{\ell,Q,<<}(B) - \lambda_{\ell,Q,<<}(A)| = \lambda_{\ell,Q,<<}(B) - \lambda_{\ell,Q,<<}(A) = |A,B| \ .$$

•

Because of this proposition, statements about betweenness and distance among points on a fixed lines are equivalent to corresponding statements for real numbers. The following is an example for this.

Proposition 3.6 *Let A,B,C,D be four pairwise different points on the same line in an absolute geometry so that C lies between A,B and B lies between A,D. Assume that*

$$|A,C|\,|D,B| = |A,D|\,|C,B| \ ,$$

then

$$\frac{1}{|A,C|} + \frac{1}{|A,D|} = \frac{2}{|A,B|} \ .$$

Proof: Let $\leq$ be the natural order on (AB) so that $A \leq B$. Then $A \leq C \leq B$ and $A \leq B \leq D$. By transitivity of the total order $\leq$, we have

$$A \leq C \leq B \leq D \ .$$

Let $x, y, z \in \mathbb{R}_0^+$ be the positive real numbers

$$x = |A, C| \quad , \quad y = |C, B| \quad , \quad z = |B, D| \ .$$

Because of the additivity of the distance, we have

$$|A, B| = x + y \quad \text{and} \quad |AD| = x + y + z \ .$$

Thus for real numbers $x, y, z > 0$ we have

$$xz = (x + y + z)y \tag{3.7}$$

and need to show that

$$\frac{1}{x} + \frac{1}{x + y + z} = \frac{2}{x + y} \ .$$

The claim is equivalent to

$$(x + y + z)(x + y) + x(x + y) = 2x(x + y + z) \ ,$$

$$2x^2 + 3xy + y^2 + zx + zy = 2x^2 + 2xy + 2xz \ ,$$

$$xy + y^2 + zy = xz \ ,$$

which is (3.7). •

Axiom 3.8 (Axioms for the Angle Measure) *Below, A, X, B are points in an absolute geometry, so that $A \neq X \neq B$.*

0. *The* **angle measure of an angle** $\angle AXB$ *is a real number.*

   Notation: We denote the angle measure of the angle $\angle AXB$ by $|\angle AXB|$.
   Note that since the angle $\angle AXB$ is a set of two rays, we have $\angle AXB = \angle BXA$. Hence $|\angle AXB| = |\angle BXA|$.

1. $|\angle AXB| \in [0, 180]$ *and*

   (a) $|\angle AXB| = 0$ *if and only if $\angle AXB$ is a zero angle,*

   (b) $|\angle AXB| = 180$ *if and only if $\angle AXB$ is a straight angle.*

2. *("Additivity") If $\angle AXB$ is not a straight angle and $Z \in IR(\angle AXB)$, then*

$$|\angle AXB| = |\angle AXZ| + |\angle ZXB|$$

3. *("Additivity") If $\angle AXB$ is a straight angle and Z is a point not on AB, then*

$$180 = |\angle AXB| = |\angle AXZ| + |\angle ZXB|$$

4. *("Uniqueness") If P, Q are points in the same half-plane of the line XA, then*

$$|\angle AXP| = |\angle AXQ| \quad \Longrightarrow \quad [X, P = [X, Q \ .$$

5. *("Construction") Let H be a half plane of the line XA. Then for every $\lambda \in (0, 180)$ there is a point Q in the half-plane H so that*

$$|\angle QXA| = \lambda \ .$$

There is a bijection similar to that of Proposition 3.2 between rays and the interval $[0, 180]$,

Proposition 3.9 *Let A, X be points in an absolute geometry so that $A \neq X$, and let H be a half-plane of the line XA. Then the map*

$$\alpha_{X,A,H} \colon \{[XQ \mid Q \text{ lies in } H \text{ or on } XA, Q \neq X\} \longrightarrow [0, 180]$$
$$\alpha_{X,A,H}([X, Q) = |\angle AQX|$$

is a bijection.

Proof: By the axiom 1 for the angle measure we have $|\angle AXA| = 0$ and $|\angle AXQ| = 0$ for a point $Q \neq X$ only if $[X, A = [X, Q$.

Let now $<<$ be the natural order on XA so that $X << A$. By the construction axiom for the distance, there is a point $Q \neq X$ on XA so that $Q << X$ (the one with $|Q, X| = 1$ for instance). In particular X lies between Q, A and $\angle QXA$ is a straight angle. By axiom 1 for the angle measure this is equivalent to $|\angle AXQ| = 180$.

If $\lambda \in (0, 180)$ then by the construction axiom for the angle measure, there is Q in the half-plane H so that $|\angle AXQ| = \lambda$. By the uniqueness axiom for the angle measure, if Q' is a second point in the same half-plane H with $|\angle AXQ'| = |\angle AXQ| = \lambda$ we must have $[X, Q = [X, Q'$. •

Theorem 3.10 (Opposite Angles) *Let A, B, X, Y, Q be points in an absolute geometry so that Q is between A, X and also between B, Y. Then the* **opposite angles** *$\angle AQB$ and $\angle XQY$ have the same measure.*

Proof: By the additivity of the angle measure in straight angles, we have

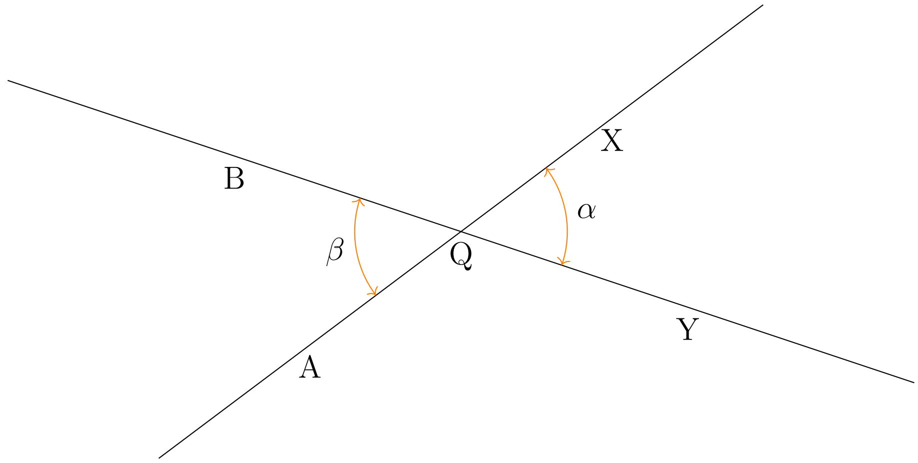
$$180 = |\angle AQX| = |\angle AQB| + |\angle BQX| \ ,$$

$$180 = |\angle BQY| = |\angle BQX| + |\angle XQY| \ .$$

Subtracting the two equations eliminates $|\angle BQX|$ to yield

$$0 = |\angle AQB| - |\angle XQY| \ .$$

•



α and β are opposite angles.

3.1 Congruence

Two sets of points in an absolute geometry are called **congruent** if all the distances and angles arising in one of the sets are equal to these quantities in the other under some correspondence between the sets.

Definition 3.11 *A* **polygon***, more specifically a n*-**gon** *for some $n \in \mathbb{N}$, is a n-tuple $(P_1, \dots, P_n)$ of pairwise different points in an absolute geometry, i.e. $P_i \neq P_j$ for $i \neq j$. The points P_i are referred to as the* **vertices** *of the polygon.*

Two n-gons $(P_1, \dots, P_n)$ and $(Q_1, \dots, Q_n)$ are said to be **congruent***, and we write*

$$(P_1, \dots, P_n) \cong (Q_1, \dots, Q_n)$$

if

$$\forall i, j = 1, \dots, n : |P_i, P_j| = |Q_i, Q_j|$$

and

$$\forall i, j, k = 1, \dots, n : |\angle P_i P_j P_k| = |\angle Q_i Q_j Q_k| \ \ .$$

Two sets M and N of points in an absolute geometry are congruent if there is a bijection $\phi\colon M \to N$ so that for all $A, B, C \in M$ we have

$$\begin{aligned} |\phi(A), \phi(B)| &= |A, B| \\ |\angle \phi(A)\phi(B)\phi(C)| &= |\angle ABC| \quad \textit{if} \quad A \neq B \neq C \end{aligned}$$

Note a subtle difference between these two notions of congruence: If A, B, C are pairwise different points in an absolute geometry, then the triangles (i.e. 3-gons) (A, B, C) and (B, A, C) need not be congruent whereas the sets $\{A, B, C\}$ and $\{B, A, C\}$ always are (because they are the same sets). We will almost exclusively be concerned with congruence of polygons.

The **The Congruence Axiom SAS for Triangles** requires that two triangles are congruent if they agree in two sides and the enclosed angle. Thus

Axiom 3.12 (Congruence Axiom SAS) *If* $A, B, C, A', B', C' \in \mathcal{P}$, $A \neq B \neq C \neq A, A' \neq B' \neq C' \neq A'$ *are so that*

$$|A, B| = |A', B'| \quad , \quad |A, C| = |A', C'| \quad \textit{and} \quad |\angle BAC| = |\angle B'A'C'|$$

then

$$|\angle ABC| = |\angle A'B'C'| \quad , \quad |\angle ACB| = |\angle A'C'B'| \quad \textit{and} \quad |B, C| = |B', C'|$$

Thus in an absolute geometry

$$|A, B| = |A', B'| \quad \text{and} \quad |A, C| = |A', C'| \quad \text{and} \quad |\angle BAC| = |\angle B'A'C'| \quad \Longrightarrow \quad (A, B, C) \cong (A', B', C') .$$

The congruence

$$(A, B, C) \cong (B, C, A)$$

means that

$$|A, B| = |B, C| = |C, A| \tag{3.13}$$

i.e. all the sides are equal, the triangle is **equilateral**, and

$$|\angle ABC| = |\angle BCA| = |\angle BAC| \tag{3.14}$$

i.e. all angles are also equal.

We will later see that for a triangle (A, B, C) in an absolute geometry, (3.13) and (3.14) are equivalent.

The congruence

$$(A, C, B) \cong (B, C, A)$$

implies that

$$|A, C| = |B, C| \ , \tag{3.15}$$

the sides containing C are equal. Such triangles are called **isosceles**. For the angles this congruence gives

$$|\angle CAB| = |\angle CBA| \tag{3.16}$$

i.e. the angle at A equals the angle at B.

Proposition 3.17 *Let (A, B, C) be an isosceles triangle, the sides containing C being equal,*

$$|A, C| = |B, C| \ .$$

Then

$$(A, C, B) \cong (B, C, A) \ .$$

In particular

$$|\angle CAB| = |\angle CBA| \ ,$$

the angles at A and B are equal.

Proof:

Since the angle at C is

$$|\angle ACB| = |\angle BCA|$$

and

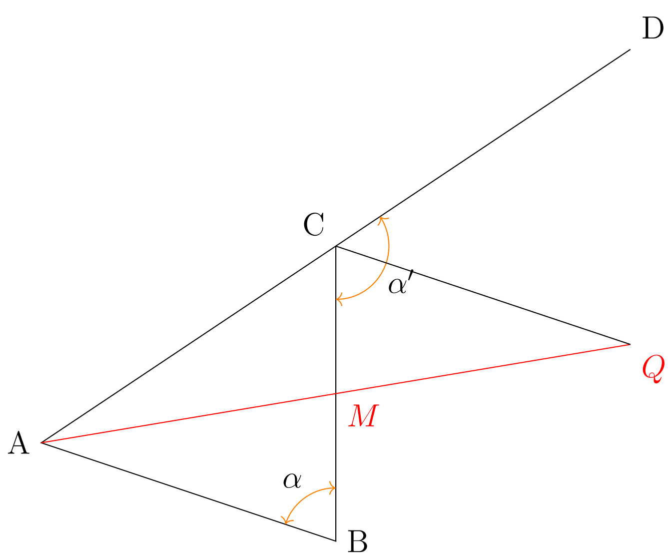
$$|A, C| = |B, C|$$

by assumption, the claim follows from the congruence axiom. •

Theorem 3.18 (Alternate Angles Theorem of Absolute Geometry) *Let A, B, C be non-collinear points in an absolute geometry, and D a point on AC, so that C lies between A, D. Then*

$$|\angle ABC| < |\angle BCD|$$

$|\angle ABC|, |\angle BCD|$ as in the Theorem are called "alternate angles".



α and α' are alternate angles, $\alpha' > \alpha$.

Do not confuse this theorem with the Theorem 3.38 about corresponding and alternating angles in a Euclidean plane, i.e. an absolute geometry where parallels are unique.

Proof: Let A, B, C, D be as in the Theorem. Let M be the midpoint of the segment $[B, C]$, i.e. the point M on (BC) with $|M, B| = |M, C|$. Let $Q \in (AM)$ be so that M is the midpoint of the segment $[A, Q]$. The triangles AMB and QMC are congruent by the congruence axiom because

$$|\angle AMB| = |\angle QMC|$$

because the angles are opposite and

$$|A, M| = |Q, M| \quad \text{and} \quad |B, M| = |C, M|$$

because M is the midpoint of the two segments $[A, Q]$ and $[B, C]$. In particular, because of the congruence axiom,

$$|\angle ABC| = |\angle ABM| = |\angle QCM| = |\angle QCB| \ .$$

We will show that $Q \in IR(\angle BCD)$. Then by the additivity axiom for the angle measure,

$$|\angle BCD| = |\angle BCQ| + |\angle QCD| > |\angle BCQ| = |\angle ABC| \ .$$

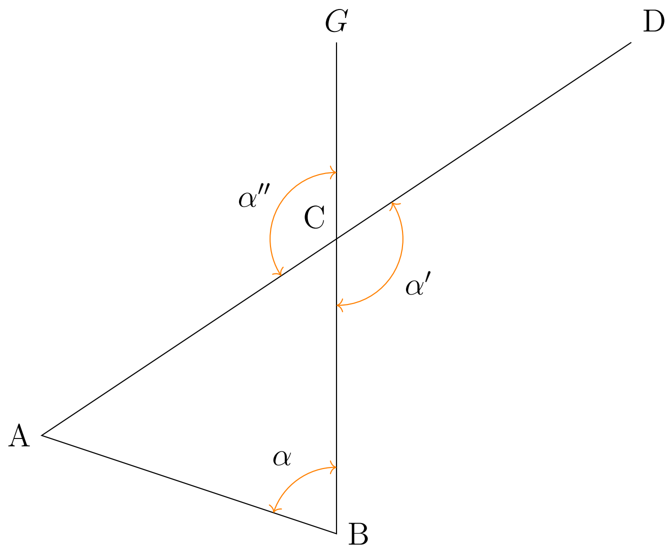
To see that Q lies in the interior region of $\angle BCD$ we need to show

1. *D and Q lie in the same half-plane of BC:* Both D and Q do not lie in the same half-plane of BC with the point A, because M is between A, Q and C is between A, D and M and C lie on BC.

2. *B and Q lie in the same half-plane of CD:* B and M lie in the same half-plane of CD and Q and M lie in the same half-plane of CD. By the transitivity of "lie in the same half-plane", B and Q lie in the same half-plane of CD.

•

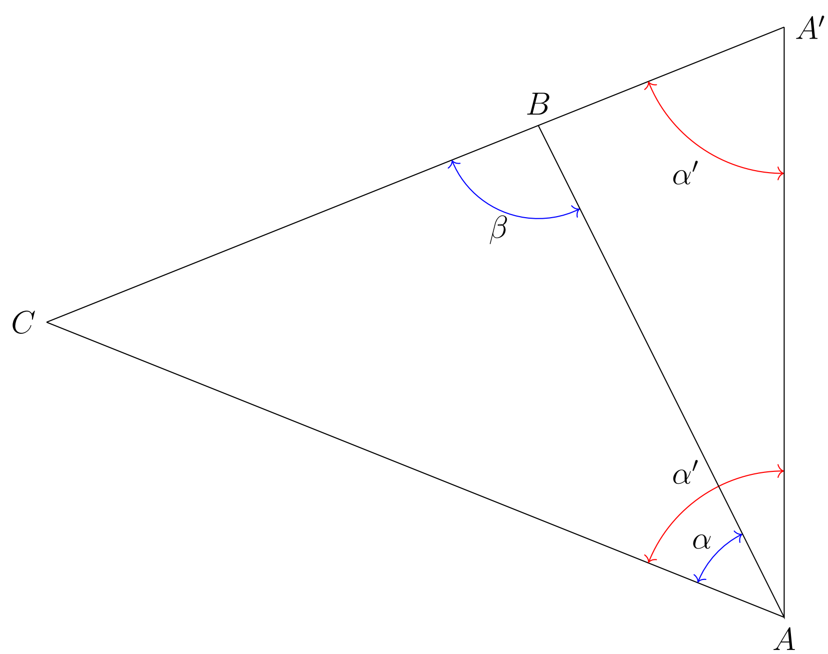
Corollary 3.19 *In the notation of Theorem 3.18, let $G \in (BC)$ so that C is between B and G. Then $|\angle ACG| > |\angle ABG|$*



α and α' are alternate angles, α' and α'' are opposite, hence equal. $\alpha'' = \alpha' > \alpha$.

Theorem 3.20 (The longer side lies opposite the larger angle) *If A, B, C are non-collinear points in an absolute geometry, then*

$$|\angle ABC| < |\angle BAC| \quad \Longleftrightarrow \quad |A,C| < |B,C|$$



$$|A,C| = |A',C| > |C,B| \Rightarrow B \in \mathrm{IR}\,(\angle CAA') \Rightarrow |\angle CBA| > |\angle CA'A| = |\angle CAA'| > |\angle CAB|$$

Proof: We know from proposition 3.17 that $|A,C| = |B,C| \Rightarrow |\angle ABC| = |\angle BAC|$. We will now show the implication

$$|A,C| > |B,C| \quad \Longrightarrow \quad |\angle ABC| > |\angle BAC| \tag{3.21}$$

and

$$|A,C| < |B,C| \quad \Longrightarrow \quad |\angle ABC| < |\angle BAC| \ . \tag{3.22}$$

The Theorem then follows, since the cases $|A,C| < |B,C|$, $|A,C| = |B,C|$, $|A,C| > |B,C|$ are mutually exclusive. It suffices to prove (3.21), the proof of (3.22) is then obtained by interchanging the roles of A and B.

Thus assume $|A,C| > |B,C|$. By the construction axiom for the distance, there is a unique point $A' \in (CB)$ so that $|CA'| = |CA|$ and C is not between B and A'. Since

$$|C,A'| = |CA| > |CB|$$

we have that B lies between C and A', hence $B \in \mathrm{IR}\,(\angle A'AC)$. By angle additivity, we have that

$$|\angle A'AC| = |\angle A'AB| + |\angle BAC| > |\angle BAC| \ . \tag{3.23}$$

Since $|AC| = |A'C|$ we have

$$|\angle A'AC| = |\angle AA'C| \tag{3.24}$$

from proposition 3.17. Now $\angle AA'C$ and $\angle ABC$ are opposite-alternate angles as in Corollary 3.19, hence

$$|\angle AA'C| < |\angle ABC| \ . \tag{3.25}$$

From (3.23), (3.24), (3.25) together we now get $|\angle ABC| > |\angle BAC|$, i.e. the right hand side of (3.21). •

Corollary 3.26 (Isosceles Triangles) *Let A, B, C be points in an absolute geometry. Then*

$$|A,C| = |B,C| \quad \Longleftrightarrow \quad |\angle BAC| = |\angle ABC| \ .$$

Proof: The implication $|A,C| = |B,C| \quad \Longrightarrow \quad |\angle BAC| = |\angle ABC|$ is established in proposition 3.17. If $|A,C| \neq |B,C|$ we must have either $|A,C| > |B,C|$, which implies $|\angle BAC| < |\angle ABC|$, or else $|A,C| < |B,C|$, which implies $|\angle BAC| > |\angle ABC|$. Thus we can have equality of the angles only if the sides are equal. •

The side-angle relation in Theorem 3.20 turns an absolute geometry into a **metric space**: The distance is positive and symmetric by axiom. We now show that it also satisfies the **triangle inequality**.

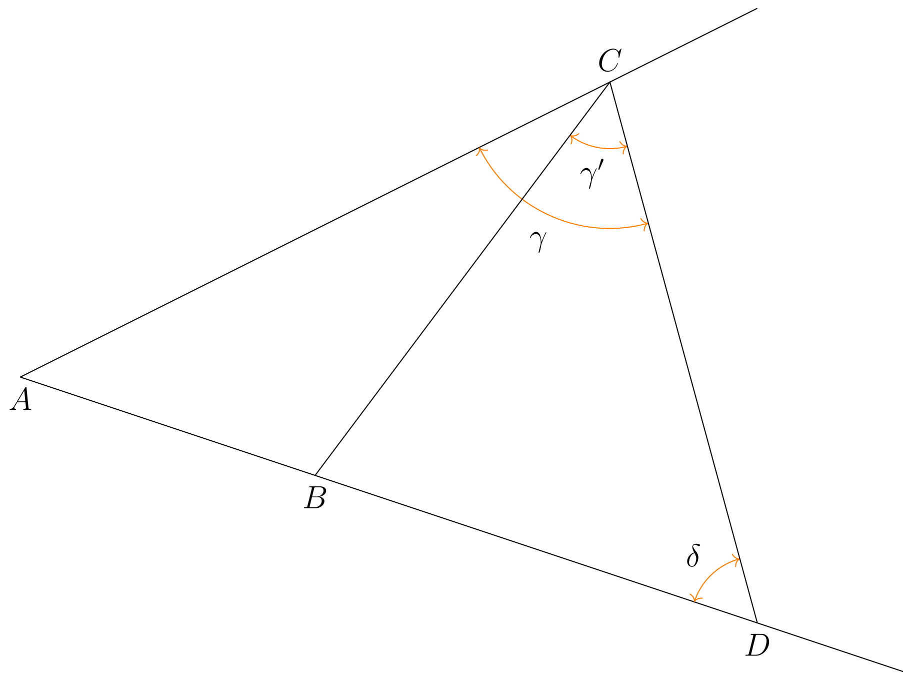
Theorem 3.27 *For points A, B, C in an absolute geometry*

$$|A,C| \leq |A,B| + |B,C| \tag{3.28}$$

with equality if and only if B lies on the segment $[A,C]$.

Proof: If the three points are collinear, i.e. on a common line, then (3.28) becomes the triangle inequality for the absolute value on the real numbers. If two of the points are equal (3.28) becomes a trivial equality. If the points are pairwise different but still collinear, then one of the point lies between the other two, and by additivity of the distance, the longer distance of the three is the sum of the other two.

If the points are not collinear we will show that we have strict inequality. On the line (AB), construct a point D so that $|A, D| = |A, B| + |B, C|$. Look at the triangle (D, B, C).



$|B, D| = |B, C|$, $\delta = \gamma' \leq \gamma$.

Then B is between A and D, hence $B \in \mathrm{IR}\,(\angle ACD)$. We must have $|B, C| = |B, D|$, the triangle (C, D, B) is isosceles, hence $|\angle BCD| = |\angle BDC|$ by proposition 3.17. By additivity of the angle measure we have

$$|\angle ACD| = |\angle ACB| + |\angle BCD| > |\angle BCD| = |\angle BDC| = |\angle ADC| \ .$$

Thus in the triangle (ADC) the angle at C is larger than the angle at D. By Theorem 3.20 the side opposite C, i.e. $[A, D]$ is longer than the side opposite D, i.e. $[A, C]$. Thus

$$|A, C| < |A, D| = |A, B| + |B, C| \ .$$

•

Theorem 3.29 (Distance of a Point from a Line, Perpendicular) *Let P be a point and ℓ be a line in an absolute geometry. The* **distance of P from ℓ** *is*

$$d(P, \ell) := \inf \{ |P, Q| \mid Q \in \ell \} \ .$$

There is a unique point $F(P,\ell) \in \ell$ so that

$$|F(P,\ell), P| = d(P,\ell) \ . \tag{3.30}$$

If $P \notin \ell$, then at $F(P,\ell)$ the segment $[P, F(P,\ell)]$ makes a right angle with ℓ, i.e. for every point Z on ℓ and different from $F(P,\ell)$ we have

$$|\angle Z F(P,\ell) P| = 90 \ . \tag{3.31}$$

We also have

$$|\angle F(P,\ell) Z P| < 90 \quad \textit{for every point} \quad Z \in \ell \setminus \{F(P,\ell)\} \tag{3.32}$$

Proof: We may assume $P \notin \ell$. For the construction of $F(P,\ell)$, choose any points $A, B \in \ell$, $A \neq B$. By the construction axiom for the angle measure and the distance there is a point Q in the half plane of $\ell = (AB)$ not containing P such that

$$|A,P| = |A,Q| \quad \text{and} \quad |\angle PAB| = |\angle QAB| \ .$$

Since P and Q lie in different half planes of ℓ, there is a unique point $F(P,\ell)$ on ℓ and between P and Q.

We show that $F = F(P,\ell)$ has the properties outlined above: If $A = F$, then $A \in [P,Q]$ and hence $|\angle ZFP| = 90$ by the construction above. If $A \neq F$, then SAS yields

$$(P, A, F) \cong (Q, A, F)$$

hence

$$|\angle PFA| = |\angle QFA| \ .$$

But since these two angles add up to 180, they must both be 90 which proves (3.31). The estimate (3.32) is an imediate consequence of the alternate angles theorem. To show that F realizes the distance uniquely, let Z be a point on ℓ different from F, and look at the triangle (Z, P, F). The angle at F is 90 and by the alternate angles theorem, the angle at Z is < 90. By Theorem 3.20 the side opposite F, i.e. $[Z,P]$ is strictly longer than the side opposite Z, i.e. $[P,F]$. Thus $|P,Z| > |P,F|$ proving (3.30). •

3.2 Congruence Theorems

Theorem 3.33 (SSS) *Two triangles are congruent if and only if corresponding sides are equal: For triangles (A,B,C), (A',B',C') in an absolute geometry, we have*

$$|A,B| = |A',B'|\,,\ |A,C| = |A',C'|\,,\ |B,C| = |B',C'| \quad \Longleftrightarrow \quad (A,B,C) \cong (A',B',C') \ .$$

Proof: Let (A,B,C), (A',B',C') be as in the theorem. Let C'' be a point in the half plane of (AB) not containing C', so that $|\angle C''AB)| = |\angle C'A'B'|$. Thus C and C'' lie in opposite half planes of

(AB). By the congruence axiom SAS, $(C'', A, B) \cong (C', A', B')$. Since C and C'' lie in different half planes of (AB) but not on (A, B) thee is exactly one point X so that $(AB) \cap [CC'']$.

By the assumption of the theorem and the construction of C'', $|A, C| = |A'C'| = |A, C''|$ and $|B, C| = |B'C'| = |B, C''|$. In particular the triangles (A, C, C'') and (B, C, C'') are isosceles. By 3.26 we therefore have

$$|\angle ACX| = |\angle ACC''| = |\angle AC''C| = |\angle AC''X| \quad \text{and}$$
$$|\angle BCX| = |\angle BCC''| = |\angle BC''C| = |\angle BC''X|$$

Let $\leq$ be the natural order on (AB) so that $A \leq B$. As to the position of X relative to A, B we now have three cases

1. *X lies between A, B* Then $X \in \mathrm{IR}(\angle ACB)$ and $X \in \mathrm{IR}(\angle AC''B)$. By the additivity of the angle measure we have

$$|\angle ACB| = |\angle ACX| + |\angle XCB| = |\angle AC''X| + |\angle XC''B| = |\angle AC''B| \ .$$

   By the congruence axiom SAS, applied at the angles at C respectively C'', we have $(A, C, B) \cong (A, C'', B)$.

2. $A \leq B \leq X$ If $X = B$ then

$$|\angle ACB| = |\angle ACX| = |\angle AC''X| = |\angle AC''B|$$

   and we finish as in the first case. If $X \neq B$, then B lies between X and A, hence $B \in \mathrm{IR}(\angle ACX)$ and $B \in \mathrm{IR}(\angle AC''X)$. By additivity of the angle measure,

$$|\angle ACB| = |\angle ACX| - |\angle BCX| = |\angle AC''X| - |\angle BC''X| = |\angle AC''B| \ .$$

   Again the congruence axiom SAS applied at the angles at C and C'' yields the congruence $(A, C, B) \cong (A, C'', B)$.

3. $X \leq A \leq B$ This case is analogous to the previous one, interchanging A with B.

•

Theorem 3.34 (ASA) *For triangles (A, B, C), (A', B', C') in an absolute geometry, we have*

$$|\angle CAB| = |\angle C'A'B'|\,,\ |A, B| = |A', B'|\,,\ |\angle ABC| = |\angle A'B'C'| \quad \Longleftrightarrow \quad (A, B, C) \cong (A', B', C') \ .$$

Proof: Given such triangles (A, B, C), (A', B', C'), let $C'' \in [AC$ be such that $|A'C'| = |A, C''|$ (construction axiom for the distance). By the congruence axiom SAS, $(A'C'B') \cong (A, C'', B)$. We may assume that $|A, C| \leq |A, C''| = |A'C'|$. If not we interchange the triangles (A, B, C), (A', B', C'). If $C \neq C''$ then additivity of the distance on the line (AC) yields that $|A, C| < |A, C| + |C, C''| = |A, C''| = |A'C'|$, hence C lies between A, C'' and therefore $C \in \mathrm{IR}(\angle ABC'')$. By the additivity of the angle measure, we must have

$$|\angle ABC| < |\angle ABC| + |\angle CBC''| = |\angle ABC''| = |\angle A'B'C'|$$

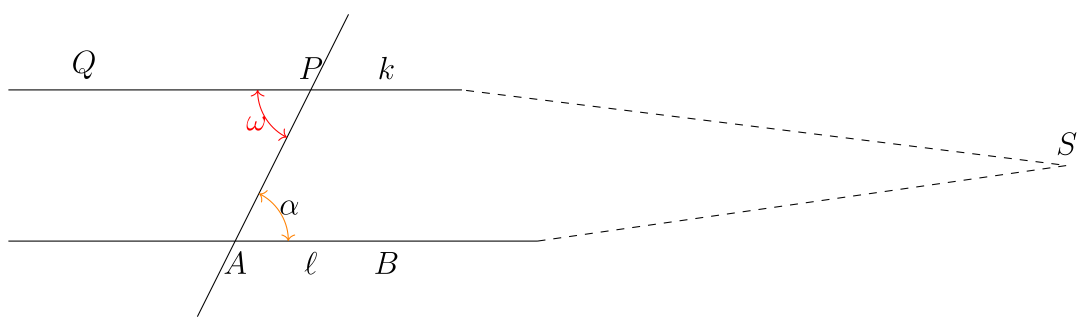
but this contradicts the assumption of the theorem. •

3.3 Parallels, Euclid's 5th postulate

Recall Definition 1.4, that two lines in a plane are **parallel** if they are equal or if they do not intersect. By the axioms 1.1, two lines in a plane either intersect in exactly one point or else are parallel. In planes, even in planes with segments, parallels through given points need not exist. The 2-set plane on three points is a simple example. However, because of the construction axiom for the angle measure, paralles through given points exist in absolute geometry.

Theorem 3.35 *In an absolute geometry, for every line ℓ and every point P, there is a line k so that P lies on k and k is parallel to ℓ.*

Proof: If P lies on ℓ we can simply take $k = \ell$. Thus we may assume the point P does not lie on ℓ. Let $A \neq B$ be two points on ℓ and $Q \neq P$ so that Q and B lie on opposite sides of (AB) and $|\angle APQ| = |\angle PAB|$. Such a point Q exists by the construction axiom for the angle measure. Let $k := (PQ)$. If there were a point $S \in \ell \cap k$, then $|\angle APQ|$ and $|\angle PAB|$ would be alternate angles, and by the alternate angles theorem 3.18 the one of the two lying on the opposite side of S with respect to (AP) would be bigger than the one lying on the same side as S.



If there were a point S, then $\omega > \alpha$.

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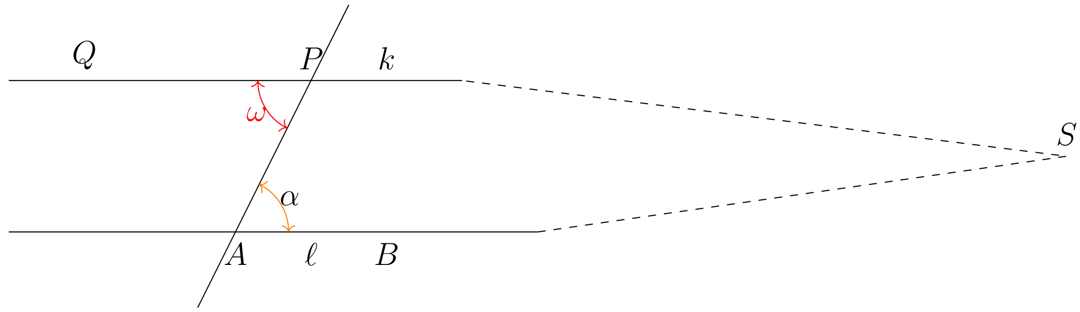
We will see later that in an absulute geometry, paralles need not be unique. We will use two equivalent ways stating this uniqueness.

Theorem 3.36 *In an absolute geometry the following are equivalent:*

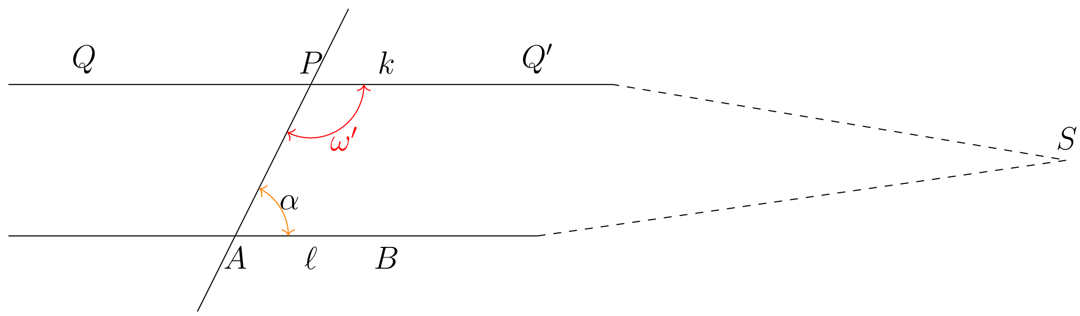
(A). (Playfair's axiom) For every line ℓ and every point P there is a **unique** *line k through P and parallel to ℓ.*

(B). (Euclid's fifth postulate) Let A, B, P, Q be points, $A \neq P$, and so that P and Q lie in different half-planes of AP. If $|\angle APQ| \neq |\angle BAP|$, then the lines (AB) and PQ intersect.



If $\omega \neq \alpha$ then k and ℓ intersect.

Often Euclid's fifth postulate is stated as in problem 26 in terms of an angle complementary to $\angle QPA$.



If $\omega' + \alpha \neq 180$ then k and ℓ intersect.

Proof: By Theorem 3.35, a parallel (at least one) always, i.e. in any absolute geometry, exists. Thus this theorem is about the equivalence of the uniqueness statement in (A) with (B).

Let A, B, P be non-collinear points $\ell := AB$, and let g be the parallel to ℓ through P constructed in Theorem 3.35. If Q is any point on g in the half-plane of AP opposite B, then $|\angle APQ| = |\angle BAP|$. Now let h be any line through P different from AP and let X be a point on h in the same half-plane as Q, i.e. also in the half-plane of AP opposite B.

First we assume Euclid's fifth postulate. If $|\angle APX| = |\angle BAP|$ we must have $g = h$. If $|\angle APX| \neq |\angle BAP|$ then h intersects ℓ, i.e. is not parallel. Thus we have only one parallel through P.

Now assume Playfair's axiom holds, i.e. g is the only parallel to ℓ through P. Then, if $|\angle APX| \neq |\angle BAP|$ we have $h \neq g$. Thus h can not be parallel, i.e. must intersect ℓ, which is the claim of Euclid's fifth postulate. •

Definition 3.37 *An absolute geometry which satisfies Euclid's fifth postulate, equivalently Playfair's parallel axiom, is a* **Euclidean Plane**.

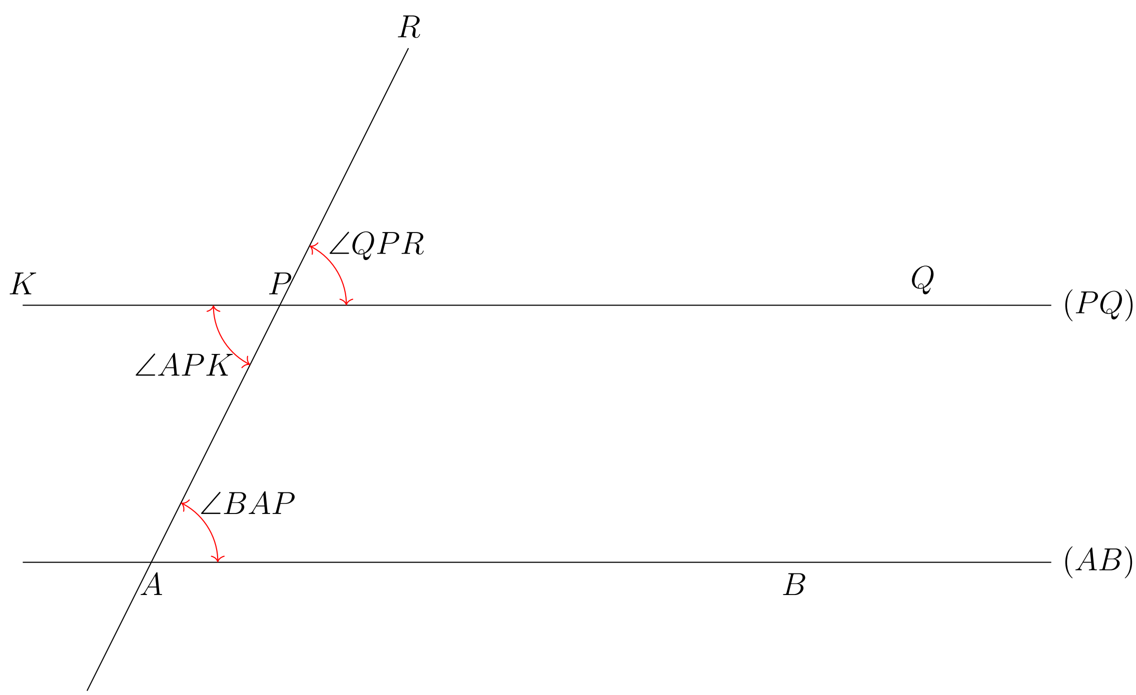
In a Euclidean plane, parallelism can be characterzed by angles. Thus

Theorem 3.38 (Corresponding Alternate Angles Theorem of Euclidean Geomety) *Let A, B, K, P, Q, R be points in a Euclidean plane, so that A, B, P are not collinear, P lies between A and R, P lies between K and Q, and B, Q lie in the same half-plane of AP. Then the following are equivalent*

1. *AB and PQ are parallel.*
2. *$|\angle BAP| = |\angle QPR|$ "corresponding angles are equal".*
3. *$|\angle BAP| = |\angle APK|$ "alternate angles are equal".*

Note that the implications $2 \Rightarrow 1$, $3 \Rightarrow 1$ and $2 \Leftrightarrow 3$ in this theorem already hold in absolute geometry. The implications $1 \Rightarrow 2$, $1 \Rightarrow 3$ however require Euclidean geoemtry i.e. one of (A), (B).

Proof: This is immediate from Euclid's fifth postulate and the Opposite Angles Theorem 3.10. •



Theorem 3.39 *(Angle Sum in Triangles) Let A, B, C be non-collinear points in a Euclidean plane with unique parallels. Then*

$$180 = |\angle ABC| + |\angle BCA| + |\angle CAB| \ .$$

Proof: Let A' be a point in the half-plane of (BC) opposite to A and so that $|\angle BCA'| = |\angle ABC|$. Analogously, let B' be in the halp-plane of (AC) opposite to B and so that $|\angle ACB'| = |\angle BAC|$.

As in the proof of Theorem 3.35, it follows that $(A'C)$ and $(B'C)$ are parallels to (AB) through C. Since parallels are unique in this plane, we have $(A'C) = (B'C)$. Hence $\angle A'CB'$ is a straight angle and therefore

$$180 = |\angle B'CA| + |\angle ACB| + |\angle BCA'| = |\angle CAB| + |\angle ACB| + |\angle ABC| \ .$$

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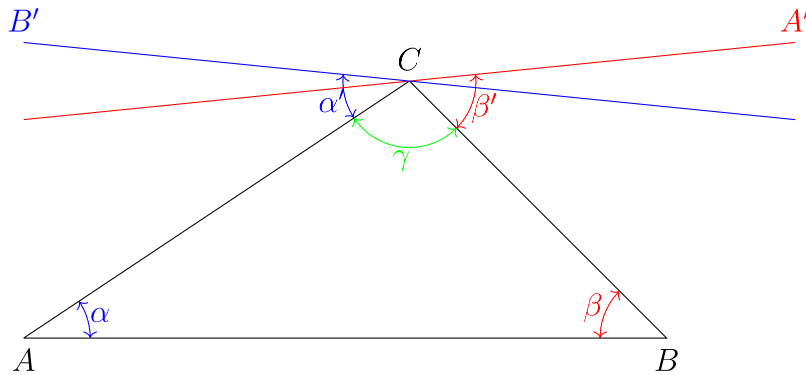


By uniqueness of parallels, the red and the blue lines coincide, $\alpha + \beta + \gamma = 180$.

4 Riemannian metrics on $\mathbb{R}^2$

4.1 Scalar Products on $\mathbb{R}^n$

Definition 4.1 *A scalar product s on $\mathbb{R}^n$ is a positive definite, symmetric, bilinear function $\mathbb{R}^n \times \mathbb{R}^n \xrightarrow{s} \mathbb{R}$. We often write $s(x,y) = \langle x \mid y \rangle = xy$ if a scalar product is fixed. This means that for all $x, y, z, w \in \mathbb{R}^n$ and $\xi, \upsilon, \zeta, \omega \in \mathbb{R}$ we must have*

$$\langle x \mid x \rangle \geq 0 \quad \text{and} \quad \langle x \mid x \rangle = 0 \text{ only if } x = 0$$
$$\langle x \mid y \rangle = \langle y \mid x \rangle$$
$$\langle \xi x + \upsilon y \mid \zeta z + \omega w \rangle = \xi\zeta \langle x \mid z \rangle + \xi\omega \langle x \mid w \rangle + \upsilon\zeta \langle y \mid z \rangle + \upsilon\omega \langle y \mid w \rangle$$

Given a scalar product s on $\mathbb{R}^n$, we can define the **norm of a vector** and the **angle between two nonzero vectors**,

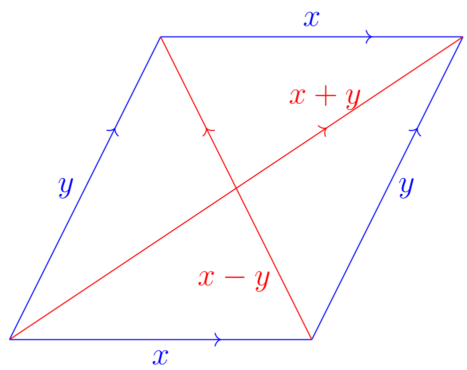
$$\|x\|_s = \sqrt{s(x,x)} \quad \text{for} \quad x \in \mathbb{R}^n$$
$$\angle_s(x,y) = \arccos\left(\frac{s(x,y)}{\|x\|_s \|y\|_s}\right) \quad \text{for} \quad x, y \in \mathbb{R}^n \setminus \{0\}$$

A scalar product s is determined by its norm $\|\cdot\|_s$, because of the **polarization formula**

$$2s(x,y) = \|x+y\|_s^2 - \|x\|_s^2 - \|y\|_s^2 \ , \tag{4.2}$$

or the **parallelogram identity**

$$4s(x,y) = \|x+y\|_s^2 - \|x-y\|_s^2 \ . \tag{4.3}$$



The **standard scalar product** on $\mathbb{R}^n$, or the **dot product**, is defined by

$$x \bullet y = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \bullet \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = x_1 y_1 + \cdots + x_n y_n \ .$$

Via matrix multiplication this can also be written as $x \bullet y = x^t y = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$. Every scalar product s on $\mathbb{R}^n$ can be expressed in terms of the standard scalar product. Let $S \in \mathrm{Mat}(\mathbb{R}, n \times n)$,

$$S = (s(e_i, e_j))_{\substack{i=1\ldots n \\ j=1\ldots n}}$$

be the matrix of the scalar products of the standard basis vectors $e_i = \begin{matrix} {\scriptstyle 1} \\ \\ \\ {\scriptstyle i} \\ \\ \\ {\scriptstyle n} \end{matrix}\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $i = 1, \ldots, n$, of $\mathbb{R}^n$. Note that this is a symmetric matrix. In particular S can be diagonalized. We can rewrite the scalar product s in terms of this matrix as

$$s(v,w) = v \bullet Sw \quad \text{for all} \quad v, w \in \mathbb{R}^n \tag{4.4}$$

Problem 4.4 Check (4.4).

The norm $\|\cdot\|_s$ is given by the formula

$$\|v\|_s^2 = s(v,v) = v \bullet Sv = \sum_{1\leq i,j\leq n} s(e_i,e_j)v_iv_j = \sum_{1\leq i\leq n} s(e_i,e_i)v_i^2 + 2\sum_{1\leq i<j\leq n}^{n} s(e_i,e_j)v_iv_j$$
$$= \sum_{1\leq i\leq j\leq n} s_{i,j}v_iv_j \ .$$

This is a **quadratic form**, i.e. a homogeneous quadratic polynomial on $\mathbb{R}^n$. Note that because of the factor 2 the $s_{i,j}$ are not exactly the matrix coefficients of the matrix S. By the formula (4.3), this quadratic form determines the scalar product.

If we name the coordinates of $\mathbb{R}^n$ as $x_1, \ldots x_n$, then we write this somewhat shorter as

$$s = \sum_{1\leq i\leq j\leq n} s_{i,j}dx_idx_j \tag{4.6}$$

The $\binom{n}{2} = \frac{n(n-1)}{2}$ coefficients $s_{i,j}$ are uniquley determined by the scalar product s and conversely, any family $(s_{i,j})_{i,j}$ of $\binom{n}{2}$ real numbers determines a scalar product provided (4.6) is positive definite, i.e.

$$\sum_{1\leq i\leq j\leq n} s_{i,j}v_iv_j > 0 \quad \text{whenever} \quad (v_1,\ldots,v_n) \neq (0,\ldots,0) \ .$$

4.2 Riemannian metrics on $\mathbb{R}^n$

Definition 4.7 *A* **Riemannian metric on** $U \subset \mathbb{R}^n$ *is a function assigning to each point* $p \in U$ *a scalar product* g_p *on* $\mathbb{R}^n$*. The norm associated with this scalar product will also be denoted by* $\sqrt{g_p}$ *or* $\|\cdot\|_{g_p}$*, i.e. we will write*

$$\|v\|_{g_p}^2 = g_p(v) = g_p(v,v) \quad \textit{for} \quad p \in U,\ v \in \mathbb{R}^n \ .$$

If we name the coordinates as $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$*, then we write*

$$g_p = \sum_{1\leq i\leq j\leq n}^{n} g_{i,j}(p)dx_idx_j \tag{4.8}$$

for the Riemannian metric with

$$g_p\left(\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}\right) = g_p\left(\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}, \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}\right) = \sum_{1\leq i\leq j\leq n} g_{ij}(p)v_iv_j$$

for some functions g_{ij} *on* U*.*

If $c\colon [a,b] \to \mathbb{R}^n$ *is a continuously differentiable curve, then the* **length of** c **with respect to the metric** g *is defined to be the integral*

$$\mathbf{length}_g(c) = \int_a^b \|c'(t)\|_{g_{c(t)}}\ dt = \int_a^b \sqrt{g_{c(t)}(c'(t))}\ dt = \int_a^b \sqrt{\sum_{1\le i\le j\le n} g_{ij}(c(t))\, c_i'(t)\, c_j'(t)}\ dt \tag{4.9}$$

if $c(t) = (c_1(t), \ldots, c_n(t))$ *and the metric is given as in* (4.8).

The **(geodesic) distance** *on* (U, g) *is given by*

$$\mathrm{d}^g(p,q) = \inf\{\mathbf{length}_g(c)\,\|\, c\colon [a,b] \to U \text{ a continuously differentiable curve, } c(a) = p, c(b) = q\} \tag{4.10}$$

A **geodesic** *in* (U, g) *is a* C^1*-curve* $c\colon [a, b \to U$ *which locally realizes this infimum, i.e. so that*

$$\forall t \in [a,b]\ \exists \delta_t > 0 : \mathrm{d}^g(c(t-\delta_t), c(t+\delta_t)) = \mathbf{length}\,(c|[t-\delta_t, t+\delta_t])\ .$$

In particular, if c is so that $\mathbf{length}\,(c) = \mathrm{d}^g(c(a), c(b))$, then c is a geodesic. Such geodesics are called **minimizing**.

Example 4.11 *The* **standard metric** *on* $\mathbb{R}^n$, *is*

$$g_p^{std}(v,w) = v \bullet w$$

independent of p. *In the style of formula* (4.8) *this becomes*

$$g_p^{std} = \sum_i dx_i^2\ .$$

Example 4.12 *On the upper half plane*

$$\mathcal{H} = \{z \in \mathbb{C}\,\|\, \Im(z) > 0\}$$

we define the **hyperbolic metric** *by*

$$g_p^{hyp}(v,w) = \frac{v \bullet w}{\Im(p)^2}$$

i.e.

$$g_p^{hyp} = \frac{dx^2 + dy^2}{\Im(p)^2}$$

if we denote the coordinates $z = x + iy = (x, y)$.

Example 4.13 *On the punctured sphere*

$$S_p^2 = \left\{(x, y, z) \in \mathbb{R}^3 \,\middle|\, x^2 + y^2 + z^2 = 1 \ , \ z \neq \pm 1\right\}$$

we denote by r the distance on the spere from the north pole, and by α the longitude. Thus we write (r, α) for the point

$$(x, y, z) = (\sin(r)\cos(\alpha), \sin(r)\sin(\alpha), \cos(r)) = (r, \alpha) \ .$$

These coordinates are called **polar coordinates**. *The usual metric on the sphere thus becomes the metric*

$$g^S_{r,\alpha} = dr^2 + \sin(r)^2 d\alpha^2 \quad \text{on} \quad \{(r, \alpha) \,|\, 0 < r < \pi \ , \ \alpha \in \mathbb{R}\} \ .$$

This is positive definite only if $\sin(r) \neq 0$, i.e. away from the poles.

Problem 4.13 Compute the length of the curve

$$c \colon [0, 2\pi] \to \{(r, \alpha) \,|\, 0 < r < \pi \ , \ \alpha \in \mathbb{R}\} \quad , \quad c(t) = (2, t) \ .$$

with respect to the spherical metric from example 4.13. Sketch this curve on the sphere and in $\mathbb{R}^2$.

Solution: By definition (4.9), we have

$$\textbf{length}\,(c) = \int_0^{2\pi} \sqrt{g^S_{c(t)}(c'(t), c'(t))}\ dt = \int_0^{2\pi} \sqrt{g^S_{(2,2t)}((0,1),(0,1))}\ dt = \int_0^{2\pi} \sqrt{\sin(2)^2}\ dt$$
$$= 2\pi \sin(2) \ .$$

4.3 Isometries

Definition 4.15 *Let $U \subset \mathbb{R}^n$ and g be a Riemannian metric on U. Let $\phi \colon U \to U$ be a differentiable map. We say that ϕ is an* **isometry** *of (U, g), if for all $p \in U$ and $v, w \in \mathbb{R}^n$ we have*

$$g_p(v, w) = g_{\phi(p)}(\mathrm{d}_p\phi v, \mathrm{d}_p\phi w) \ . \tag{4.16}$$

This is equivalent to preserving the length of curves. Thus $\phi \colon U \to U$ is an isomoetry if and only if for all differentiable curves c in U, we have

$$\textbf{length}\,(c) = \textbf{length}\,(\phi \circ c) \ .$$

Because of the polarization formula or the parallelogram identity, It suffices to check (4.16) only for equal vectors $v = w$, i.e. ϕ is an isometry already if

$$g_p(v) = g_{\phi(p)}(\mathrm{d}_p\phi v)$$

for all $p \in U$ and $v \in \mathbb{R}^n$.

Example 4.17 *Recall that the* **transpose** *of a matrix*

$$A = (a_{i,j})_{\substack{i=1\dots n\\ j=1\dots n}} \qquad = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix}$$

is the matrix

$$A^t = (a_{j,i})_{\substack{i=1\dots n\\ j=1\dots n}} \qquad = \begin{pmatrix} a_{1,1} & \cdots & a_{n,1} \\ \vdots & & \vdots \\ a_{1,n} & \cdots & a_{n,n} \end{pmatrix} .$$

In terms of the standard scalar product on $\mathbb{R}^n$ *this is equivalent to*

$$\langle Av \mid w\rangle = \left\langle v \,\middle|\, A^t w\right\rangle \quad \textit{for all} \quad v, w \in \mathbb{R}^n .$$

A matrix is called **orthogonal** *if its transpose is its inverse, i.e. if* $A^t = A^{-1}$*. In terms of the scalar product this means that* A *preservers the scalar product,*

$$\langle Av \mid Aw\rangle = \left\langle A^t A v \,\middle|\, w\right\rangle = \langle v \mid w\rangle \quad \textit{for all} \quad v, w \in \mathbb{R}^n .$$

We denote by $\mathrm{O}\,(n)$ *the* **orthogonal group** *of* $(n \times n)$*-matrices, i.e.*

$$\mathrm{O}\,(n) := \left\{A \in \mathrm{Mat}(n \times n, \mathbb{R}) \,\middle|\, A^t = A^{-1}\right\} .$$

Every affine linear map

$$\alpha\colon \mathbb{R}^n \to \mathbb{R}^n \quad , \quad x \overset{\alpha}{\longmapsto} Ax + b ,$$

with $A \in \mathrm{O}\,(n)$*,* $b \in \mathbb{R}^n$*, is an isometry of the standard metric on* $\mathbb{R}^n$*. In fact,* $\mathrm{d}_p\alpha = A$ *and*

$$g^{std}_{\alpha(p)}(\mathrm{d}_p\alpha v, \mathrm{d}_p\alpha w) = Av \bullet Aw = v \bullet w = g^{std}_p(v, w)$$

because $A \in \mathrm{O}\,(n)$*. In fact one can show that every isometry of the standard metric on* $\mathbb{R}^n$ *is of this form.*

Example 4.18 *In the hyperbolic upper half plane* $(\mathcal{H}, g^{hyp})$ *we have isometries*

1. *Translations* $\tau_a(z) = z + a$ *for* $a \in \mathbb{R}$
2. *Scaling, homothety* $\sigma_\alpha(z) = \alpha z$ *for* $\alpha \in \mathbb{R}^+$
3. *Inversion* $\iota(z) = \frac{-1}{z}$

We check this for the inversion: Writing complex numbers as pairs $x+iy=(x,y)$, we have

$$\iota(x,y)=\frac{-1}{x+iy}=\frac{-x+iy}{x^2+y^2}=\left(\frac{-x}{x^2+y^2},\frac{y}{x^2+y^2}\right) ,$$

$$\mathrm{d}_{(x,y)}\iota=\begin{pmatrix}-\frac{1}{x^2+y^2}+\frac{2x^2}{(x^2+y^2)^2} & \frac{2xy}{(x^2+y^2)^2}\\ -\frac{2xy}{(x^2+y^2)^2} & \frac{1}{x^2+y^2}-\frac{2y^2}{(x^2+y^2)^2}\end{pmatrix}=\frac{1}{(x^2+y^2)}\underbrace{\begin{pmatrix}\frac{x^2-y^2}{x^2+y^2} & \frac{2xy}{x^2+y^2}\\ \frac{-2xy}{x^2+y^2} & \frac{x^2-y^2}{x^2+y^2}\end{pmatrix}}_{A} .$$

The matrix on the right hand side is orthogonal. This means that $Av\bullet Aw=v\bullet w$ for all vectors $v,w\in\mathbb{R}^2$. Hence

$$g^{hyp}_{\iota(x,y)}\left(\mathrm{d}_{(x,y)}\iota v,\mathrm{d}_{(x,y)}\iota w\right)=\frac{1}{\left(\frac{y}{x^2+y^2}\right)^2}\left(\frac{1}{(x^2+y^2)}\right)^2 Av\bullet Aw$$

$$=\frac{1}{y^2}v\bullet w=g^{hyp}_{(x,y)}(v,w) .$$

Of course we could have computed in complex numbers. The derivative of $\iota(z)=\frac{-1}{z}$ is

$$\iota'(z)=\frac{1}{z^2} .$$

The hyperbolic metric can be expressed in terms of complex multiplication as

$$g^{hyp}_z(v)=\frac{|v|^2}{\Im(z)} .$$

Hence

$$g^{hyp}_{\iota(z)}(\iota'(z)v)=g^{hyp}_{-1/z}\left(\frac{1}{z^2}v\right)=\frac{\left|\frac{v}{z^2}\right|^2}{\Im(-1/z)^2}=\frac{|v|^2\left|\frac{1}{z^2}\right|^2}{\left(\Im(z)\left|\frac{1}{z^2}\right|\right)^2}=\frac{|v|^2}{\Im(z)^2}=g^{hyp}_z(v,w)$$

because for a complex number $z\in\mathcal{H}$ we have $\Im(-1/z)=\Im(z)/|z|^2$.

4.4 Möbius transformations

A **Möbius transformation** is a function M (on some domain, e.g. $\mathcal{H}$, of $\mathbb{C}$) of the form

$$M_A(z)=\frac{az+b}{cz+d} \quad , \quad A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\mathrm{GL}\left(2,\mathbb{C}\right) .$$

$\mathrm{GL}\left(n,\mathbb{C}\right)$ *is the group of invertible complex $(n\times n)$-matrices.* Let $\mathcal{M}$ denote the set of Möbius transformations.

A **Möbius transformation of the upper half-plane** is a Möbius transformation mapping the upper half-plane to itself. The set of such Möbius transformations is denoted by $\mathcal{M}^+$, hence

$$\mathcal{M}^+:=\{M\in\mathcal{M}\mid\mathcal{M}(\mathcal{H})=\mathcal{H}\}=\{M\in\mathcal{M}\mid\Im(z)>0\Rightarrow\Im(M(z))>0\} .$$

Note that there is no unique matrix belonging to a given Möbius transform. For every $\lambda \in \mathbb{C}$, we have $M_A = M_{\lambda A}$.

Under the correspondence $A \overset{\mu}{\longmapsto} M_A$, matrix multiplication becomes composition of Möbius transforms, the map $\mu\colon \mathrm{GL}(2,\mathbb{C}) \to \mathcal{M}$ is a **group homomorphism**. To see this, let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, and compute

$$M_A(M_B(z)) = \frac{a_{11}\frac{b_{11}z+b_{12}}{b_{21}z+b_{22}} + a_{12}}{a_{21}\frac{b_{11}z+b_{12}}{b_{21}z+b_{22}} + a_{22}} = \frac{(a_{11}b_{11} + a_{12}b_{21})z + a_{11}b_{12} + a_{12}b_{22}}{(a_{21}b_{11} + a_{22}b_{21})z + a_{21}b_{12} + a_{22}b_{22}} = M_{AB}(z) \ .$$

Theorem 4.19 *The Möbius transformations of the upper half plane correspond to real matrices of positive determinant. Thus for $M \in \mathcal{M}$,*

$$M(\mathcal{H}) = \mathcal{H} \quad \Longleftrightarrow \quad \exists A \in \mathrm{GL}^+(2,\mathbb{R}),\ M = M_A \ .$$

hence

$$\mathcal{M}^+ = \left\{ M_A \,\middle|\, A \in \mathrm{GL}^+(2,\mathbb{R}) \right\} \ .$$

Proof: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{C})$. If $c = 0$, then $d \neq 0$ and

$$M_A = M_{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}} = M_{\begin{pmatrix} a/d & b/d \\ 0 & 1 \end{pmatrix}}$$

so we may assume $d = 1$. But

$$M_{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}}(z) = az + b$$

and this maps $H \to H$ bijectively if and only if $a \in \mathbb{R}^+$ and $b \in \mathbb{R}$.

If $c \neq 0$, we can assume that $c = 1$ because

$$M_A = M_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = M_{\begin{pmatrix} a/c & b/c \\ 1 & d/c \end{pmatrix}} \ .$$

Then

$$M_A(z) = \frac{az+b}{z+d} = \frac{a(z+d) - (ad+b)}{z+d} = a + \det(A)\iota(z+d) \ .$$

If $\Im(d) < 0$ then $M_A(z)$ would not be defined for $z = -d \in H$. If $\Im(d) > 0$ then we would have

$$|z+d| > |\Im(d)| \quad \text{hence} \quad |\iota(z+d)| < \frac{1}{|\Im(d)|}$$

for all $z \in \mathcal{H}$. But this implies that

$$|M_A|(z) \leq |a| + \frac{1}{|\Im(d)|} ,$$

M_A would not be surjective. Thus we must have $\Im(d) = 0$, $d \in \mathbb{R}$. Furthermore, for all $x \in \mathcal{H}$ we must have

$$a + \det(A)x \in \mathcal{H}$$

But his forces $a \in \mathbb{R}$, $\det(A) \in \mathbb{R}^+$ and hence $b = \det(A) - ad \in \mathbb{R}$. •

Theorem 4.20 *The set $\mathcal{M}^+$ of Möbius transformations M_A with $A \in \mathrm{GL}^+(2,\mathbb{R})$ is the same as the set of all compositions of of translations, scalings and the inversion. More precisely,*

$$\mathcal{M}^+ = \left\{ \tau_u \circ \sigma_\alpha \,\middle|\, u \in \mathbb{R}, \alpha \in \mathbb{R}^+ \right\} \cup \left\{ \tau_u \circ \sigma_\alpha \circ \iota \circ \tau_w \,\middle|\, u, w \in \mathbb{R}, \alpha \in \mathbb{R}^+ \right\} .$$

Proof: Clearly, the inversion, the translations and scalings as in example 4.18 are such Möbius transformations:

$$\tau_b = \mu \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad , \quad \sigma_\alpha = \mu \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad \iota = \mu \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

To show the converse, let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}^+(2,\mathbb{R}) = \{A \in \mathrm{GL}\,(2,\mathbb{R}) \mid \det(A) > 0\}$. If $c = 0$, then we must have $d \neq 0$, $a/d > 0$, and

$$M_A(z) = \frac{az + b}{cz + d} = \frac{a}{d} z + \frac{b}{d} = \tau_{b/d}(\sigma_{a/d}(z))$$

If $c \neq 0$, then since $\det(A) = ad - bc > 0$,

$$\frac{az + b}{cz + d} = \frac{\frac{a}{c}\left(z + \frac{d}{c}\right) - \frac{ad - bc}{c^2}}{z + \frac{d}{c}} = \frac{a}{c} + \frac{ad - bc}{c^2} \frac{-1}{z + \frac{d}{c}} = \tau_{a/c}(\sigma_{\det(A)/c^2}(\iota(\tau_{d/c}(z))))$$

In all cases, M_A is a composition of translations, scalings and inversions. •

Via this Theorem, mapping properties of Möbius transformations can be proved by showing them for translations, scalings and the inversion.

A **hyperbolic line** is a subset $C(a,b) \subset \mathcal{H}$, $a, b \in \mathbb{R} \cup \{\infty\}$, $a < b$ of the form

$$C(a,b) = \left\{ z \in \mathcal{H} \,\middle|\, \left| z - \frac{a+b}{2} \right| = \frac{b-a}{2} \right\} = \left\{ \frac{a+b}{2} + \frac{b-a}{2} e^{it} \,\middle|\, 0 < t < \pi \right\} \quad \text{if} \quad b < \infty$$

and

$$C(a,\infty) = \left\{ a + it \,\middle|\, t \in \mathbb{R}^+ \right\} .$$

The **ideal points** a, b of $C(a,b)$, respectively a, ∞ of $C(a,\infty)$ are not points of the hyperbolic plane.

Corollary 4.21 *A Möbius transform maps hyperbolic lines to hyperbolic lines.*

Proof: This is immediately clear for the translations and the scalings. We check it for the inversion. We will show that

$$\iota(C(a,b)) = C(1/a, 1/b)$$

where the cases $b = \infty$ or $a = 0$ have to be read benevolently. For $p, r \in \mathbb{R}$, $r > 0$, the equation of the circle $|z - p| = r$. i.e. the circle with center p and radius r, is

$$z\overline{z} - p(z + \overline{z}) + p^2 - r^2 = 0 \tag{4.22}$$

hence

$$\frac{p^2 - r^2}{z\overline{z}} - p\left(\frac{1}{z} + \frac{1}{\overline{z}}\right) + 1 = 0$$

which is equivalent to

$$z\overline{z} - \underbrace{\frac{p}{p^2 - r^2}}_{q}\left(\frac{1}{z} + \frac{1}{\overline{z}}\right) + \underbrace{\frac{1}{p^2 - r^2}}_{q^2 - s^2} = 0 \ .$$

But this is of the form (4.22) with

$$\text{center} \quad q = \frac{p}{p^2 - r^2} \quad \text{and radius} \quad s^2 = \frac{p^2}{(p^2 - r^2)^2} - \frac{1}{p^2 - r^2} = \frac{r^2}{(p^2 - r^2)^2}$$

i.e.

$$s = \frac{r}{|p^2 - r^2|} \ .$$

The ideal endpoints of this circle are

$$q \pm s = \frac{p}{p^2 - r^2} \pm \frac{r}{|p^2 - r^2|} = \frac{p \pm r}{p^2 - r^2} = \frac{a \text{ or } b}{ab} = \frac{1}{a} \text{ or } \frac{1}{b} \ .$$

(The $\pm$ signs here need not correspoind.) •

Since translations, scalings and the inversion are isometries of the hyperbolic upper half plane, we immedately get

Corollary 4.23 *Möbius transformations in $\mathcal{M}^+$ are isometries of $(\mathcal{H}, g^{hyp})$.*

The hyperbolic plane is homogeneous, i.e. for every two points there is an isometry mapping one to the other. We have an even stronger transitivity:

Theorem 4.24 *In the hyperbolic upper half plane let (P_i, ℓ_i, H_i), $i = 1, 2$, be triples consisting of a point, a line through this point and a half plane of this line. Such triples are called* **flags**. *Then there is a Möbius transformation M_A in $\mathcal{M}^+$ (in particular an isometry) of $(\mathcal{H}, g^{hyp})$ so that*

$$M_A(P_1) = P_2 \quad , \quad M_A(\ell_1) = \ell_2 \quad , \quad M_A(H_1) = H_2 \ .$$

Proof: We will show that we can map any flag (Q,k,H) to the flag $(i, i\mathbb{R}^+, \{z \,|\, \Im(z) > 0, \Re(z) > 0\})$ consisting of i, the positive real axis and the right quadrant. We first map k to $i\mathbb{R}^+ = C(0,\infty)$. If $k = C(a,b) = \frac{a+b}{2} + \frac{b-a}{2}e^{i(0,2\pi)}$, $a,b \in \mathbb{R}$, $a < b$, we map the ideal endpoints $b \mapsto 0$ and $a \mapsto \infty$,

$$M_{\begin{pmatrix}\alpha & -a\alpha \\ 1 & -b\end{pmatrix}}(z) = \alpha\frac{z-b}{z-a}$$

with any $\alpha \in \mathbb{R}^+$. If $k = C(b,\infty) - b + i\mathbb{R}^+$ we can take

$$M_{\begin{pmatrix}\alpha & -b\alpha \\ 0 & 1\end{pmatrix}}(z) = \alpha(z-b) \ .$$

In both cases we can choose α so that the point P is mapped to i.

Finally, if the Möbius transform constructed above maps the given half plane to the left upper quadrant, we adjust by composing with the inversion map ι, which maps

$$\{z \in \mathbb{C} \,|\, \Re(z) < 0 < \Im(z)\} \stackrel{\iota}{\longleftrightarrow} \{z \in \mathbb{C} \,|\, 0 < \Im(z),\ 0 < \Re(z)\}$$

•

Problem 4.25 *Find a Möbius transformation M_A of the upper half-plane so that*

$$M_A(9) = \infty \quad , \quad M_A(-1) = 0 \quad , \quad M_A(1+4i) = i \ .$$

Check that $1+4i$ lies on the hyperbolic line with ideal points -1 and 9.

Solution: The first two conditions are easy to satisfy. Simply create a pole at 9 and a zero at -1, i.e. mapping

$$z \mapsto \pm\frac{z+1}{z-9} \ .$$

We need to choose the sign so that this preservers the upper half-plane, i.e. so that the determinant of the corresponding matrix is positive,

$$\det\begin{pmatrix}\pm 1 & \pm 1 \\ 1 & -9\end{pmatrix} = \mp 9 \mp 1 = \mp 10 > 0 \ .$$

Thus we need to choose the negative sign and map

$$z \mapsto \frac{-z-1}{z-9} \ . \tag{4.26}$$

The crucial observation is that all scalings fix 0 and ∞ and therefore map the hyperbolic line $C(0,\infty)$ to itself. We are therefore free to insert a factor α into (4.26), which we can use to satisfy the third condition. We thus try

$$M_A(z) = \alpha\frac{-z-1}{z-9} \ .$$

This maps -1 to 0 and 9 to ∞ for all α. The third condition,

$$M(1+4i) = \alpha\frac{-(1+4i)-1}{1+4i-9} = \alpha\frac{-2-4i}{-8+4i} \stackrel{!}{=} i \ ,$$

leads to $\alpha = 2$,

$$M(z) = \frac{-2z-2}{z-9} = M_A(z) \quad \text{with} \quad A = \begin{pmatrix} -2 & -2 \\ 1 & -9 \end{pmatrix} \in \mathrm{GL}^+(2,\mathbb{R})$$

Problem 4.27 *Find a Möbius transformation M of the upper half-plane so that*

$$M(C(1,3)) = C(0,\infty) \quad , \quad M\left(\frac{3}{2}+\frac{\sqrt{3}}{2}i\right) = i$$

and M maps the interior of $C(1,3)$ to $\{z \in H \mid \Re(z) > 0\}$.

Solution: As in problem 4.25 we satisfy the first condition with a Möbius transformation with a zero at 1 or 3 and a pole at 3 or 1, giving us two choices

$$M_A(z) = \frac{-z+1}{z-3} \quad \text{or} \quad M_B(z) = \frac{z-3}{z-1} \ .$$

The sign is chosen here so that both M_A and M_B preserve the upper half-plane.

We choose the one that maps the half-planes as required. To check this, we can simply map points in the half-planes, even ideal points. The center of the semi-circle $C(1,3)$ is 2 and is an ideal point of the interior of $C(1,3)$. This is mapped to

$$M_A(2) = \frac{-2+1}{2-3} = \frac{-1}{-1} = 1 \in \overline{\{z \in H \mid \Re(z) > 0\}} \ ,$$

$$M_B(2) = \frac{2-3}{2-1} = \frac{-1}{1} = -1 \notin \overline{\{z \in H \mid \Re(z) > 0\}} \ .$$

where $\overline{X}$ denotes the union of the half-plane X with its ideal points. Thus we have to choose M_A. As in problem 4.25, we use that scalings preserve the line $C(0,\infty)$ and also its half-planes. We look for $\alpha > 0$ so that

$$i \stackrel{!}{=} \alpha M_A\left(\frac{3}{2}+\frac{\sqrt{3}}{2}i\right) = \alpha\frac{-\left(\frac{3}{2}+\frac{\sqrt{3}}{2}i\right)+1}{\left(\frac{3}{2}+\frac{\sqrt{3}}{2}i\right)-3} = \alpha\frac{-\frac{\sqrt{3}}{2}i-\frac{1}{2}}{-\frac{3}{2}+\frac{\sqrt{3}}{2}i} \ ,$$

hence

$$\alpha = \frac{-\frac{3}{2}i-\frac{\sqrt{3}}{2}}{-\frac{\sqrt{3}}{2}i-\frac{1}{2}} = \sqrt{3} \ .$$

Thus the Möbius transformation we look for is M_T,

$$M_T(z) = \sqrt{3}M_A(z) = \frac{-\sqrt{3}z+\sqrt{3}}{z-3} \quad \text{with} \quad T = \begin{pmatrix} -\sqrt{3} & \sqrt{3} \\ 1 & -3 \end{pmatrix} \in \mathrm{GL}^+(2,\mathbb{R})$$

Homogeneity greatly facilitates calculations in $(\mathcal{H}, g^{\text{hyp}})$. Thus, for instance, computing distances via the definiton (4.10) by taking the infimum can be very difficult. We therefore first compute distances only along $C(0,\infty) = i\mathbb{R}^+$ and then use a Möbius transformation. Thus let $p = i$, $q = \lambda i$, and $c\colon [a,b] \to \mathcal{H}$, $c(a) = i$, $c(b) = \lambda i$ be a C^1-curve with coordinates $c(t) = (x(t), y(t))$. Then

$$\mathbf{length}\,(c) = \int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)}\,dt \geq \int_a^b \frac{\sqrt{y'(t)^2}}{y(t)}\,dt \geq \left| \int_a^b \frac{y'(t)}{y(t)}\,dt \right| = |\ln(y(b)) - \ln(y(a))| = |\ln(\lambda)| \ .$$

But this is the length of the curve $c\colon [0,\lambda] \to \mathcal{H}$, $c(t) = it$. We thus have shown

Theorem 4.28 *The curve $c_\lambda\colon [1,\lambda] \to \mathcal{H}$, $t \mapsto it$, is a minimizing geodesic in $(\mathcal{H}, \mathrm{d}^{hyp})$. Its lenght is*

$$\mathbf{length}\,(c_\lambda) = \mathrm{d}^{hyp}(i, \lambda i) = |\ln(\lambda)| \ .$$

In order to compute the distance $\mathrm{d}^{\text{hyp}}(P,Q)$ of arbitrary points $P, Q \in \mathcal{H}$, we find a Möbius transformation M mapping $P \mapsto i$ and $Q \mapsto i\lambda \in i\mathbb{R}^+$ with some $\lambda \in \mathbb{R}^+$. Then $\mathrm{d}^{\text{hyp}}(P,Q) = |\ln(\lambda)|$.

Example 4.29 *We will compute the hyperbolic distance between*

$$P = 2 + i \quad \textit{and} \quad Q = 2 + \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \ .$$

First observe that $(PQ) = C(1,3) = \{2 + e^{it} \mid t \in \mathbb{R}\}$. Then the curve

$$\gamma\colon [\pi/4, \pi/2] \to \mathcal{H} \quad , \quad \gamma(t) = 2 + e^{it}$$

is a minimizing geodesic from $Q = \gamma(i/4)$ to $P = \gamma(\pi/2)$.

We compute its length.

$$\gamma'(t) = ie^{it} \quad , \quad g^{hyp}(\gamma'(t)) = 1 \quad \textit{and} \quad \Im(\gamma(t)) = \sin(t) \ ,$$

$$\begin{aligned}
\mathrm{d}^{hyp}(P,Q) = \mathbf{length}\,(\gamma) &= \int_{\pi/4}^{\pi/2} \sqrt{g^{hyp}_{\gamma(t)}(\gamma'(t))}\,dt \\
&= \int_{\pi/4}^{\pi/2} \sqrt{\frac{\gamma'(t) \bullet \gamma'(t)}{\Im(\gamma(t))^2}}\,dt \\
&= \int_{\pi/4}^{\pi/2} \frac{1}{\sin(t)}\,dt \\
&= \left[\frac{1}{2} \ln\left(\frac{1 - \cos(u)}{1 + \cos(u)} \right) \right]_{u=\pi/4}^{u=\pi/2} = -\frac{1}{2} \ln\left(\frac{1 - \frac{1}{\sqrt{2}}}{1 + \frac{1}{\sqrt{2}}} \right) = \frac{1}{2} \ln(3 + 2\sqrt{2}) \ .
\end{aligned}$$

Alternatively, we can use Möbius transformations to move the line (PQ) to $C(0,\infty)$: To find such a Möbius transformation we map the ideal endpoints, for instance $1 \mapsto 0$, $3 \mapsto \infty$, via

$$M(z) = \alpha \frac{z-1}{z-3}$$

where only the sign of α needs to be chosen correctly, i.e. so that M preserves the upper half plane, equivalently so that the corresponding matrix has positive determinant. In the case at hand, we need to choose $\alpha < 0$, say -1. Then

$$M(z) = \frac{-z+1}{z-3} \quad \textit{maps} \quad C(1,3) \to C(0,\infty)$$

and since M is an isometry of g^{hyp},

$$\mathrm{d}^{hyp}(P,Q) = \mathrm{d}^{hyp}(M(P),M(Q)) = \left|\ln\left(\frac{M(P)}{M(Q)}\right)\right| = \ln(\sqrt{2}+1)$$

because

$$M(P) = \frac{-(2+i)+1}{2+i-3} = \frac{-i-1}{i-1} = i\ ,$$

$$M(Q) = \frac{-\left(2+\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)+1}{2+\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}-3} = \frac{-1-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}}{-1+\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}} = \frac{\sqrt{2}+1+i}{\sqrt{2}-1-i} = \frac{i2\sqrt{2}}{4-2\sqrt{2}} = \frac{i}{\sqrt{2}-1} = i(\sqrt{2}+1)\ .$$

Check that this coincides with the result computed above.

We determine the **isotropy group** $\mathcal{M}_i^+$ of i, the group of all Möbius transformations $M \in \mathcal{M}^+$ fixing i, i.e. with $Mi = i$.

Proposition 4.30

$$\begin{aligned}\mathcal{M}_i^+ &= \left\{M_A \,\middle|\, A \in \mathrm{GL}^+(2,\mathbb{R}), M_A(i) = i\right\}\\ &= \left\{\mu\begin{pmatrix} a & -b \\ b & a\end{pmatrix} \,\middle|\, a,b \in \mathbb{R}, a^2+b^2 > 0\right\}\\ &= \left\{\mu\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta)\end{pmatrix} \,\middle|\, \theta \in \mathbb{R}\right\} = \mu\left(\mathrm{SO}\left(2\right)\right)\end{aligned}$$

Proof: This is because

$$\frac{ai+b}{ci+d} = i \quad \Longleftrightarrow \quad ai+b = di-c \quad \Longleftrightarrow \quad a = d \text{ and } b = -c\ .$$

The last identity in 4.30 holds because for any $\alpha \in \mathbb{R}^+$ the matrices $A, \alpha A \in \mathrm{GL}^+(2,\mathbb{R})$ yield the same Möbius transform, i.e. for all $\alpha > 0$ we have $M_A = M_{\alpha A}$. •

Problem 4.31 *Sketch the hyperbolic circle of radius* $\ln(2)$ *around* i*, i.e.*

$$C_{\ln(2)}(i) = \left\{p \in \mathcal{H} \,\middle|\, \mathrm{d}^{hyp}(i,p) = \ln(2)\right\}\ .$$

This is a euclidean circle. Compute its euclidean center and radius.

Solution: The hyperbolic circle intersects the imaginary axis i the two points 2 and 1/2. Thus the euclidean center of this circle must be

$$Z = i\frac{5}{4} \ .$$

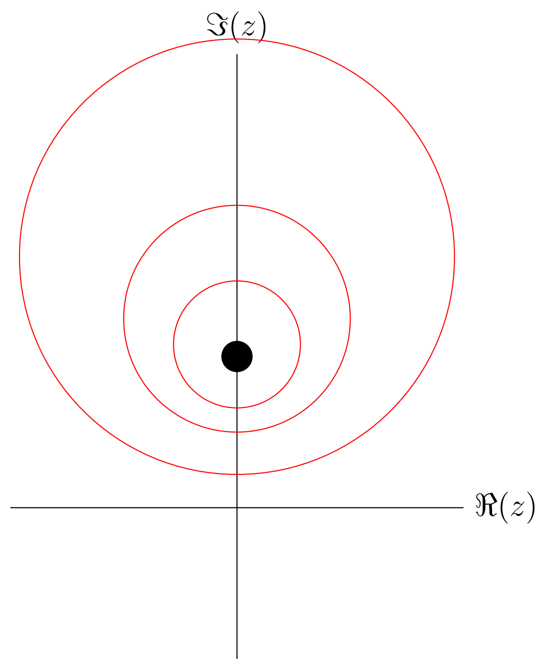
By Proposition 4.30 the hyperbolic circle with radius r around i consists of the points

$$C_{\ln(2)}(i) = \{M(2i) \mid M \in \mathcal{M}_i\} = \left\{ M_{\begin{pmatrix} \cos\theta) & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}(2i) \;\middle|\; \theta \in \mathbb{R} \right\}$$

$$= \left\{ \frac{2i\cos\theta - \sin\theta}{2i\sin\theta + \cos\theta} \;\middle|\; \theta \in \mathbb{R} \right\}$$

We compute the euclidean distance of these points from i to show that this is constant:

$$\left| \frac{2i\cos\theta - \sin\theta}{2i\sin\theta + \cos\theta} - \frac{5i}{4} \right|^2 = \frac{36\sin\theta^2 + 9\cos\theta^2}{4\sin\theta^2 + \cos\theta^2} \, \frac{1}{16} = \frac{9}{16} \ .$$



concentric hyperbolic circles around i.

5 The Gauss-Bonnet Theorem

5.1 Area and Angle Sum

The **Gauss curvature of a triangle** ΔABC **in a surface** is

$$K(\Delta ABC) = \frac{\alpha + \beta + \gamma - \pi}{\textbf{area}\,\Delta ABC}$$

where α, β, γ are the inner angles of ΔABC, and the Gauss curvature at a point p on this surface is the limit

$$K(p) = \lim_{A \to p, B \to p, C \to p} K(\Delta ABC) \ .$$

A triangulation of a surface is a decomposition of the surface in triangles so that two triangles meet in one vertex or in an entire edge or not at all. Then every edge lies in two triangles. If T is a triangulation T of a surface S, then the integral of the Gauss curvature over S is

$$\begin{aligned}
\int_S K &= \sum_{\Delta ABC \in T} K(\Delta ABC) \times \textbf{area}\, \Delta ABC \\
&= \sum_{\Delta ABC \in T} (\alpha + \beta + \gamma - \pi) \\
&= \sum \text{all interior angles} - \pi \times \#\text{triangles} \\
&= 2\pi \times \#\text{vertices} - \pi \times \#\text{triangles} \\
&= 2\pi \times \left(\#\text{vertices} - \frac{1}{2} \#\text{triangles} \right) \ .
\end{aligned}$$

With the notation

$$v = \#\text{vertices} \quad , \quad e = \#\text{edges} \quad , \quad t = \#\text{triangles}$$

this becomes

$$\int_S K = 2\pi \left(v - \frac{1}{2}\, t \right) \ . \tag{5.1}$$

By the conditions set out above, in a triangulation we always have

$$3t = 2e$$

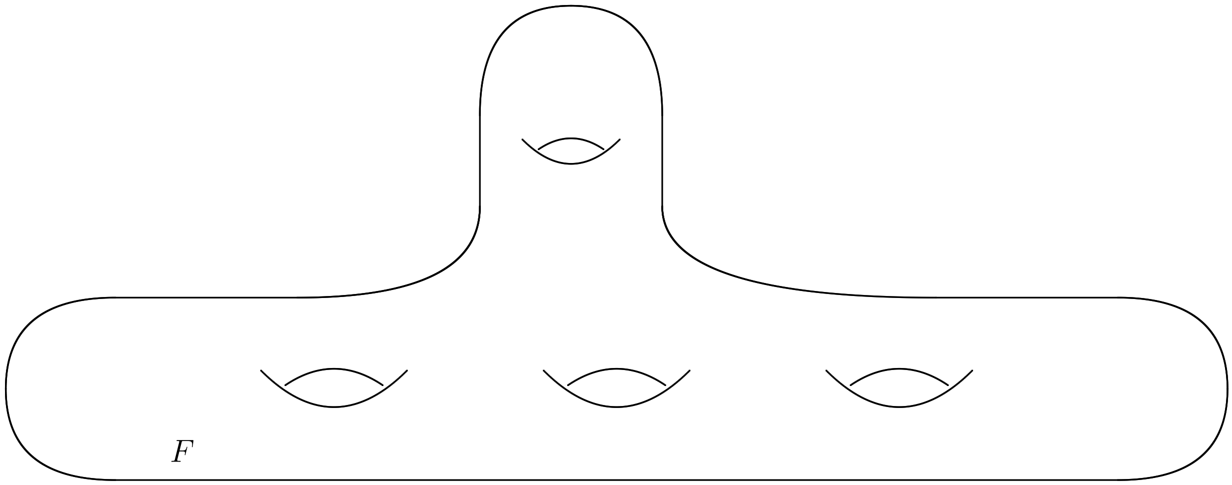
Since every triangle has three edges, and each edge is in two triangles. We can thus rewrite (5.1) as

$$\int_S K = 2\pi \ (\underbrace{v - e + t}_{\chi(S)}) \ . \tag{5.2}$$

This is the **Gauss Bonnet Theorem** for closed surfaces. The number $\chi(S)$ is called the **Euler characteristic** of the surface. Since the left hand side of (5.2) does not depend on the triangulation, the Euler characteristic does is the same for all triangulations of a given surface. Since the right hand side of (5.2) is independent on th choice of a metric on a given surface, the curvature integral is the same for all metrics on a given surface.

One advantage, and our reason for rewriting (5.1) in the form (5.2) is that (5.2) still holds if we work with decompositions of a surface in polygons.

Example 5.3 *Let $F \subset \mathbb{R}^3$ be the surface with boundary shown in the picture below. Compute the integral of the Gauss curvature over F.*



Summing over the triangles of a triangulation of F, we get the Gauss Bonnet Theorem,

$$\int_F K = \sum_{\substack{\Delta ABC \text{ a triangle} \\ \text{in the triangulation}}} K(\Delta ABC) \times \mathbf{area}\, \Delta ABC = \sum (\textit{interior angles} - \pi)$$

$$= 2\pi \# \textit{vertices} - \pi \times \# \textit{triangles}$$

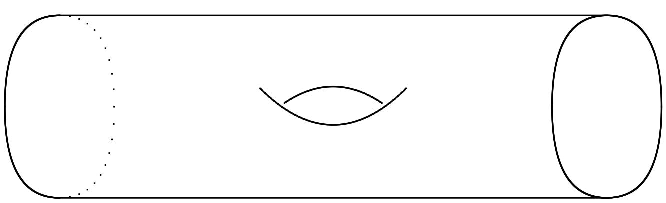
$$= 2\pi \# \textit{vertices} - 2\pi \times \# \textit{edges} + 2\pi \times \# \textit{triangles}$$

$$2\pi \times (\# \textit{vertices} - \# \textit{edges} + \# \textit{triangles}) = 2\pi \chi(F) \quad \textit{Euler Characteristic}$$

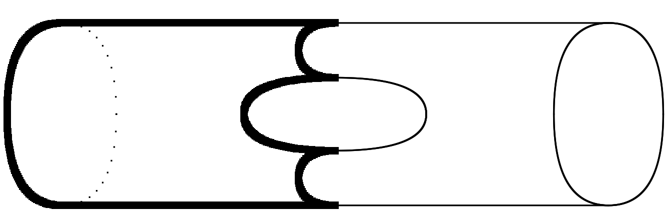
since each edge is in two triangles, we also have

$$2 \times \# \textit{edges} = 3 \times \# \textit{triangles}\ .$$

We now need to decompose the surface F into triangles. The basic block is

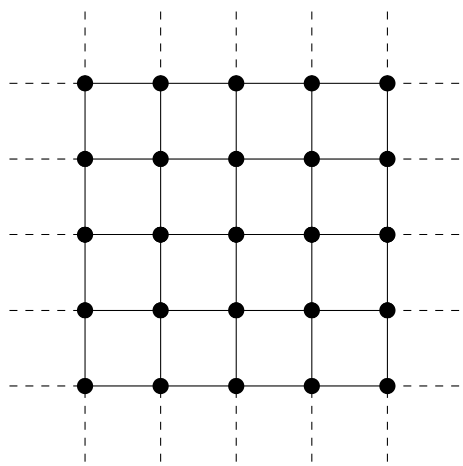


This can be split into four hexagons,



Problem 5.3 Show that a surface must have Euler characteristic 0 if it can be decomposed into squares so that

1. Each edge lies in two squares,
2. Two squares meet in a edge (and two vertices, in one vertex, or not at all.
3. Each vertex is common to four squares.



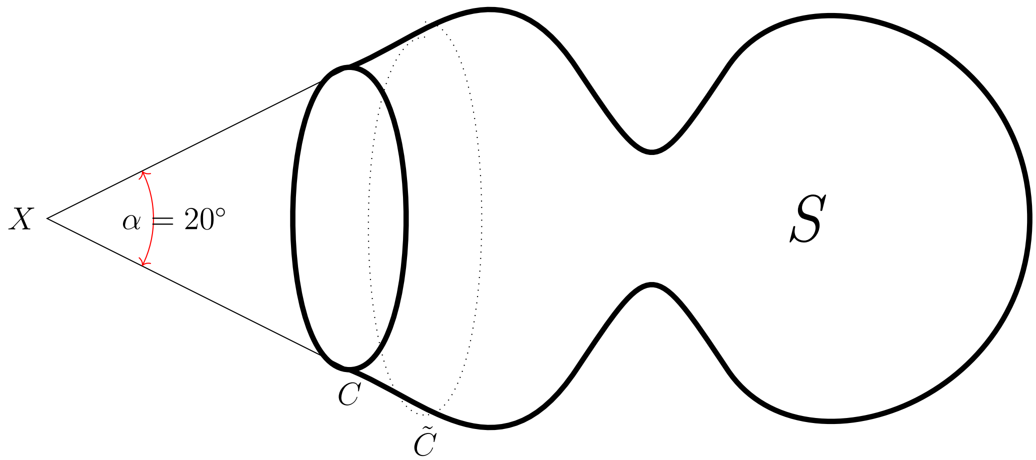
locally the

Solution: The number f of squares determines all the others: We have four vertices per square and each vertex is common to four squares, hence $v = f$. Similarly for the edges, we have four per square and each edge is shared by two squares, hence $e = 4f/2 = 2f$. Thus the Euler characteristic is

$$\chi = f - e + v = f - 2f + f = 0 \ .$$

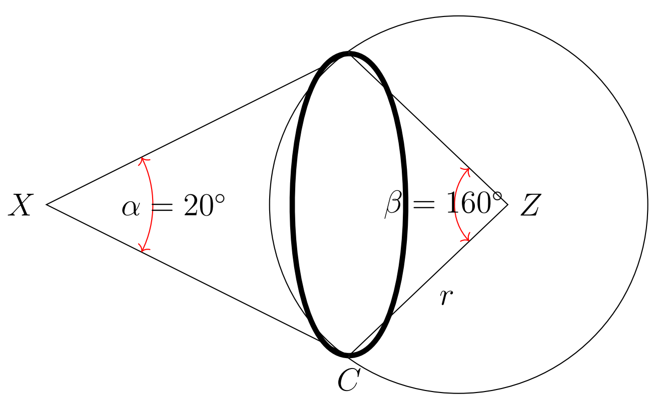
By the classification of closed surfaces, this shows that the surface must be a torus, and the torus obviously has such a "quadrangulation".

Problem 5.5 *Compute the integral of the Gauss curvature over the surface S show in the picture.*



The boundary curve C of S is a circle. Near this circle the metric of the surface is that of a cone, shown in the picture between the vertex X and the circle $\tilde{C}$. The angle of the cone is $20°$.

Solution: We close the surface adding a sperical cap.



By the Gauss-Bonnet Theorem, the curvature integral over S is that over the full sphere, i.e. 4π, minus the integral over the cap.

We are left with the computation of the curvature integral over the cap. We may assume the radius is 1. Then the area of the cap is

$$\int_0^{80°} 2\pi \sin(\theta)\ d\theta = 2\pi(1 - \cos(80°))\ .$$

The curvature integral over S is therefore

$$\int_S K = 4\pi - \int_{\text{cap}} K = 4\pi - 2\pi(1 - \cos(80°)) = 2\pi(1 + \cos(80°)))\ .$$

6 Topics for repetition

1. Incidence Axioms for a Plane
2. Ordered Geometry, planes with segments, half-planes, natural order, betweenness
3. Absolute Geometrty: axioms for the absolute plane: distance, angle measure, alternate angles theorem
4. Absolute Geometrty: triangles, congruence axiom and theorems (SAS, SSS, SSA, ASA).
5. (Uniqueness of) Parallels, euclidean and hyperbolic parallel postulate.
6. Riemannian metrics on $\mathbb{R}^2$, curve length
7. Poincaré model for the hyperbolic plane, upper half plane, hyperbolic metric
8. Isometries of the hyperbolic metric, Möbius transformations, ideal points
9. Euler characteristic, Gauss-Bonnet Theorem, Triangulations

7 Problems with Solutions

7.1 Planes, Planes with Segments - Ordered Geometry

1. Let A, B be points in a plane with segments, $A \neq B$. Prove that

$$[A, B \cup [B, A = (AB)\ .$$

Solution: We may assume that $A \underset{(A,B)}{\leq} B$. Then

$$[A, B = \left\{ X \in (A, B) \,\middle|\, A \underset{(A,B)}{\leq} X \right\}$$

$$[B, A = \left\{ X \in (A, B) \,\middle|\, X \underset{(A,B)}{\leq} B \right\}\ .$$

By definition,

$$[A, B \cup [B, A \subset (A, B)\ .$$

Since $\underset{(A,B)}{\leq}$ is a total order on (A, B),

$$(A, B) = \left\{ X \in (A, B) \,\middle|\, A \underset{(A,B)}{\leq} X \right\} \cup \left\{ X \in (A, B) \,\middle|\, X \underset{(A,B)}{\leq} A \right\}$$

$$= [A, B \cup \left\{ X \in (A, B) \,\middle|\, X \underset{(A,B)}{\leq} A \right\}$$

By transitivity of the total order $\underset{(A,B)}{\leq}$, since $A \underset{(A,B)}{\leq} B$, we have that

$$X \underset{(A,B)}{\leq} A \quad \text{implies} \quad X \underset{(A,B)}{\leq} B \,.$$

hence

$$\left\{X \in (A,B) \,\middle|\, X \underset{(A,B)}{\leq} A\right\} \subset \left\{X \in (A,B) \,\middle|\, X \underset{(A,B)}{\leq} B\right\} = [B, A$$

and therefore

$$(A,B) = [A, B \cap [B, A$$

as claimed.

2. Consider the plane with

$$\text{Points:} \quad \mathcal{P} = \{1,2,3,a,b,c\}$$

$$\text{Lines:} \quad \mathcal{L} = \{\{1,2,3\}, \{a,b,c\}\} \cup \{\{i,j\} \mid i = 1,2,3,\ j = a,b,c\}$$

(assuming that $a,b,c,1,2,3$ are pairwise different). In order to define a plane with segments $(\mathcal{P},\mathcal{L},\leq)$ having this as underlying plane, we only need to specify total orders $\underset{\{1,2,3\}}{\leq}$ and $\underset{\{a,b,c\}}{\leq}$ because the other elements of $\mathcal{L}$ have only 2 elements and therefore a unique pair of mutually reverse total orders.

Is there a order function $\leq$ so that $(\mathcal{P},\mathcal{L},\leq)$ is a plane with segments?

Hint: Wlog, we may assume that 2 is between $1,3$ and b is between a,c (why?). Look at the half planes of the line $2b$. Show that 1 and 3 must be in the same half plane, and show that they must be in different half planes. A contradiction.

Solution: The segments $[1,a]$ and $[a,3]$ do not intersect the line $2b$, hence $1,a,3$ lie in the same half plane of $2b$. On the other hand, the segment

$$[1,3] = \{1,2,3\}$$

intersects $2b$ in the point 2, hence $1,3$ lie in different half planes of $2b$.

3. Let l,k be lines in a plane with segments intersecting in a point P. Prove that there are four pairwise disjoint convex subsets A,B,C,D of points in that plane so that the set of point lying neither on l nor on k is $A \cup B \cup C \cup D$.

Solution: Let $\mathcal{P}$ and $\mathcal{L}$ denote the set of points respectively the set of lines in this plane. Let $L_\pm$ and $K_\pm$ the the half planes of l and k respectively. Then

$$\mathcal{P} \setminus l = L_- \cup L_+ \quad , \quad \mathcal{P} \setminus k = K_- \cup K_+$$

hence

$$\mathcal{P} \setminus (k \cup l) = \underbrace{(L_- \cap K_-)}_{A} \cup \underbrace{(L_- \cap K_+)}_{B} \cup \underbrace{(L_+ \cap K_-)}_{C} \cup \underbrace{(L_+ \cap K_+)}_{D} \,.$$

The sets A,B,C,D are convex because they are intersections of convex sets. Thus, for instance, if $X,Y \in A$ then $X,Y \in L_-$ and $X,Y \in K_-$. Since L_- and K_- are convex this implies that $[X,Y] \subset L_-$ and $[X,Y] \subset K_-$, hence $[X,Y] \subset L_- \cap K_- = A$.

If L_-, L_+ are the half planes of $l \in \mathcal{L}$, then $L_+ \cup l$ is also convex. To see this, assume that $A, B \in L_+ \cup l$. If $A, B \in L_+$ then $[AB] \subset L_+$ by the convexity of L_+. Also if $A, B \in l$ then $[AB] \subset l$ by the definition of the segments. Thus we only need to consider the case $A \in L_+$, $B \in l$. In this case let $X \in [AB]$. If $X \in L_-$ then $X \neq B$ and then since $A \in L_+$, there is $Y \in [A, X] \cap l$. But then $Y, B \in l$, and since $Y \neq B$ we have

$$l = (YB) = (AB)$$

which is impossible since $A \in L_+$ hence $A \notin l$. As before, intersections like $(L_+ \cup l) \cap K_+$ are convex, and

$$l = (l \cap K_+) \cup \{X\} \cup (l \cap K_-)$$

$$k = (k \cap L_+) \cup \{X\} \cup (k \cap L_-)$$

$$\mathcal{P} \setminus (k \cap l) = \mathcal{P} \setminus \{X\} = (\mathcal{P} \setminus (k \cup l)) \cup l \cup k \setminus \{X\}$$

$$\mathcal{P} \setminus (k \cap l) = \mathcal{P} \setminus \{X\} = \underbrace{((L_- \cup l) \cap K_-)}_{A} \cup \underbrace{(L_- \cap (K_+ \cup k)}_{B} \cup \underbrace{(L_+ \cap (K_- \cup k))}_{C} \cup \underbrace{((L_+ \cup l) \cap K_+)}_{D} \ .$$

4. Let k, ℓ be lines intersecting in one point X, $k \cap l = \{X\}$, and let H be a half plane of ℓ. Show that $(k \cap H) \cup \{X\}$ is a ray.

   Solution: If $k \cap H = \emptyset$ then, since $\#k \geq 2$, there is $P \in k$, $P \neq X$, and $P \in H' = \mathcal{P} \setminus (\ell \cup H)$, the other half plane of ℓ. Let $\leq$ be the natural order on k so that $X \leq P$. We claim that

   $$(k \cap H) \cup \{X\} = \{X\} = \{Y \in k \mid Y \leq X\} \ .$$

   To see this, let $Y \in k$, $Y \leq X$, $Y \neq X$. Then $[Y, P] \cap \ell = \{X\}$, hence P and Y are in different half planes of ℓ. But this forces $Y \in H \cap k$ which can therefore not be empty, impossible.

   We may thus assume that there is $Q \in k \cap H \neq \emptyset$, and choose the natural order $\leq$ on k so that $X \leq Q$. We claim that

   $$(k \cap H) \cup \{X\} = \{R \in k \mid X \leq R\} \ .$$

   In order to see this, first assume that $R \in k$, $R \neq X$ and $X \leq R$. Then X is not between R and Q. If the intersection $[RQ] \cap \ell$ were not empty, then we would have $k = \ell$, impossible. Thus $[RQ] \cap \ell = \emptyset$ and R and Q lie in the same half plane of ℓ, i.e. in H. For the reverse inclusion, assume that $R \in k \cap H$. If we had $R \leq X$, then $[R, Q] \cap \ell \ni X$, hence R and Q would have to lie in different half planes of ℓ, hence $R \notin H$.

5. Show that every ray r in a plane with segments is of the form

   $$r = (H \cap k) \cup \{X\}$$

   where $\{X\} = \ell \cap k$ is the vertex of the ray and ℓ, k are lines, H a half plane of ℓ.

   Solution: Let $r = \{R \in k \mid X \leq R\}$ for some $X \in k$, k a line in the plane and $\leq$ one of its natural orders. Let Q be a point not in k, and let H, H' be the half planes of $\ell = (XQ)$. Then $r = (H \cap k) \cup \{X\}$ or $r = (H' \cap k) \cup \{X\}$

6. In a plane with segments, can there be four points such that

   (a) no three of the points lie on one line, and

(b) each of the four points lies in the interior region of any angle formed by the other three points.

Prove your answer.

Hint: Thus, if the points are named A, B, C, D, in any order, we must have $D \in IR(\angle ABC)$, $D \in IR(\angle ACB)$, $D \in IR(\angle BAC)$, $A \in IR(\angle BCD)$, $A \in IR(\angle BDC)$, $A \in IR(\angle CBD)$ ...

Solution: This is true for the plane of 2 sets on 4 points. In this plane every point not on the rays of an angle lies in the interior region of that angle. This is because one of the two half planes of any line is empty.

7. Let $\mathcal{P}$, $\mathcal{L}$ be the sets of points respectively lines in a plane. Assume that $\mathcal{P}$ is finite and that

   (a) For each $Q \in \mathcal{P}$ and $l \in \mathcal{L}$ with $Q \notin l$ there is exactly one $a \in \mathcal{L}$ with $Q \in a$ such that $a \cap l = \emptyset$ ("l and a do not intersect").

   (b) For each $l, k \in \mathcal{L}$ there is $P \in \mathcal{P}$ such that $P \notin l$ and $P \notin k$,

   where we write $Q \in \ell$ to mean Q lies on ℓ. Prove that all $l \in \mathcal{L}$ have the same number of points, i.e. that for $l, k \in \mathcal{L}$,

$$\# \{Q \in \mathcal{P} \,|\, Q \in l\} = \# \{Q \in \mathcal{P} \,|\, Q \in k\} \ .$$

Hint: Count the lines through a point.

Solution: Let $k, l \in \mathcal{L}$. By assumption there is $Q \in \mathcal{P}$ such that $Q \notin k$ and $Q \notin l$. Let $\mathcal{L}_Q$ be the set of lines through Q, i.e.

$$\mathcal{L}_Q := \{h \in \mathcal{L} \,|\, Q \in h\} \ .$$

By assumption, exactly one of the lines in $\mathcal{L}_Q$ does not intersect l and each of the other lines intersect l in exactly one point. Applying this to k in turn shows that

$$\# \{X \in \mathcal{P} \,|\, X \in l\} = \#\mathcal{L}_Q - 1 = \# \{X \in \mathcal{P} \,|\, X \in k\} \ .$$

8. Let $(\mathcal{P}, \mathcal{L}, \leq)$ be a plane with segments. Let $A, B, C, D \in \ell \in \mathcal{L}$, $A \neq B$, and $C, D \notin [A, B]$. Prove that either $[A, B] \subset [C, D]$ or $[C, D] \cap [A, B] = \emptyset$.

   Solution: Wlog. we may assume that $A \underset{\ell}{\leq} B$. Since $C \notin [A, B]$ we have 2 possibilities as to the position of C and 2 for D:

$$C \underset{\ell}{\leq} A \quad \text{or} \quad B \underset{\ell}{\leq} C$$
$$D \underset{\ell}{\leq} A \quad \text{or} \quad B \underset{\ell}{\leq} D$$

   which gives four cases:

   (a) $C, D \underset{\ell}{\leq} A \underset{\ell}{\leq} B : [C, D] \cap [A, B] = \emptyset$

   (b) $C \underset{\ell}{\leq} A \underset{\ell}{\leq} B \underset{\ell}{\leq} D : [A, B] \subset [C, D]$

   (c) $D \underset{\ell}{\leq} A \underset{\ell}{\leq} B \underset{\ell}{\leq} C : [A, B] \subset [C, D]$

   (d) $A \underset{\ell}{\leq} B \underset{\ell}{\leq} C, D : [C, D] \cap [A, B] = \emptyset$

9. Let A, B, $A \neq B$, be points in a plane with segments. Prove that

$$[A, B \cap [B, A = [A, B] \ .$$

Solution: We may assume that $A \underset{(A,B)}{\leq} B$. Then

$$[A, B = \left\{ X \in \ (A, B) \,\middle|\, A \underset{(A,B)}{\leq} X \right\}$$

$$[B, A = \left\{ X \in \ (A, B) \,\middle|\, X \underset{(A,B)}{\leq} B \right\}$$

hence

$$[A, B] = \left\{ X \in (A, B) \,\middle|\, A \underset{(A,B)}{\leq} X \underset{(A,B)}{\leq} B \right\} = [A, B \cap [B, A$$

as claimed.

10. Let ℓ, k be lines in a plane with segments, and let A, B, C, D be points on ℓ so that $A \neq B$ and C, D are not in $[A, B]$. Prove that if $\ell \cap k = \emptyset$ ("ℓ and k are parallel") then one half plane of ℓ lies entirely in a half plane of k.

Solution: Let H_+, H_- be the half planes of ℓ, hence $\mathcal{P} = H_+ \cup \ell \cup H_-$. Since $k \cap \ell = \emptyset$, $k \subset H_+ \cup H_-$. We choose the half planes so that $k \cap H_+ \neq \emptyset$, say $X \in k \cap H_+$.

Then $k \cap H_- = \emptyset$ because otherwise, say if we had $Y \in k \cap H_-$, we would have $X \in H_+$, $Y \in H_-$, hence $[X, Y] \cap \ell \neq \emptyset$. But since $[X, Y] \subset k$ this contradicts $\ell \cap k = \emptyset$.

We thus have $k \cap H_- = \emptyset$. From the convexity of the half planes it follows that if $P, Q \in H_-$, then $[P, Q] \subset H_-$, hence $[P, Q] \cap k = \emptyset$, and P, Q lie in the same half plane of k. It follows that all points of H_- lie in the same half plane of k.

11. Let $\mathcal{P}$ be the set of points and $\mathcal{L}$ be the set of lines of a plane with segments.

  (a) If $X \in \mathcal{P}$, what is

$$\bigcup_{X \in \ell \in \mathcal{L}} \ell$$

  ("the union of all lines through X")? Prove your answer!

  Solution: By definition of a plane, for every $Y \in \mathcal{P}$ there is $\ell \in \mathcal{L}$ so that $Y, X \in l$. It follows that

$$Y \in \bigcup_{X \in \ell \in \mathcal{L}} l = \mathcal{P}$$

  (b) Let $A, B, C, D \in \mathcal{P}$ be so that $C \in [A, B]$ and $D \in [A, C]$. Prove that $C \in [D, B]$.

  Solution: If $A = C$ then the statemnet is trivial. We may therefor assume that $A \neq C$, hence $A \neq B$. Since $C \in [A, B]$, we have $C \in (A, B)$, hence $(A, B) = (A, C) = (C, B)$. Let $\leq := \underset{(A,B)}{\leq}$. By the definition of the segments, $C \in [A, B]$ and $D \in [A, C]$ mean that

$$A \leq C \leq B \quad \text{or} \quad B \leq C \leq A \quad \text{and}$$

$$A \leq D \leq C \quad \text{or} \quad C \leq D \leq A \ .$$

By transitivity of a total order we can only always have the first cases, or always the second cases.
In the first case, $A \leq C \leq B$ and $A \leq D \leq C$, we get $D \leq C \leq B$, hence $C \in [D, B]$.
In the second case, $B \leq C \leq A$ and $C \leq D \leq A$, we get the same.

12. Show that in a plane with segments, the intersection of a line with a half plane is empty, all of the line, or the interior of a ray, i.e. of the form $[X, Y \setminus \{X\}$ for some points X, Y of the plane.

Solution: Let ℓ, k be lines and H a half plane of the line ℓ. Recall from Definition 2.9 that two points X, Y not on ℓ lie in the same half-plane of ℓ if no point between X, Y lies on ℓ We consider a number of cases for the intesection of the line k with the half-plane H of ℓ:

(a) *The lines k and ℓ do not intersect:* If X, Y are points on k, then, since no point on k lies on ℓ, there is no point between X, Y on ℓ. Hence X and Y are in the same half-plane of ℓ. Thus all points of k lie in the same half-plane of ℓ. Since by Axiom 2.10 there are at most two mutually disjoint half-planes of ℓ, the intersection of k with H is either empty or k.

(b) $k = \ell$: In this case, since the half-planes of ℓ are equivalence classes on points not on ℓ, the intersection of H with k is the intersection of H with ℓ, empty.

(c) *k and ℓ intersect in exactly one point X:* If there is no point on k and in H, the intersection of k with H is empty, which is one of the possibilities claimed. Assume now that a point Y on k lies in H. We use the Definition (2.15), and show that the intersection $k \cap H$ of k with H is the interior $[X, Y \setminus \{X\}$ of the ray $[X, Y$. To this end, let $Z \in [X, Y \setminus \{X\}$ be a point on the ray $[X, Y$ different from X. Hence X is not between Z and Y, and hence no point on ℓ is between Z and Y. It follows that Z and Y lie in the same half-plane of ℓ. Since Y is in H, so is Z. For the converse, assume Z on k lies in H. Then no point between Z and Y lies on ℓ. In particular, the point X is not between Y and Z. But this again means that Z lies on the ray $[X, Y$. Since the point Z is in H, it is not on ℓ and hence different from X.

By the incicence axioms 1.1 two different lines cannot intersect in more than one point, the above three cases exhaust all possibilities.

13. Let A, B, C be points in a plane with segments, not on a common line, and let ℓ be a line of that plane. Let Δ be the intersection of half planes,

$$\Delta = H((AB), C) \cap H((BC), A) \cap H((CA), B) \ .$$

Show that $\ell \cap \Delta$ is the interior of a segment, i.e. there are points X, Y of the plane such that

$$\ell \cap \Delta = [X, Y] \setminus \{X, Y\} \ .$$

Hint: You might use problem 12.

Solution: By problem 12 the intersection in question is the intersection of interiors of rays, lines and possibly the empty set.

14. On the sphere

$$S^2 = \left\{(x, y, z) \in \mathbb{R}^3 \,\middle|\, x^2 + y^2 + z^2 = 1\right\}$$

consider the set of great circles, i.e.

$$\mathcal{L} = \left\{ l_{a,b,c} \,\middle|\, (a,b,c) \in \mathbb{R}^3 \setminus \{0\} \right\}$$

$$l_{a,b,c} = \left\{ (x,y,z) \in S^2 \,\middle|\, ax + by + cz = 0 \right\} \ , \ (a,b,c) \in \mathbb{R}^3 \setminus \{0\} \ .$$

Is $(S^2, \mathcal{L})$ a plane?

Hint: Draw $\mathcal{P}$ and some of these "lines".

Solution: This is not a plane because any two great circles intersect in a pair of antipodal points. Thus for instance,

$$(1,0,0), (-1,0,0) \in \mathcal{P}, (1,0,0) \neq (-1,0,0) \ ,$$

$$\{(1,0,0), (-1,0,0)\} = l_{0,1,0} \cap l_{0,0,1} \quad \text{and}$$

$$l_{0,1,0} \neq l_{0,0,1}$$

15. Let A, B, C, D be points in a plane with segments and assume that

   (a) B lies between A and C,

   (b) C lies between B and D.

   Prove that B lies between A and D.

   Solution: First we show that the four points lie on one line. To see this, infer from the first condition that A, B, C are pairwise different and that B lies on the line AC. Since B is different from C, there is a unique line BC, and, since the two points B, C lie on AC as well as on BC, the two lines are equal, $AC = BC$. With the same argument applied to the second condition we see that $BC = BD$, hence $\ell := AC = BC = BD$ is a line containing all the four points.

   Wlog we may assume that $\underset{\ell}{\leq}$ is the one of the two natural orders on ℓ so that $A \underset{\ell}{\leq} C$. By the first condition, we have

$$A \underset{\ell}{\leq} B \underset{\ell}{\leq} C \quad \text{or} \quad C \underset{\ell}{\leq} B \underset{\ell}{\leq} A \tag{7.1}$$

   but by transitivity of the total order, the second would imply that $C \underset{\ell}{\leq} A$, hence $C = A$, a contradiction. We can therefore rule out the second alternative in (7.1) and have

$$A \underset{\ell}{\leq} B \underset{\ell}{\leq} C \ . \tag{7.2}$$

   From the second condition we could have

$$B \underset{\ell}{\leq} C \underset{\ell}{\leq} D \quad \text{or} \quad D \underset{\ell}{\leq} C \underset{\ell}{\leq} B \ . \tag{7.3}$$

   From (7.2) we have already $B \underset{\ell}{\leq} C$. Since $B \neq C$, this rules out the second alternative in (7.3) which leaves

$$B \underset{\ell}{\leq} C \underset{\ell}{\leq} D \ . \tag{7.4}$$

   Together with (7.2) this proves

$$A \underset{\ell}{\leq} B \underset{\ell}{\leq} C \underset{\ell}{\leq} D \ ,$$

   hence B lies between A and D.

16. Let ℓ be a line in a plane with segments and let $A_0, A_1, A_2, \ldots, A_n$ be pairwise different points in that plan not on the line ℓ. Assume that there is a point X lying on ℓ and between A_0 and A_1. Show that there is $i = 2, \ldots n$, and a point Y so that Y lies on ℓ and between A_{i-1}, A_i or between A_0, A_n.

    Hint: Prove this by induction over n. The case $n = 2$ is Pasch's Theorem 2.13.

    Solution: Induction starts at $n = 2$, where the statement is Pasch's Theorem. We now assume the statement for $n = k$, $k \in \mathbb{N}$, $k \geq 2$, "induction hypothesis", and prove it for $n = k+1$. To this end, let $A_0, A_1, A_2, \ldots, A_n, A_{n+1}$ be point in a plane with segments and ℓ be a line containing none of these points and intersecting the segment $[A_0, A_1]$. By the induction hypothesis, there are two cases.

    (a) *The line ℓ intersects $[A_{i-1}, A_i]$ for some i, $2 \leq i \leq n$:* Then trivially, ℓ intesects $[A_{i-1}, A_i]$ for some i, $2 \leq i \leq n+1$.

    (b) *The line ℓ intersects $[A_n, A_0]$:* Then Pasch's Theorem 2.13 for the triangle A_0, A_n, A_{n+1} gives that ℓ intersects $[A_n, A_{n+1}]$ or $[A_0, A_{n+1}]$

17. Let A, X, B be non-collinear points in a plane with segments. Let D and Z be points in that plane so that Z lies on the ray $[X, D$ and between A, B. Show that D lies in the interior region of $\angle AXB$.

    Solution: We need to show that A and D lie in the same half-plane of XB and that B and D lie in the same half-plane of XA. The two are analogous, exchanging A with B. We prove the first.

    If A and D were in different half-planes of XB, then a point U on XB would lie between A, D. We apply Pasch's Theorem to the line XB and the triangle ADZ:

    First, none of the points A, D, Z lies on XB. A is not on XB since A, X, B are not collinear. D, Z, X all lie on the line $XD = XZ$. If any of D, Z were also on XB then the lines XB and XZ would be equal. Since $Z \neq B, A$, we would have $XB = XZ = BZ = AZ$ and A, B, X would be collinear.

    Since X is the only intersection point of XB with $XD = XZ$, which is the vertex of the ray $[X, D$, the point X does not lie between D, Z. By Pasch's Theorem, XB must therefore contain a point V between A, Z. This point V is different from B, since Z lies between A, B. Hence $AB = VB = XB$ which is impossible since A, X, B are not collinear.

18. Let A, X, B be non-collinear points in a plane with segments, and let P be a point between A and B. Show that P lies in the interior region of $\angle AXB$.

    Solution: We need to show that

    (a) B and P lie in the same half-plane of XA, and

    (b) A and P lie in the same half-plane of XB.

    Since the problem is symmetric in A, B, it suffices to prove the first. Since P is between A, B, the points A, P, B lie on the same line, in particular, $BP = AB$. Since A, X, B are not collinear, the lines AX and AB are different and intersect only in the point A. Since P is between A, B the point A is not between P, B. Hence there is no point on the line AX between P, B, which means that P, B lie in the same half-plane of AX.

7.2 Absolute Geometry, Euclidean Geometry

19. Let A, B, C, D be points of an absolute plane so that $B \neq D$ and A, C lie in different half planes of (BD). Assume that $|A, B| = |A, D|$ and $|C, B| = |C, D|$. Let X be the midpoint of $[B, D]$, i.e. the unique point on $[B, D]$ so that $|B, X| = |D, X|$.

   (a) Prove that all the angles at X are right angles , i.e.

   $$90 = |\angle BXA| = |\angle AXD| = |\angle DXC| = |\angle CXB| \ .$$

   Solution: The triangle (ABD) is isosceles, hence $|\angle ABX| = |\angle ABD| = |\angle ADB| = |\angle ADX|$ By SAS (angles at B resp. D), $(ABX) \cong (ADX)$, hence $|\angle AXB| = |\angle AXD|$. But $\angle BXD$ is straight, hence

   $$180 = |\angle BXD| = |\angle BXA| + |\angle AXD| = 2\,|\angle BXA|$$

   and therefore

   $$|\angle AXB| = |\angle AXD| = 90 \ .$$

   Replacing A with C in the above argument shows that also

   $$|\angle CXB| = |\angle CXD| = 90 \ .$$

   (b) Prove that $[A, C] \cap [B, D] = \{X\}$.
   Solution: By the additivity of the angle measure,

   $$|\angle CXA| = |\angle CXB| + |\angle BXA| = 90 + 90 = 180$$

   hence $\angle CXA$ is a straight angle, the points C, X, A lie on a common line and X is between A, C. In particular $X \in [AC]$.

20. Let ℓ be a line in an absolute geometry and let P be a point not on ℓ. Let A, X be points on ℓ, $A \neq X$ and

   $$|\angle AXP| = 90 \ .$$

   Prove that

   $$|P, X| \leq |P, Y| \quad \text{for all} \quad Y \in \ell \ .$$

   We then call $|P, X|$ the distance of the point P from the line ℓ,

   $$d(P, \ell) = \inf\{|P, Y| \mid Y \in \ell\} \ . \tag{7.5}$$

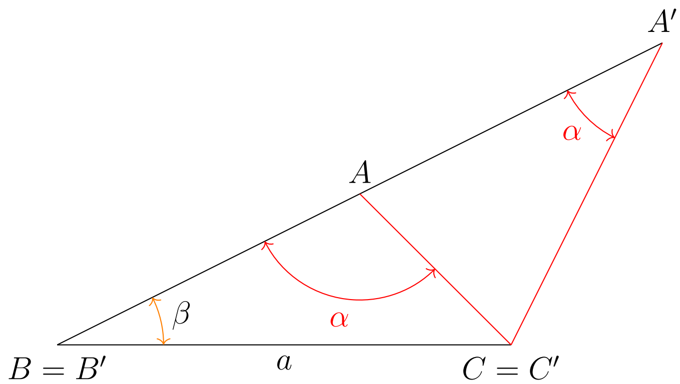
   Hint: First show that any angle between the lines ℓ and (XP) has measure 90. Then apply the theorem on alternate angles 3.18.

   Solution: For any $Y \in \ell$, $Y \neq X$ we have that $|\angle YXP| = 90$. By the theorem on alternate angles 3.18, $|\angle PYX| \leq |\angle PXY|$. Thus, in the triangle (PXY) the side $[P, X]$ lies opposite the angle $\angle PYX$ and must therefore be shorter than the side $[P, Y]$ which is opposite $\angle PXY$ by Theorem 3.20.

21. Prove AAS: Two triangles (ABC) and $(A'B'C')$ in an absolute geometry are congruent if

$$a = |B, C| = |B', C'| \quad , \quad \beta = |\angle ABC| = |\angle A'B'C'| \quad \text{and} \quad \alpha = |\angle BAC| = |\angle B'A'C'| \ .$$

Hint: You may assume that $B = B'$, $C = C'$ and that $A' \in [B, A$, even $A \in [BA']$. Why?



Solution: Let (ABC) and $(A'B'C')$ be two triangles (i.e. triples of points of a Euclidean plane) as in the assumption of the theorem. We can assume that $|BA| \leq |BA'|$, otherwise we interchange $(ABC) \leftrightarrow (A'B'C')$. Let A'' be the point in the same half plane of (BC) as A so that $|\angle CBA''| = \beta = |\angle C'B'A'|$. By SAS, $(A''BC) \cong (A'B'C')$, and by the choice we made at the start, $|BA| \leq |BA'| = |BA|$, hence $A \in [B, A'']$.

It thus remains to show that $(ABC) \cong (A''BC)$, i.e. $A = A''$. Assuming that $A \neq A''$ we have A between B, A''. Let $X \in (CA)$ be so that A lies between X, C. Then by the opposite angles theorem 3.18 and the alternate angles theorem,

$$= |\angle BAC| = |\angle XAA''| > |\angle AA''C| = |\angle BA''C|$$

contradicting the assumption that $|\angle BAC| = |\angle B'A'C'| = |\angle BA''C|$.

22. Let A, B, C, D be points in an absolute geometry so that no three of them lie on a common line and so that A and D lie in different half-planes of the line (BC). Also assume that

$$(A, B, C) \cong (D, C, B) \ .$$

Prove that

$$(A, C, D) \cong (D, B, A) \ .$$

Solution: The given congruence implies that

$$|\angle ABC| = |\angle DCB| \quad \text{and} \quad |\angle ACB| = |\angle DBC| \ .$$

As in the proof of theorem 3.35, because of the alternate angles theorem 3.18, the first equation implies that $(AB)||(CD)$ and the second equation implies that $(AC)||(BD)$. In particular, $B \in \mathrm{IR}\,(\angle ACD)$, because B lies in the same half-plane of (AC) as D and B lies in the same halp-plane of (CD) as A. Analogously, $C \in \mathrm{IR}\,(\angle ABD)$. By additivity of the angle

$$|\angle ACD| = |\angle ACB| + |\angle BCD| = |\angle DBC| + |\angle ABC| = |\angle DBA| \ .$$

Also we have

$$|A, B| = |D, C| \quad \text{and} \quad |A, C| = |B, D| \ .$$

By the congruence axiom SAS,

$$(A, C, D) \cong (D, B, A) \ .$$

23. Let A, X, B, P, Q, R be points in an absolute geometry so that X lies between A, B, the point Q is not on (AB) and $P \in IR(\angle AXQ)$, $R \in IR(\angle QXB)$. Assume that

$$|\angle AXP| = |\angle PXQ| \quad \text{and} \quad |\angle QXR| = |\angle RXB|$$

What is $|\angle PXR|$? (a number!)

Solution: Because of the additivity axiom for angles and since $\angle AXB$ is a straight angle, we have

$$180 = |\angle AXP| + |\angle PXQ| + |\angle QXR| + |\angle RXB| = 2\,|\angle PXR|$$

hence

$$|\angle PXR| = 90 \ .$$

24. Let A, B, C, X and A', B', C', X' be points in an absolute geometry so that

$$|A, C| = |A', C'| \quad , \quad |X, C| = |X', C'| \quad , \quad |B, C| = |B', C'| \ .$$

Also assume that X lies between A, B and X' lies between A', B', and that the angles at X respectively X' are right angles, i.e.

$$|\angle AXC| = |\angle BXC| = |\angle A'X'C'| = |\angle' X'C'| = 90^\circ \ .$$

Prove that $(A, B, C) \cong (A', B', C')$. Can we drop the assumption that X lies between A, B and X' between A', B'?

Solution: By SSA the we have congruences of the triangles

$$(A, C, X) \cong (A', C', X') \quad \text{and} \quad (B, C, X) \cong (B', C', X')$$

hence

$$|AX| = |A'X'| \quad \text{and} \quad |BX| = |B', X'| \ .$$

Since X lies between A, B and X' lies between A', B' this implies that

$$|A, B| = |A, X| + |X, B| = |A', X'| + |X', B'| = |A', B'| \ .$$

By SSS the triangles ΔABC and $\Delta A'B'C'$ are congruent.

25. Let A, B, X, C, D be points in an absolute geometry so that A, B, X are not collinear and X is the midpoint of the segments $[A, C]$ and $[B, D]$. Show that the lines AB and CD are parallel.

Hint: It may help to look at the proof of Theorem 3.35. This holds in absolute geometry.

Solution: By the opposite angles theorem, $|\angle AXB| = |\angle DXC|$, and since X is the midpoint of segment $[A, C]$, we have $|A, X| = |C, X|$. Similarly $|B, X| = |D, X|$. By the congruence axiom SAS,

$$(A, X, B) \cong (C, X, D)) \ . \tag{7.6}$$

Assume that Z were an intersection point of AB and CD. The point Z can not lie on AC. Otherwise, $AX = AC = AB$ because $A \neq Z$ and A, B, X would lie on a common line. Wlog we may assume that Z and B lie in the same half-plane of AC, because otherwise, since B and D lie in different half-planes of AC, we can interchange B and D (and also A with C). By the alternate angles theorem,

$$|\angle BAC| = |\angle ZAC| < |\angle DCA| \ . \tag{7.7}$$

Since X lies on the ray $[A.C = [A, X$ and on the ray $[C, A = [C, X$, the congruence (7.6) yields

$$|\angle BAC| = |\angle BAX| = |\angle DCX| = |\angle DCA|$$

contradicting (7.7).

26. Show that in an absolute geometry Euclid's fifth postulate is equivalent to

$$|\angle BAP| = |\angle XPA| = |\angle YPZ| = 180 - |\angle APZ| \tag{7.8}$$

for all points A, B, P, X, Y, Z so that

(a) A, B, P are not collinear,

(b) P lies between X, Z,

(c) P is between A, Y,

(d) X and B lie in opposite half-planes of AP,

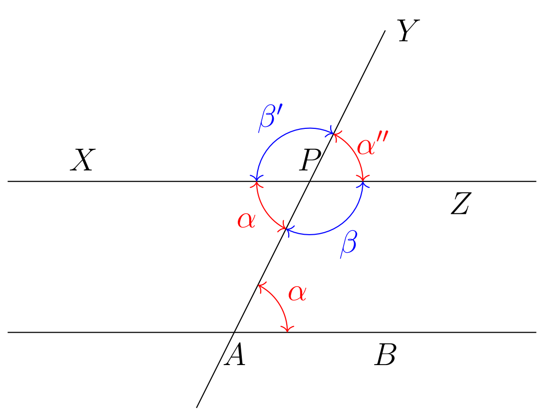
(e) AB is parallel to XZ.

Hint: You only need to say why the two rightmost identities in (7.8) hold.

Solution: Let A, B, P, X, Y, Z be such points. Since P is between X, Z, we have $XZ = XP = PZ$. Hence Euclid's fifth postulate is equivalent to the first identity $|\angle BAP| = |\angle XPA|$ being true whenever the lines AB and $XZ = XP = PZ$ are parellel.

The second identity in (7.8) always holds in absolute geometry, because the angles $\angle XPA$ and $\angle YPZ$ are opposite.

The third identity in (7.8) also holds in absolute geometry, because the angle $\angle APY$ is a straight angle. By additivity in straight angles, $180 = |\angle APZ| + |\angle ZPY|$.



In a **Euclidean Plane**, if $AB||XZ$, then $\alpha = \alpha' = \alpha''$, $\beta = \beta'$ and $180 = \alpha + \beta$.

27. Let A, B, C be noncollinear points in an absolute geometry. Assume that there are two different parallels to (AB) through C. Show that the sum of the interior angles of the triangle (A, B, C) is strictly less than 180,

$$|\angle ABC| + |\angle CAB| + |\angle BCA| < 180 \ .$$

Solution: This is the converse of problem 9.

7.3 Calculations in the Hyperbolic Upper Half-Plane

28. Let M be a Möbius transformations of the upper half plane so that $M(i\mathbb{R}^+) = i\mathbb{R}^+$, i.e. for all $\lambda \in \mathbb{R}^+$ there is $\mu \in \mathbb{R}^+$ so that $M(i\lambda) = i\mu$. Furthermore assume that $M(1+i) = 2i - 2$. Determine M (i.e. a matrix $A \in \mathrm{GL}^+(2, \mathbb{R})$ so that $M = M_A$ as in (7.9)).

Solution: Since the ideal end points of $i\mathbb{R}$ are 0 and ∞ there are two cases. Either M fixes these points or interchanges them. In the first case we must have

$$M(z) = \alpha z \quad \text{for some} \quad \alpha \in \mathbb{R}^+ \ ,$$

in the second

$$M(z) = \frac{\alpha}{z} \quad \text{for some} \quad \alpha \in \mathbb{R}^- \ .$$

In the first case the sign of the real part is preserved, hence such a Möbius transformation can not map $1 + i$ to $-2 + 2i$, and we are left with the second case. Now

$$\frac{\alpha}{1+i} = -2 + 2i$$

if and only if $\alpha = (1 + i)(-2 + 2i) = -4$, hence $M_A(z) = M(z) = \frac{-4}{z}$ with $A = \begin{pmatrix} 0 & -4 \\ 1 & 0 \end{pmatrix}$.

29. For $i = a, b, c$, find a Möbius transformation $T_i \in \mathcal{M}^+$ so that

(a) $T_a(C(-1, 2)) = C(3, 4)$

Solution: This can be done by scaling and translation only. We first adjust the radius by scaling and then translate to map the correct center. Thus $T_a = \tau_{20/6}\sigma_{1/3}$, i.e.

$$T_a(z) = \frac{z}{3} + \frac{20}{6}$$

(b) $T_b(C(6, \infty)) = C(3, 4)$

Solution: Clearly the translation τ_{-6} maps $C(6, \infty)$ to $C(0, \infty)$. A Möbius transformation taking $C(3, 4)$ to $C(0, \infty)$ is $N(z) = \frac{z-4}{z-3}$. Since the inverse of the corresponding matrix is

$$\begin{pmatrix} 1 & -4 \\ 1 & -3 \end{pmatrix}^{-1} = \begin{pmatrix} -3 & 4 \\ -1 & 1 \end{pmatrix} \ ,$$

the Möbius transformation

$$N^{-1}(z) = \frac{-3z + 4}{-z + 1}$$

takes $C(0,\infty)$ to $C(3,4)$. We need to compose this with τ_{-6}. Thus

$$T_b(z) = \frac{-3(z-6)+4}{-(z-6)+1} = \frac{-3(z-6)+4}{-(z-6)+1} = \frac{-3z+22}{-z+7}$$

maps $C(6,\infty))$ to $C(3,4)$.

(c) $T_c(C(-5,-1)) = C(6,\infty)$

Solution: We first map to $C(0,\infty)$ and then translate. Thus

$$T_c(z) = \frac{z+1}{z+5} + 6 = \frac{7z+31}{z+5}$$

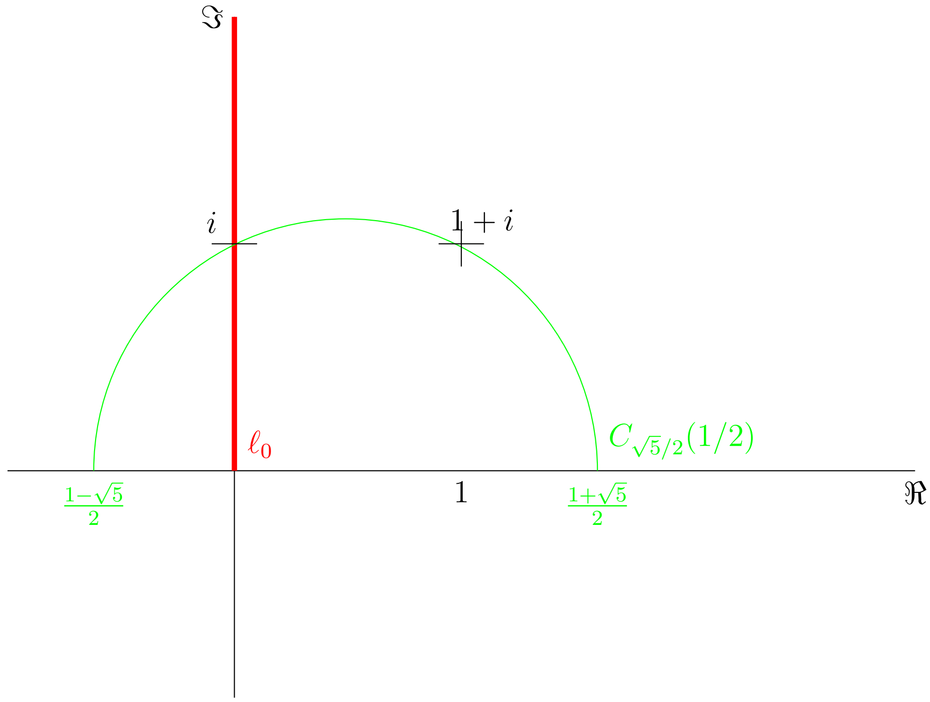
30. In the upper half plane model of the hyperbolic plane, find C so that $\Delta(i, 1+i, C)$ is equilateral. Compute the interior angles of your triangle.

    Hint: There are two possible solutions.

31. Compute the hyperbolic distance $d(i, i+1)$.

    Hint: The hyperbolic line through i and $i+1$ is the semicircle with center $1/2$ and radius $\frac{\sqrt{5}}{2}$. Find a Möbius transform $M(z) = \frac{az+b}{cz+d}$ that maps this circle $C_{\sqrt{5}/2}(1/2)$ to the positive imaginary axis $\ell_0 = \{it \,|\, t \in \mathbb{R}^+\}$ and use that the distance on the upper imaginary axis is given by

$$d(it, is) = \int_s^t \frac{1}{u}\, du = \ln(t) - \ln(s) = \ln(t/s) \quad \text{for} \quad 1 \le s \le t\ .$$



Solution: We want to map $C_{\sqrt{5}/2}(1/2)$ to ℓ_0. Thus we try mapping $\frac{1+\sqrt{5}}{2} \to \infty$ and $\frac{1-\sqrt{5}}{2} \to 0$. This can be done by the Möbius transform

$$m_\pm \colon z \mapsto \pm\frac{z - \frac{1-\sqrt{5}}{2}}{z - \frac{1+\sqrt{5}}{2}} = \pm\frac{2z-1+\sqrt{5}}{2z-1-\sqrt{5}}$$

which takes

$$i \mapsto \pm\frac{2i-1+\sqrt{5}}{2i-1-\sqrt{5}} = \pm\frac{(2i-1+\sqrt{5})(-2i-1-\sqrt{5})}{4+(1+\sqrt{5})^2} = \pm\frac{(4+1-5)+i(-4\sqrt{5})}{10+2\sqrt{5}}$$

$$= \pm\frac{-2\sqrt{5}}{5+\sqrt{5}}i = \pm\frac{-2}{1+\sqrt{5}}i$$

We need the one which maps the upper half plane to itself, i.e.

$$m = m_- : z \mapsto -\frac{2z-1+\sqrt{5}}{2z-1-\sqrt{5}}$$

This takes

$$i \mapsto \frac{2}{1+\sqrt{5}}i$$

and

$$\begin{aligned} 1+i \mapsto -\frac{2(i+1)-1+\sqrt{5}}{2(i+1)-1-\sqrt{5}} &= -\frac{2i+1+\sqrt{5}}{2i+1-\sqrt{5}} \\ &= -\frac{(2i+1+\sqrt{5})(-2i+1-\sqrt{5})}{4+(1-\sqrt{5})^2} \\ &= -\frac{(4+1-5)+i(-4\sqrt{5})}{10-2\sqrt{5}} \\ &= -\frac{-2\sqrt{5}}{5-\sqrt{5}}i = \frac{2}{\sqrt{5}-1}i\ . \end{aligned}$$

Hence the distance is

$$d\,(i, 1+i) = d\,(m(i), m(1+i)) = d\left(\frac{2}{\sqrt{5}+1}i, \frac{2}{\sqrt{5}-1}i\right)$$

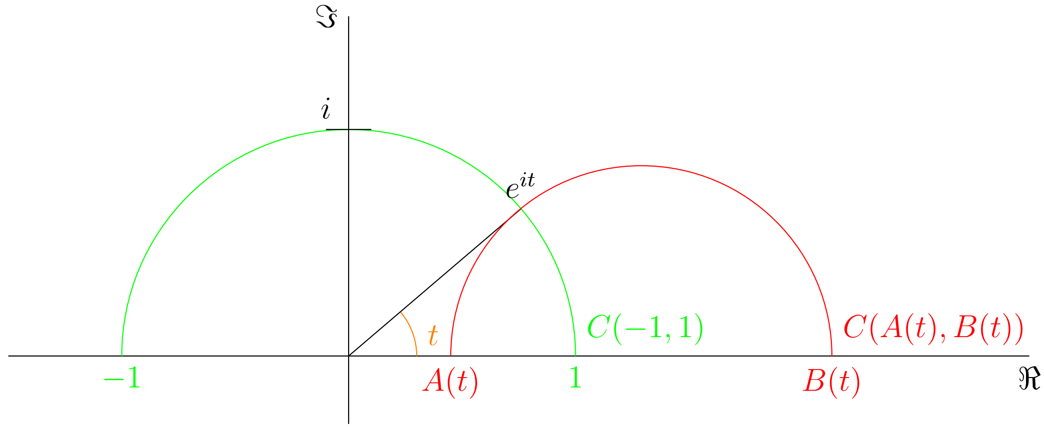
$$= \left|\ln\left(\frac{2}{\sqrt{5}-1}\right) - \ln\left(\frac{2}{\sqrt{5}+1}\right)\right| = \ln\left(\frac{\sqrt{5}+1}{\sqrt{5}-1}\right)$$

32. Let $C(A(t), B(t))$ be the hyperbolic line intersecting the hyperbolic line $C(-1,1) = C_1(0)$ perpendicularly in the point $x = e^{it} = (\cos(t), \sin(t)) \in C(-1,1) = C_1(0)$. Compute $A(t), B(t) \in \mathbb{R} \cup \{\infty\}$.

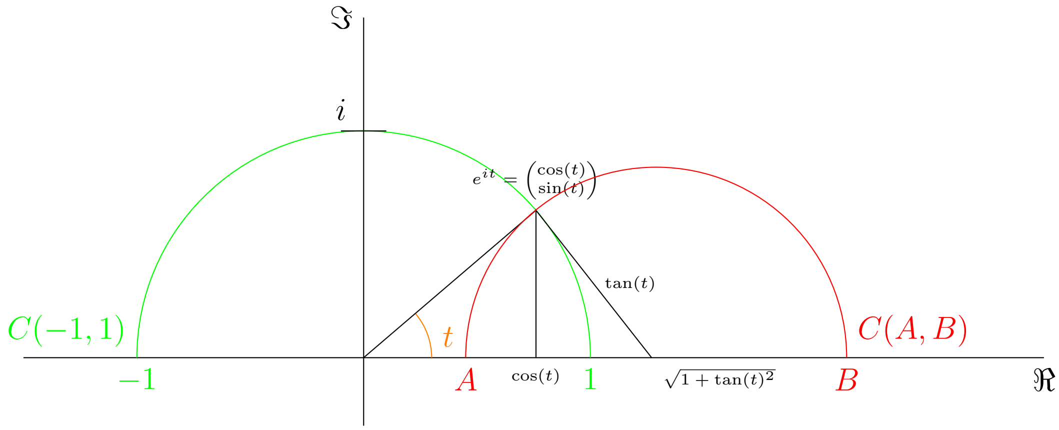
Hint: Recall the notation for hyperbolic lines. We write $C(A,B)$ for the hyperbolic line joining the ideal points A, B (which are not points in the hyperbolic plane). If $A, B \in \mathbb{R}$ (i.e. $\neq \infty$) then

$$C(A,B) = C_{\frac{|A-B|}{2}}\left(\frac{A+B}{2}\right)\ ,$$

the semicircle in the upper half plane of radius $\frac{|A-B|}{2}$ with centre $\frac{A+B}{2}$. If one of the ideal endpoints of the line is ∞, the hyperbolic line is a parallel to the imaginary axis, $C(A,\infty) = C(\infty, A)$. See also problem 35.



Solution: With a little trigonometry,



Thus

$$A = \sqrt{1+\tan(t)^2} - \tan(t) = \frac{1-\sin(t)}{\cos(t)} \quad \text{and} \quad B = \frac{1+\sin(t)}{\cos(t)}$$

which must be read as

$$A = 0 \quad \text{and} \quad B = \infty$$

in case of $t = \pi/2$.

33. Let M be the Möbius transform with

$$M(0) = 1 \quad , \quad M(1) = 2 \quad \text{and} \quad M(\infty) = 0 \; .$$

Find $M(i)$.

Solution: For $M(z) = \frac{az+b}{cz+d}$ to have these properties, we must have $a = 0$ and $b = d = 1$, without loss of generality. Then the second equations gives

$$M(1) = \frac{1}{c+1} = 2 \quad \text{hence} \quad c = \frac{-1}{2} \quad , \quad M(z) = \frac{2}{-z+2}$$

in particular

$$M(i) = \frac{2}{2-i} = \frac{4+2i}{5}$$

34. By $C_r(p)$, $p \in \mathbb{C}$, $r \in \mathbb{R}_0^+$, we denote the circle in $\mathbb{C} = \mathbb{R}^2$ with center p and radius r, NOT the hyperbolic line with ideal points p, r.

Recall that when identifying $\mathbb{R}^2$ with $\mathbb{C}$, the circle of radius $r > 0$ around $p = (a, b) \in \mathbb{R}^2 = \mathbb{C}$ is given by

$$C_r(a,b) = \left\{(x,y) \in \mathbb{R}^2 \,\middle|\, (x-a)^2 + (y-b)^2 = r^2\right\} = \left\{z \in \mathbb{C} \,\middle|\, |z-p|^2 = r^2\right\} .$$

The inversion $I \colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, $I(z) = \frac{1}{z}$ maps circles to circles, i.e. $I(C_r(p)) = C_s(q)$ for some $q = q(p,r)$, $s = s(p,r)$ (if $0 \notin C_r(p)$, i.e. $|p| \neq r$).

Let $T \colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be the map with

$$T(z) = \frac{z+1}{z-1} .$$

Compute center and radius of $T(C_1(3))$.

Hint: The center of this circle is not $T(3)$.

Solution: If $z \in \mathbb{R}$ then $T(z) \in \mathbb{R}$. Also $C_1(3)$ is preserved by complex conjugation. Hence $T(C_1(3))$ is the circle which contains the points $T(3-1) = T(2) = 3$ and $T(3+1) = T(4) = 5/3$ and is symmetric about the real axis, hence

$$T(C_1(3)) = C_{\frac{3-5/3}{2}}\left(\frac{3+5/3}{2}\right) = C_{\frac{2}{3}}\left(\frac{7}{3}\right)$$

Alternatively: We can decompose T,

$$T(z) = \frac{z+1}{z-1} = \frac{(z-1)+2}{z-1} = 1 + 2\frac{1}{z-1} ,$$

i.e.

$$T = S_2 \circ M_2 \circ I \circ S_{-1}$$

where $S_b(z) = z+b$, $M_a(z) = az$ and $I(z) = 1/z$. We can thus use the calculation of $I(C(p,r))$,

$$I(C_r(p)) = \left\{\frac{1}{z} \,\middle|\, (z-p)(\overline{z}-\overline{p}) = r^2\right\} = \left\{\frac{1}{z} \,\middle|\, z\overline{z} - 2\Re(\overline{p}z) + |p|^2 - r^2 = 0\right\} =$$

$$= \left\{\frac{1}{z} \,\middle|\, \frac{1}{|p|^2 - r^2} - 2\Re\left(\frac{p\frac{1}{z}}{|p|^2 - r^2}\right) + \frac{1}{z}\frac{1}{\overline{z}} = 0\right\} =$$

$$= \left\{u \,\middle|\, \frac{1}{|p|^2 - r^2} - 2\Re\left(\frac{pu}{|p|^2 - r^2}\right) + u\overline{u} = 0\right\}$$

$$= C_{\sqrt{\left|\frac{p}{|p|^2-r^2}\right|^2 - \frac{1}{|p|^2-r^2}}}\left(\frac{p}{|p|^2 - r^2}\right)$$

$$= C_{\sqrt{\frac{r^2}{(|p|^2-r^2)^2}}}\left(\frac{p}{|p|^2 - r^2}\right)$$

$$= C_{\frac{r}{||p|^2-r^2|}}\left(\frac{p}{|p|^2 - r^2}\right)$$

Thus

$$T(C_1(3)) = S_1(M_2(I(S_{-1}(C_1(3)))))$$

$$= S_1(M_2(I(C_1(2)))) = S_1\left(M_2\left(C_{\frac{1}{3}}\left(\frac{2}{3}\right)\right)\right) = S_1\left(C_{\frac{2}{3}}\left(\frac{4}{3}\right)\right) = C_{\frac{2}{3}}\left(\frac{7}{3}\right)$$

35. Recall the notation for hyperbolic lines, i.e. semicircles in the upper half plane with center on the real line or rays parallel to the imaginary axis. We denote by $C(a,b)$ the hyperbolic line with ideal points $a, b \in \mathbb{R} \cup \{\infty\}$, and by $C_r(p)$ the semicircle in the upper half-plane with center $p \in \mathbb{R}$ and radius $r \in \mathbb{R}^+$. Thus, for $p \in \mathbb{R}$, $r \in \mathbb{R}^+$, $A, B \in \mathbb{R}$,

$$C(a,b) = C_{\frac{|b-a|}{2}}\left(\frac{a+b}{2}\right) \quad \text{if} \quad a, b \in \mathbb{R}$$

which is interpreted as

$$C(a,\infty) = \left\{a + it \,\middle|\, t \in \mathbb{R}^+\right\}$$

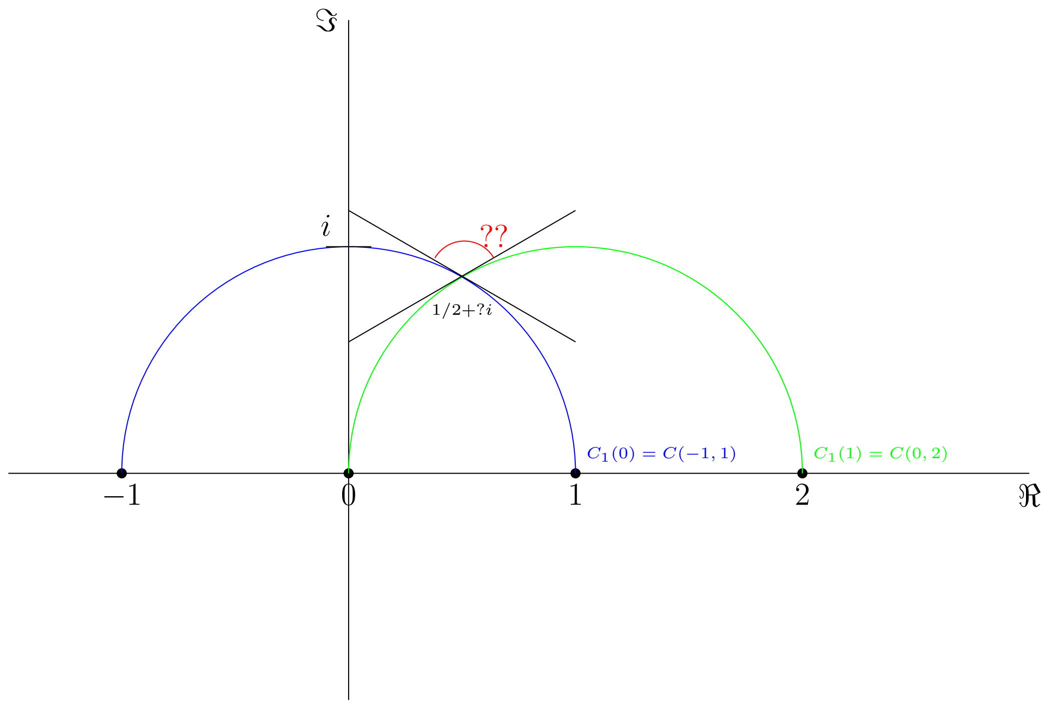
if $b = \infty$, where

$$C_r(p) = \{z \in \mathbb{C} \,|\, \Im z > 0, |z - p| = r\} = \left\{p + re^{it} \,\middle|\, 0 < t < \pi\right\} = C(p - r, p + r)\ ,$$

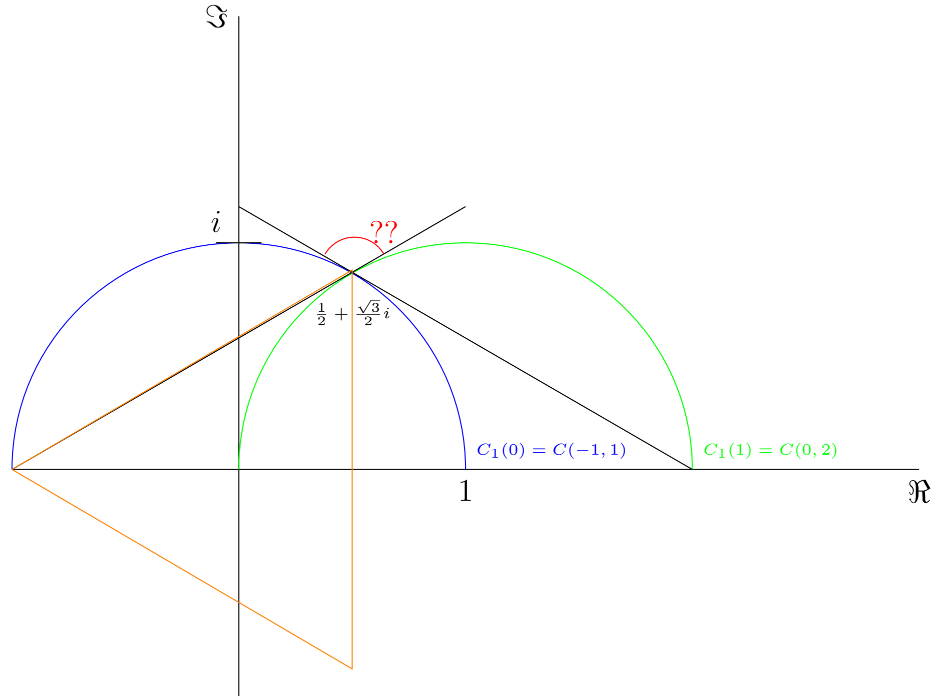
$$C(a,\infty) = C(\infty, a) = \{z \in \mathbb{C} \,|\, \Im z > 0, \Re z = a\} = \left\{a + it \,\middle|\, t \in \mathbb{R}^+\right\}\ .$$

The hyperbolic angle between two hyperbolic lines intersecting at P is the standard Euclidean angle their tangents make at the point P.

Find one of the two angles the hyperbolic line $C_1(0) = C(-1,1)$ makes with $C_1(1) = C(0,2)$:



Solution: The two tangents in the diagram extend,



producing the isosceles orange triangle. Hence the angles are $\frac{2\pi}{3} = 120°$ and $\frac{\pi}{3} = 60°$
Alternative solution: The two circles are:

$$C_1(0) = \left\{(x, y) \,\middle|\, x^2 + y^2 = 1, y > 0\right\} \quad , \quad C_1(1) = \left\{(x, y) \,\middle|\, (x-1)^2 + y^2 = 1, y > 0\right\} \ .$$

The circle intersect in the point (x_0, y_0) which must satisfy both equations, hence

$$x_0^2 + y_0^2 = 1 = (x_0 - 1)^2 + y_0^2 \quad \text{and} \quad y > 0$$

which leads to

$$(x_0, y_0) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \ .$$

Normal vectors to the circles at this point are

$$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \quad \text{and} \quad \left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$$

respectively. As tangent vectors we pick any of vectors perpendicular to these, for instance

$$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \quad \text{and} \quad \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

respectively. The angle between these is

$$\arccos\left(\frac{\left\langle \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \,\middle|\, \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \right\rangle}{\left\| \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \right\| \left\| \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \right\|}\right) = \arccos\left(-\frac{3}{4} + \frac{1}{4}\right) = \arccos\left(\frac{-1}{2}\right) = \frac{2\pi}{3} \ .$$

36. Recall that the **Möbius transformations of the upper half plane** are the maps $M\colon H \to H$, $H = \{z \in \mathbb{C} \mid \Im z > 0\}$, given by

$$M_A(z) = \frac{az+b}{cz+d} \quad \text{where} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}^+(2,\mathbb{R}) \ . \tag{7.9}$$

The group $\mathrm{GL}^+(2,\mathbb{R})$ is that of real (2×2)-matrices of positive determinant. The transformation M_A as in (7.9) maps 0 to $\frac{b}{d}$ and ∞ to $\frac{a}{c}$, if $d \neq 0 \neq c$. Also note that A and λA, $\lambda \in \mathbb{R}^+$, induce the same Möbius transformation, $M_{\lambda A} = M_A$.

(a) Find the Möbius transformation M of the upper half plane so that

$$M(C(0,\infty)) = C(0,2) \quad \text{and} \quad M(i) = 1+i \ .$$

Solution: We must have $M(0) = 0$ and $M(\infty) = 2$, hence

$$M(z) = \frac{2z}{z+d}$$

and need to adjust $d \in \mathbb{R}$ so that $M(i) = 1+i$, i.e.

$$\frac{2i}{i+d} = 1+i \quad \Longleftrightarrow \quad 2i = (i+d)(1+i) = d-1+(d+1)i \quad \Longleftrightarrow \quad d = 1 \ .$$

Thus $M(z) = \frac{2z}{z+1}$.

(b) Find the ideal points of $M(C(-1,1))$.

Solution: Since $M(-1) = \infty$ and $M(1) = 1$, we have that

$$M(C(-1,1)) = C(1,\infty) \ .$$

Alternative solution: The line $C(-1,1)$ is the perpendicular to $C(0,\infty)$ through i, hence $M(C(-1,1))$ is the perpendicualr to $M(C(0,\infty)) = C(0,2)$ through $M(i) = 1+i$. This perpendicular is $C(1,\infty)$.

(c) Sketch $M(C(-1/2,1/2))$, $M(C(-2,2))$, $M(C(-200,200))$.

Solution: In fact all the lines

$$C(-1,1)), C(-1/2,1/2)), C(-2,2)), C(-200,200))$$

are perpendicular to $C(0,\infty)$ which is mapped to $C(0,2)$. Thus the lines $M(C(-1,1))$, $M(C(-1/2,1/2))$, $M(C(-2,2))$, $M(C(-200,200))$ are all perpendicular to $C(0,2)$.

(d) Find $\lambda \in \mathbb{R}^+$ so that $M(\lambda i) = 1 + \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \in C(0,2)$.

Solution: We need to solve

$$M(i\lambda) = \frac{2\lambda i}{\lambda i + 1} = \frac{2\lambda i(1-\lambda i)}{1+\lambda^2} = \frac{2\lambda^2}{1+\lambda^2} + \frac{2i\lambda}{1+\lambda^2} \overset{!}{=} 1 + \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$

which gives $\lambda = 1+\sqrt{2}$.

Alternatively: We can invert M. Recall that M belongs to the matrix $M = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$ in $\mathrm{GL}^+(2,\mathbb{R})$ which is easy to invert,

$$M^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{-1}{2} & 1 \end{pmatrix} \ .$$

Thus the inverse of the Möbius transformation M is

$$M^{-1}(z) = \frac{\frac{1}{2}z}{\frac{-1}{2}z+1} = \frac{z}{-z+2} .$$

To compute the point on the imaginary axis $C(0,\infty)$, we compute

$$M^{-1}\left(1+\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right) = -\frac{\sqrt{2}+1+i}{\sqrt{2}+1+i-2\sqrt{2}} = -\frac{\sqrt{2}+1+i}{-\sqrt{2}+1+i} = \frac{\sqrt{2}+1+i}{\sqrt{2}-1-i}$$

$$= \frac{(\sqrt{2}+1+i)(\sqrt{2}-1+i)}{(\sqrt{2}-1)^2+1} = \frac{2-1-1+i2\sqrt{2}}{4-2\sqrt{2}} = \frac{i}{\sqrt{2}-1} = i(\sqrt{2}+1)$$

(e) What is the hyperbolic distance between

$$1+i \quad \text{and} \quad 1+\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}} \ ?$$

Solution: We have $M(i) = 1+i$ and $M(i(\sqrt{2}+1)) = 1+\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}$. Since the Möbius transformations preserve the hyperbolic distance, we have

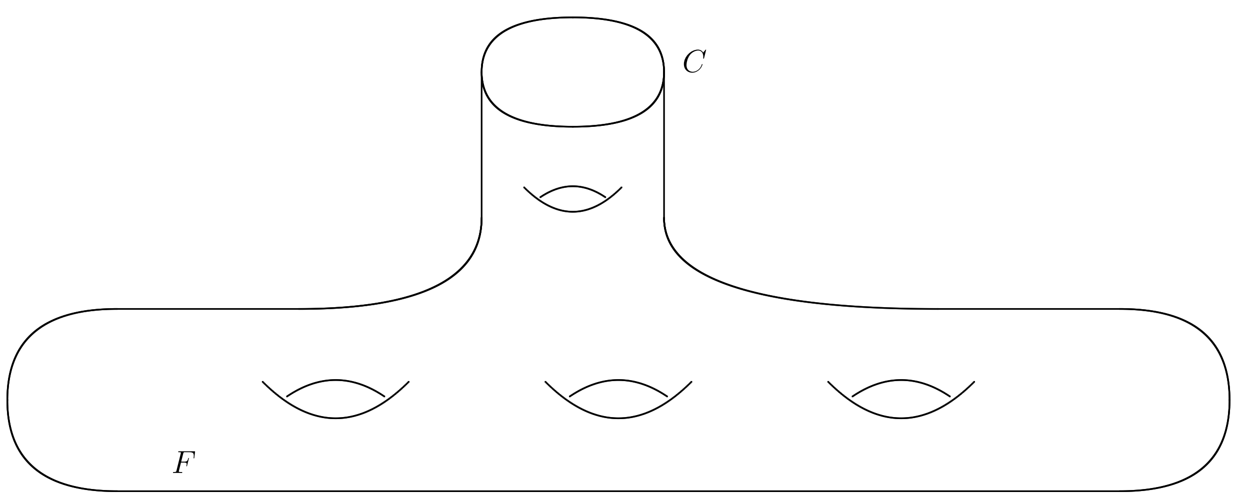
$$d\left(1+i, 1+\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right) = d(i, i(\sqrt{2}+1)) = \ln(1+\sqrt{2}) .$$

7.4 Gauss-Bonnet Theorem and Euler Characteristic

37. Recall that the Gauss curvature of a triangle ΔABC in a surface is

$$K(\Delta ABC) = \frac{\alpha+\beta+\gamma-\pi}{\mathbf{area}\,\Delta ABC}$$

where α, β, γ are the inner angles of ΔABC. The integral of the Gauss curvature K over the triangle is $K(\Delta ABC) \times \mathbf{area}\,\Delta ABC$. Let $F \subset \mathbb{R}^3$ be the surface with boundary shown in the picture below. Compute the integral of the Gauss curvature over F.



Hint: State the Gauss-Bonnet Theorem. Decompose F into triangles, count vertices, edges and triangles to compute the Euler characteristic. Note that the surface has a boundary circle C. In order to apply the Gauss-Bonnet Theorem as stated in class, you need to close F with a cap, a hemisphere attached at C.

Solution: Summing over the triangles of a triangulation of F, we get the Gauss Bonnet Theorem,

$$\int_F K = \sum_{\substack{\Delta ABC \text{ a triangle} \\ \text{in the triangulation}}} K(\Delta ABC) \times \mathbf{area}\,\Delta ABC = \sum (\text{interior angles} - \pi)$$

$$= 2\pi \#\text{vertices} - \pi \times \#\text{triangles}$$

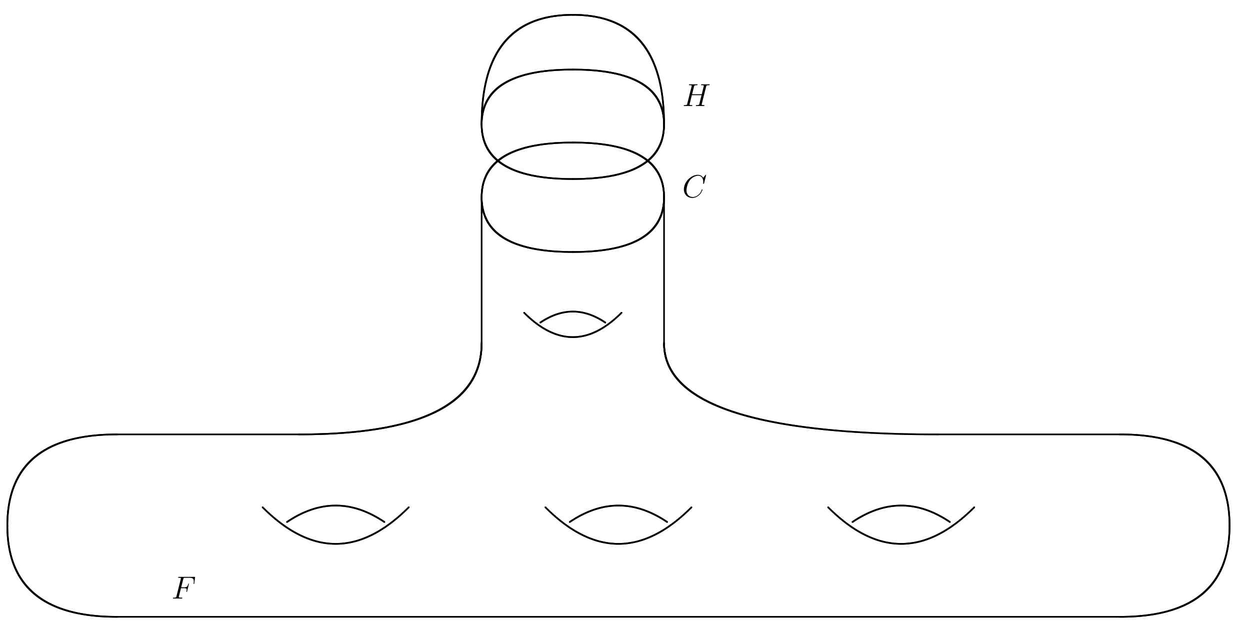
$$= 2\pi \#\text{vertices} - 2\pi \times \#\text{edges} + 2\pi \times \#\text{triangles}$$

$$2\pi \times (\#\text{vertices} - \#\text{edges} + \#\text{triangles}) = 2\pi\chi(F) \quad \text{Euler Characteristic}$$

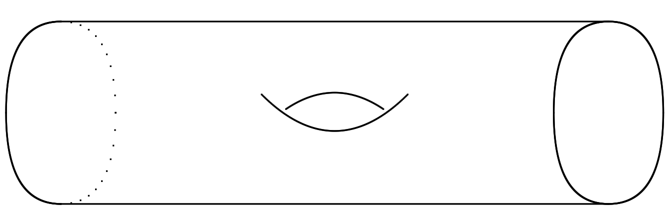
since each edge is in two triangles, we also have

$$2 \times \#\text{edges} = 3 \times \#\text{triangles}\ .$$

We first attach a hemisphere H at C to close the surface,



We now decompose the closed surface $F \cup H$ into triangles. The basic block is



This can be triangulated with 12 triangles, 21 edges (6 of which are on the boundary) and 7 vertices (6 of which are on the boundary). The three end caps can be done with one triangle each. Overall we get 2+48 triangles, 84-9 edges, 28-9 vertices, hence the Euler Characteristic becomes $\chi = 50 - 75 + 19 = -6$. By the Gauss-Bonnet Theorem,

$$\int_{F \cup H} K = 2\pi\chi(F \cup H) = -12\pi \ .$$

To this, the cap H contributes half the total curvature of a sphere, i.e.

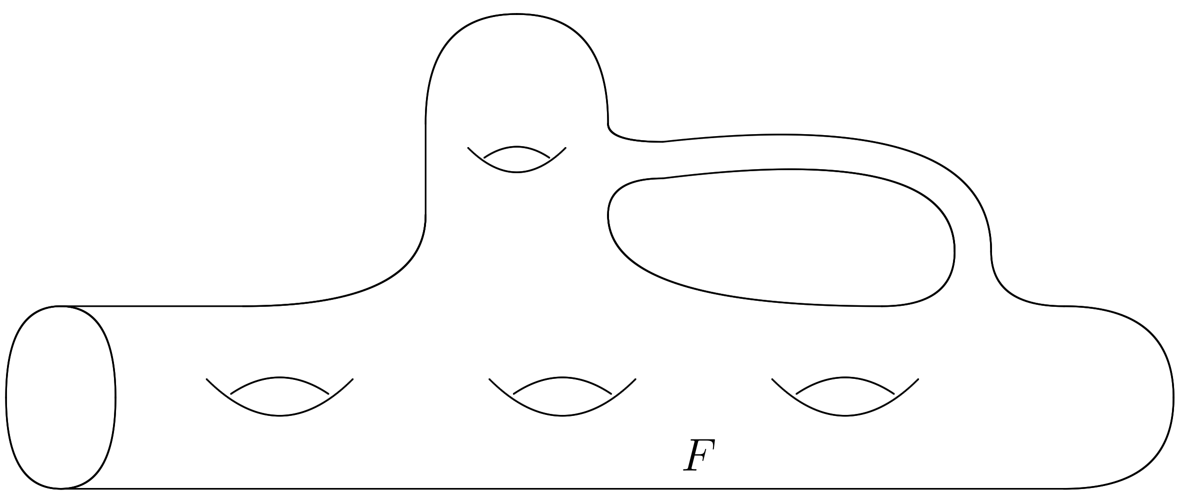
$$\int_H K = \frac{1}{2} \int_{S^2} K = \pi\chi(S^2) = 2\pi \ .$$

By the additivity of the integral,

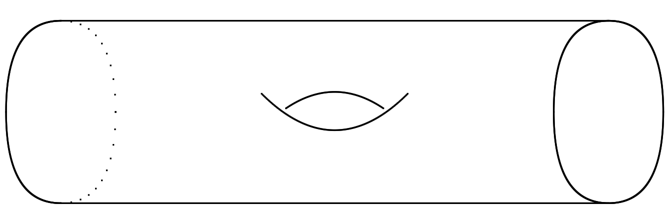
$$\int_F K = \int_{F \cup H} K - \int_H K = -12\pi - 2\pi = -14\pi$$

38. Compute the integral of the Gauß curvature over the surface F shown below.

Hint: Recall notions from example 5.3. You need to be careful at the boundary of this.



Solution: (*not using the genus*) As in 5.3 we first decompose the surface in 5 (!) pieces as



This can be triangulated with 12 triangles, 21 edges (6 of which are on the boundary) and 7 vertices (6 of which are on the boundary).

Glueing five of these together with one end-cap (a triangle) thus gives

$$5 \times (7, 21, 12) + (3, 3, 1) \quad \text{vertices, edges, triangles}$$

but we need to subtract the vertices and edges where we glued, i.e. five circles

$$-5 \times (3, 3, 0)$$

This gives a total of

$$(35 + 3 - 15, 105 + 3 - 15, 61) = (23, 93, 61) \quad \text{vertices, edges, triangles}$$

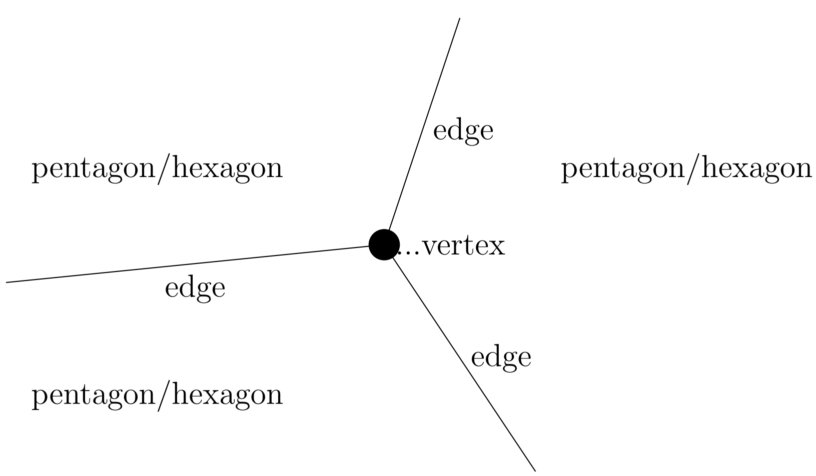
The Euler characteristic is therefore $23 - 93 + 61 = -9$. Since we have the Gauss-Bonnet Theorem only for closed surfaces, we need to cap off the boundary at the left of this surface, by glueing a triangle. This only adds one triangle, and gives the Euler characteristic -8 for the capped off surface. By the Gauss Bonnet theorem, the capped off surface has therefor curvature integral $-8 \times 2\pi = -16\pi$. Since we added a hemisphere to close the surface, we need to subtract the curvature integral of a hemisphere, which is 2π. Thus the curvature integral of the surface F as shown is -18π.

alternatively: If S denotes the closed surface obtained from F by glueing it with a hemisphere, then S has genus 5, hence Euler characteristic -8 and integral of the Gauss curvature equal to $2\pi \times (-8) = -16$. Since the hemisphere has integral 2π, we get -16π for the integral of the Gauss curvature over F.

39. Is it possible to make a football out out of polygon patches so that

   (a) there are 27 pentagons
   (b) the patches are hexagons or pentagons,
   (c) every edge belongs to exactly two patches
   (d) each vertex is met by exactly three edges
   (e) each vertex belongs to exactly three patches

   Thus the edge graph at each corner is a tripod, i.e. looks like



   Hint: First express the Euler characteristic in terms of the numbers h, p, e, v of hexagons, pentagons, edges and vertices. From this you can eliminate the numbers e and v.

   Solution: The Euler characteristic of the sphere is 2, and the number of faces of this polygonal decomposition is $h + p$. Thus

$$2 = h + p - e + v \ .$$

Since each edge lies in exactly two polygons, and each pentagon/hexagon has 5 respectively 6 edges, we have

$$e = \frac{5p + 6h}{2} \; .$$

Similarly, for the vertices, each polygon contributes 5 respectively 6 vertices and each vertex lies in three polygons. Thus

$$v = \frac{5p + 6h}{3} \; .$$

Inserting this into the formula for the Euler characteristic, we get

$$2 = h + p - \frac{5p + 6h}{2} + \frac{5p + 6h}{3} = \frac{p}{6} \; .$$

It follows that $p = 12$. Thus it is not possible to decompose the sphere as above with 27 pentagons.

40. Can the orientable surface of genus 2 be decomposed into hexagons, so that exactly three edges meet at every vertex?

Solution: If we denote by h, e, v the number of hexagons, edges and vertices respectively of such a decomposition, we must have

$$\begin{aligned} -2 &= h - e + v \quad \text{Euler characteristic} \\ 6h &= 2e \quad \text{every hexagon has 6 edges} \\ 3v &= 2e \quad \text{every vertex lies in 3 edges} \end{aligned}$$

Eliminating $e = 3h$ and $v = \frac{2}{3}e = 2h$ leads to

$$-2 = h(1 - 3 + 2) \; ,$$

impossible. Thus there is no such decomposition of F_2.