

# Complex Analysis II - MT433P

## 1 Problems

1. Let  $\mu, \gamma: [0, 1] \rightarrow \mathbb{C} \setminus \{0\}$  be closed continuous curves with same base point, i.e.  $\gamma(0) = \mu(0)$ . Show that

$$w(\gamma * \mu; 0) = w(\gamma; 0) + w(\mu; 0) .$$

**Hint:** As usual we are sloppy as to the interval bounds. In this problem,  $\gamma$  and  $\mu$  can be concatenated because they are both closed and the endpoint of  $\gamma$  is the start point of  $\mu$ :  $\mu(0) = \gamma(0) = \gamma(1)$ . By definition of the concatenation,  $\gamma * \mu$  is the curve defined on  $[0, 2]$  given by

$$(\gamma * \mu)(t) = \begin{cases} \gamma(t) & \text{if } 0 \leq t \leq 1 \\ \mu(t-1) & \text{if } 1 \leq t \leq 2 \end{cases} .$$

Closedness now means that  $(\gamma * \mu)(0) = (\gamma * \mu)(2)$ .

2. Let  $\gamma: [0, 2\pi] \rightarrow \mathbb{C}$  be the curve with

$$\gamma(t) = \cos(14t) + i \sin(42t) .$$

Check that  $\gamma(t) \neq 0$  for all  $t$  and find the winding number of  $\gamma$  around 0.

What is the winding number of the curve  $\mu: [0, 2\pi] \rightarrow \mathbb{C}$  be the curve with

$$\mu(t) = \cos(14t) + i \sin(35t)$$

around 0?

**Hint:** The curves are concatenations of simpler curves. You can use the expression of the winding number in terms of ray crossings.

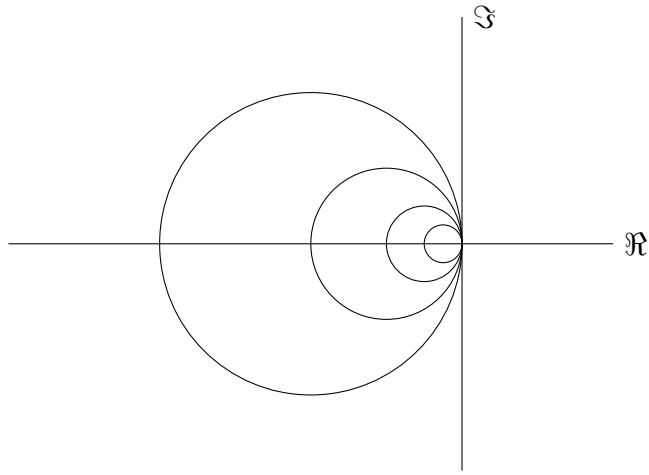
3. Is there a continuous closed curve  $\gamma: [0, 1] \rightarrow \mathbb{C}$  so that for every  $n \in \mathbb{Z}$ , there is a point  $p_n \in \mathbb{C} \setminus \gamma([0, 1])$  with  $w(\gamma; p) = n$ ?

**Solution:** It suffices to find such a curve for all positive integers,  $n \in \mathbb{N}$ . To construct such a curve we concatenate circles of length  $\frac{1}{2^n}$  (or any other summable sequence), for instance, we define

$$c(t) = -1 + e^{2\pi i t} \quad , \quad t \in [0, 1]$$

and

$$\gamma(0) := 0 \quad , \quad \gamma(t) = \frac{1}{2^{n-1}} c(2^n(t - 2^{-n})) \quad \text{if } 2^{-n} \leq t \leq 2^{-n+1} .$$



The winding number of this curve around the point  $-3/2^n$  for  $n \in \mathbb{N}$  is  $n$ .

4. Let  $\gamma: [-1, 1] \rightarrow \mathbb{C}$ ,

$$\gamma(t) = 2t + i \sin(\pi t) .$$

Compute

$$\int_{\gamma} \frac{1}{z+1} + \frac{1}{z-1} dz .$$

**Solution:** The integrand  $f(z)$  is odd, i.e.  $f(-z) = -f(z)$ . The same holds for the curve,  $\gamma(-t) = -\gamma(t)$ . It follows that

$$\int_{\gamma} f(z) dz = \int_{-1}^1 f(\gamma(t))\gamma'(t) dt = 0 .$$

5. Compute

$$\int_0^{2\pi} \frac{1}{1 + \sin(t)^2} dt .$$

**Solution:** We can rewrite this as a complex line integral over a closed curve and apply the Residue

Theorem,

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{1 + \sin(t)^2} dt &= \int_0^{2\pi} \frac{1}{1 + \left(\frac{e^{it} - e^{-it}}{2i}\right)^2} \frac{1}{ie^{it}} ie^{it} dt \\
&= \oint_{|z|=1} \frac{1}{1 + \left(\frac{z - \frac{1}{z}}{2i}\right)^2} \frac{1}{iz} dz \\
&= \oint_{|z|=1} \frac{4iz}{-4z^2 + z^4 - 2z^2 + 1} dz \\
&= \oint_{|z|=1} \frac{4iz}{z^4 - 6z^2 + 1} dz \\
&= \oint_{|z|=1} \frac{4iz}{\underbrace{(z^2 - (3 - 2\sqrt{2}))(z^2 - (3 + 2\sqrt{2}))}_{J(z)}} dz \\
&= 2\pi i \left( \operatorname{Res} \left( J(z); z = \sqrt{3 - 2\sqrt{2}} \right) + \operatorname{Res} \left( J(z); z = -\sqrt{3 - 2\sqrt{2}} \right) \right) \\
&= 2\pi i \left( \frac{4i\sqrt{3 - 2\sqrt{2}}}{2\sqrt{3 - 2\sqrt{2}} \left( (3 - 2\sqrt{2}) - (3 + 2\sqrt{2}) \right)} + \frac{-4i\sqrt{3 - 2\sqrt{2}}}{-2\sqrt{3 - 2\sqrt{2}} \left( (3 - 2\sqrt{2}) - (3 + 2\sqrt{2}) \right)} \right) \\
&= 2\pi i \left( \frac{4i}{2(-4\sqrt{2})} + \frac{-4i}{-2(-4\sqrt{2})} \right) = \pi\sqrt{2}.
\end{aligned}$$

6. Compute the integral

$$\int_{-\infty}^{\infty} \frac{\sin(t)}{t} dt.$$

**Hint:** Close the path of integration as in the picture below. Choose the relation between  $h$  and  $r$  so that the contribution of the integrals over  $\beta_*$ ,  $\omega_*$  and  $\mu_*$  can be neglected in the limit  $r \rightarrow \infty$ . Compare the integral over  $\alpha_r$  with that over the full circle.

**Solution:** This is the imaginary part of

$$\lim_{r \rightarrow \infty} \left( \int_{-r}^{-1/r} \frac{e^{it}}{t} dt + \int_{1/r}^r \frac{e^{it}}{t} dt \right) = \lim_{r \rightarrow \infty} \left( \int_{\gamma_r} \frac{e^{iz}}{z} dz + \int_{\delta_r} \frac{e^{iz}}{z} dz \right)$$

Here  $\gamma_r$ ,  $\delta_r$ ,  $\alpha_r$ ,  $\beta_{r,h}$  are the curves

$$\begin{aligned}
\gamma_r: [-r, -1/r] &\rightarrow \mathbb{C}, & \gamma_r(t) &= t, \\
\delta_r: [1/r, r] &\rightarrow \mathbb{C}, & \delta_r(t) &= t, \\
\alpha_r: [0, \pi] &\rightarrow \mathbb{C}, & \alpha(t) &= \frac{1}{r}e^{it} \\
\beta_{r,h}: [-r, r] &\rightarrow \mathbb{C}, & \beta(t) &= ih + t \\
\omega_{r,h}: [0, h] &\rightarrow \mathbb{C}, & \omega(t) &= -r + it \\
\mu_{r,h}: [0, h] &\rightarrow \mathbb{C}, & \mu(t) &= r + it
\end{aligned}$$

as in the picture below. Thus  $\gamma_r * \alpha_r^{-1} * \delta_r * \mu_{r,h} * \beta_{r,h}^{-1} * \omega_{r,h}$  is closed. We estimate the integrals over the auxiliary curves:

$$\left| \int_{\omega_{r,h}} \frac{e^{iz}}{z} dz \right|, \left| \int_{\mu_{r,h}} \frac{e^{iz}}{z} dz \right| \leq \frac{h}{r}$$

$$\left| \int_{\beta_{r,h}} \frac{e^{iz}}{z} dz \right| \leq 2re^{-h}$$

If we choose  $r = h^2$ , both of these converge to 0 for  $h \rightarrow \infty$ .

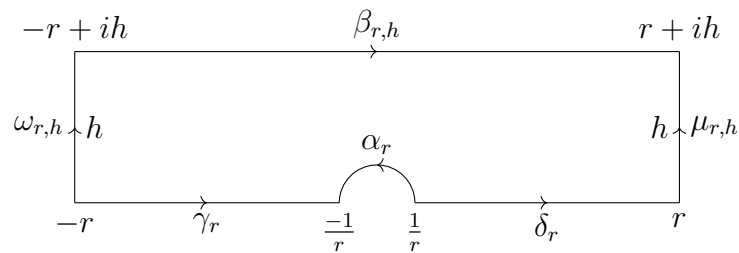
$$\lim_{r \rightarrow \infty} \int_{\alpha_r} \frac{e^{iz}}{z} dz = \frac{1}{2} \lim_{\epsilon \rightarrow 0} \int_{|z|=\epsilon} \frac{e^{iz}}{z} dz = \pi i$$

Since

$$\oint_{\gamma_r * \alpha_r^{-1} * \delta_r * \mu_{r,h} * \beta_{r,h}^{-1} * \omega_{r,h}} \frac{e^{iz}}{z} dz = 0$$

this yields

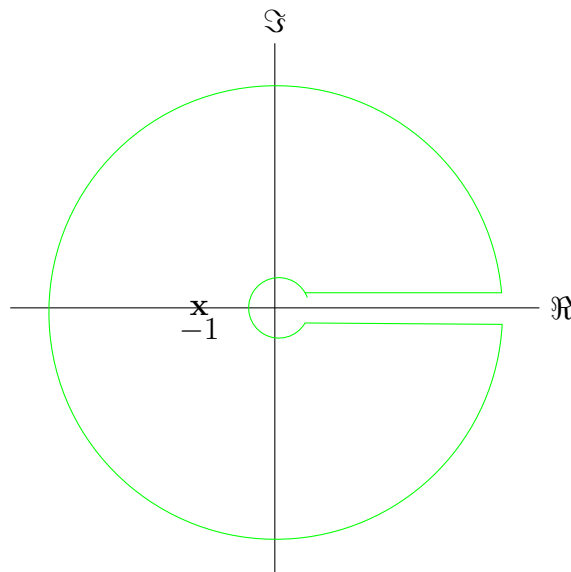
$$\int_{-\infty}^{\infty} \frac{\sin(t)}{t} dt = \pi .$$



7. Compute the integral

$$\int_0^{\infty} \frac{1}{\sqrt{t}(1+t)} dt .$$

**Hint:** Look at the curve



8. Throughout this problem, zeros are counted with multiplicity. How many zeros has the polynomial  $p(z) = z^5 + 9z^2 + 3$  in

(a)  $B_2(0)$ ?

(b)  $A_{2,3}(0) = \{z \in \mathbb{C} \mid 2 < |z| < 3\}$ ?

**Solution:** If  $|z| = 2$  then

$$|z^5 + 3| \leq 2^5 + 3 = 35 < 9|z^2| = 36 .$$

By Rouché's Theorem ??,  $p$  has as many zeros in  $B_2(0)$  as  $z^2$ , i.e. 2.

If  $|z| = 3$  then

$$|9z^2 + 3| \leq 81 + 3 = 84 < |z^5| = 3^5 = 243 .$$

By Rouché's Theorem,  $p$  has as many zeros in  $B_3(0)$  as  $z^5$ , i.e. 5. Since none of these are on  $S_2(0)$ , we have 3 zeros in the annulus  $A_{2,3}(0)$ .

9. Show that the function  $f \in \mathcal{O}(\mathbb{C})$  with  $f(z) = e^z + z^5 + 1$  has exactly three zeros with negative real part.

**Hint:** Compare  $|e^z| = e^{\Re(z)}$  and  $|z^5 + 1|$  for  $z$  imaginary and for  $\Re(z) \leq 0$ ,  $|z| = 4056$ . The estimate is in your favour in order to apply Rouché's Theorem ??, except at  $z = 0$ . Use the zero counting winding number directly.

**Solution:** Let  $\gamma$  be the boundary curve of  $\Omega = \{z \in \mathbb{C} \mid \Im(z) \leq 0, |z| < 7000\}$ . For  $z$  on  $\gamma$  we have  $|z^5 + 1| > |e^z|$  except at  $z = 0$ , so we can not use Rouché's Theorem directly. But as for the proof of that theorem, we look at the zero's counting winding number  $w(f \circ \gamma, 0)$ . Consider the homotopy

$$H(s, t) = se^{\gamma(t)} + \gamma(t)^5 + 1 .$$

For  $s \in [0, 1]$ , if  $\gamma(t) \neq 0$  then  $H(s, t) \neq 0$  because of the above estimates. If  $\gamma(t) = 0$ , then  $H(s, t) = s + 1$ , by direct computation. Thus, in all cases  $H: [0, 1] \times [0, 1] \rightarrow \mathbb{C} \setminus \{0\}$  is a homotopy relative endpoint of  $f \circ \gamma$  with  $g \circ \gamma$ ,  $g(z) = z^5 + 1$  avoiding 0. Thus

$$w(f \circ \gamma, 0) = w(g \circ \gamma, 0) = 3$$

because  $z^5 + 1$  has three simple zeros in the left half plane.

10. Let  $z_0$  be an isolated, not removable singularity of  $f(z)$ . Is it possible that  $z_0$  is a removable singularity of  $e^{f(z)}$ ?

**Solution:** If  $z_0$  were a removable singularity of  $e^{f(z)}$ , then this has a holomorphic extension  $g$ . Then

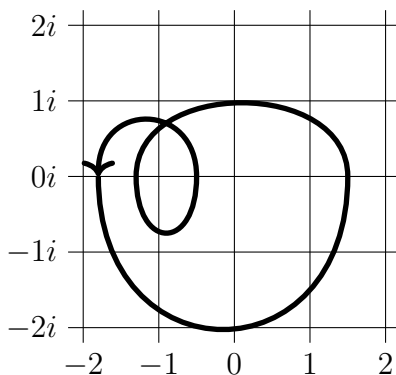
$$\frac{g'(z)}{g(z)} = \frac{f'(z)e^{f(z)}}{e^{f(z)}} = f'(z) .$$

Hence  $z_0$  is a simple pole of  $f'$ , impossible.

11. Compute the integral

$$\oint_{\gamma} \frac{z}{\sin(\pi z) \cos(i\pi z)} dz$$

where  $\gamma$  is as shown in the picture.



**Hint:** If  $f$  is odd, i.e.  $f(z) = -f(-z)$ , or even, i.e.  $f(z) = f(-z)$ , what is the relation between the residues  $\text{Res}(f, p)$  and  $\text{Res}(f, -p)$ ? Check your answer to this against  $f(z) = \frac{1}{z}$ !

What is the relation between  $\text{Res}(f(\alpha z), z = 0)$  and  $\text{Res}(f(z), z = 0)$ ?

You do not need to work out all the singularities in this problem!

**Solution:** Since

$$2i \sin(z) = e^{iz} - e^{-iz} = 0$$

if and only if  $e^{iz} = \pm 1$ ,  $z \in \pi\mathbb{Z}$  and

$$2 \cos(z) = e^{iz} + e^{-iz} = 0$$

if and only if  $e^{iz} = \pm i$ ,  $z \in \frac{\pi}{2} + \pi\mathbb{Z}$ , the integrand has simple poles at  $\mathbb{Z} \cup i\left(\frac{1}{2} + \mathbb{Z}\right)$ . The singularities surrounded by  $\gamma$  are

$$-1, 0, 1, \frac{i}{2}, -\frac{i}{2}, -\frac{3i}{2}$$

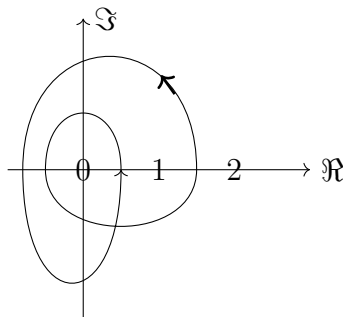
The singularity at 0 is removable. The integrand is even and therefore has opposite residues at opposite points. All singularities have winding number 1 with exception of  $-1$  where it is 2. Thus we only need the residues

$$\begin{aligned} \text{Res}\left(\frac{z}{\sin(\pi z) \cos(i\pi z)}; z = -1\right) &= \frac{-1}{\cos(-i\pi)} \text{Res}\left(\frac{1}{\sin(\pi z)}; z = -1\right) \\ &= \frac{-1}{\cos(i\pi)} \frac{-1}{\pi} \\ &= \frac{1}{\pi \cos(i\pi)} = \frac{2}{\pi(e^{-\pi} + e^{\pi})} \end{aligned}$$

$$\begin{aligned} \text{Res}\left(\frac{z}{\sin(\pi z) \cos(i\pi z)}; z = -\frac{3i}{2}\right) &= \frac{-\frac{3i}{2}}{\sin(-\frac{3\pi i}{2})} \text{Res}\left(\frac{1}{\cos(i\pi z)}; z = -\frac{3i}{2}\right) \\ &= \frac{\frac{3i}{2}}{\sin(\frac{3\pi i}{2})} \text{Res}\left(\frac{1}{\sin(\pi z)}; z = 0\right) \\ &= \frac{\frac{3i}{2}}{\sin(\frac{3\pi i}{2})} \frac{1}{\pi} = \frac{3i}{2\pi(e^{-3\pi/2} - e^{3\pi/2})} \end{aligned}$$

and the integral is  $2\pi i$  times the sum of these.

12. Let  $\gamma$  be the curve shown in the picture. Compute  $\oint_{\gamma} \frac{z^2 e^{\frac{1}{z}}}{z-1} dz$ .



**Solution:** The singularities of the integrand  $f(z) = \frac{z^2 e^{\frac{1}{z}}}{z-1}$  are at 0, essential, and 1, a pole of order 1. The winding numbers are

$$w(\gamma, 0) = 2 \quad , \quad w(\gamma, 1) = 1 .$$

Since  $z^2 e^{\frac{1}{z}}$  is holomorphic near  $z = 1$ , the residue of  $f$  and 1 is

$$\operatorname{Res} \left( \frac{z^2 e^{\frac{1}{z}}}{z-1}; z=1 \right) = z^2 e^{\frac{1}{z}} \Big|_{z=1} \operatorname{Res} \left( \frac{1}{z-1}; z=1 \right) = e .$$

The Laurent series of  $f(z)$  at  $z = 0$  is

$$f(z) = z^2 e^{\frac{1}{z}} \frac{1}{z-1} = \sum_{k=0}^{\infty} \frac{1}{k!} z^{2-k} \times (-1) \sum_{l=0}^{\infty} z^l = - \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{z^{l-k+2}}{k!} .$$

The residue is the coefficient with  $z^{-1}$ , hence

$$\operatorname{Res} \left( \frac{z^2 e^{\frac{1}{z}}}{z-1}; z=0 \right) = - \sum_{k=3}^{\infty} \frac{1}{k!} = 1 + 1 + \frac{1}{2} - e = \frac{5}{2} - e .$$

By the Residue Theorem, the integral is

$$\oint_{\gamma} \frac{z^2 e^{\frac{1}{z}}}{z-1} dz = 2\pi i \left( 2 \left( \frac{5}{2} - e \right) + e \right) = 2\pi i (5 - e)$$

13. Show that the function  $\cos(z) + z$  has  $2n$  zeros (counting multiplicities) in the domain  $\{z \in \mathbb{C} \mid 0 < \Re(z) < 2\pi n\}$ .

It follows from this that the zeros are all simple, how?

**Solution:** If  $z = 2n\pi + \lambda i$ ,  $n \in \mathbb{N}_0$ ,  $\lambda \in \mathbb{R}$ ,  $t \in [0, 1]$  then

$$f_t(z) = \cos(z) + tz = \frac{e^{iz} + e^{-iz}}{2} + 2 = \frac{e^{\lambda} + e^{-\lambda}}{2} + 2nt\pi + it\lambda \neq 0 .$$

We estimate

$$\begin{aligned} |f_t(x + iy)| &= |\cos(x + iy) + t(x + iy)| \\ &= \left| \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2} + t(x + iy) \right| \\ &\geq \frac{e^{|y|} - e^{-|y|}}{2} - t|x + iy| \\ &\geq \frac{e^{|y|}}{2} - \frac{1}{2} - |x| - |y| \end{aligned}$$

For  $0 \leq x \leq 2n$  this is positive if

$$\frac{e^{|y|}}{2} - |y| > 2n + \frac{1}{2} \tag{1.1}$$

Let  $y \in \mathbb{R}^+$  satisfy this. Such  $y$  exists because  $\frac{e^t}{2} - t \xrightarrow{y \rightarrow \infty} \infty$ . Thus for any  $h$ ,  $h > y$ , and all  $z$  on the boundary of

$$\Omega = \{z \mid 0 < \Re(z) < 2n, |\Im(z)| < h\}$$

we have that  $f_t(z) \neq 0$  for all  $t \in [0, 1]$ . By the argument principle,  $f_1$  has the same number of zeros as  $f_0(z) = \cos(z)$ , in  $\Omega$ , i.e.  $2n$ .

To see that all these are simple, first note that  $f_1$  has no real zero because for  $t \in \mathbb{R}_0^+$ ,

$$f(t) = \cos(t) + t = 1 + \int_0^t \underbrace{-\sin(u) + 1}_{\geq 0} du > 0 .$$

But  $f$  is real, i.e.  $f(\bar{z}) = \overline{f(z)}$  and therefore zeros of  $f$  come in conjugate pairs  $z, \bar{z}$ . By the above we have two zeros in each strip  $(2k\pi, 2(k+1)\pi) + i\mathbb{R}$  and these must be conjugates of each other. Since they are not real, they must be different.

*Alternatively:* A (at least) double zero is also a zero of the derivative

$$f'(z) = -\sin(z) + 1 .$$

But the zeros of this are all of the form  $z = \frac{\pi}{2} + 2k\pi$ ,  $k \in \mathbb{Z}$ . However none of these is a zero of  $f$ ,

$$f\left(\frac{\pi}{2} + 2k\pi\right) = \cos(\pi/2) + \frac{\pi}{2} + 2k\pi \neq 0 \quad \text{for all } k \in \mathbb{Z} .$$

14. Consider the polynomial

$$p(z) = 11z^9 + 15z^3 + 1 .$$

Determine the number of zeros of  $p$  in each of

$$B_1(0) \quad , \quad B_2(0) \quad , \quad B_3(0) \quad \text{and} \quad \mathbb{C} \setminus B_3(0) .$$

15. Consider the polynomial

$$p(z) = z^9 + 10z^6 + 15z^3 + 1 .$$

Determine the number of zeros of  $p$  in each of

$$B_1(0) \quad , \quad B_2(0) \quad , \quad B_3(0) \quad \text{and} \quad \mathbb{C} \setminus B_3(0) .$$

**Hint:** First count with multiplicities. Why does  $p$  not have higher order zeros?



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16. Find a biholomorphic map

$$\Omega := \{z \in \mathbb{C} \mid 0 < \Re(z), 0 < \Im(z) < \pi\} \xrightarrow{\phi} \Psi := \{z \in \mathbb{C} \mid 0 < \Im(z) < 2\pi\} .$$

**Hint:** power functions, exponential/log, Möbius transformations.

17. Classify the following domains up to biholomorphic equivalence. Briefly justify your claims.

(a)  $\Omega_a = \mathbb{C} \setminus \mathbb{R}_0^+$

(b)  $\Omega_b = \mathbb{E} \setminus \{0\}$

(c)  $\Omega_c = \mathbb{C} \setminus (\{1, 2\} \cup [3, \infty))$

(d)  $\Omega_d = \{z \in \mathbb{C} \mid \Re(z) + 3\Im(z) < 2\}$

(e)  $\Omega_e = \{z \in \mathbb{C} \mid \Re(z) > \Im(z)^4\}$

(f)  $\Omega_f = \mathbb{E} \setminus \{0, \frac{1}{2}\}$

(g)  $\Omega_g = \{z \in \mathbb{C} \mid 2 < |z| < 3\}$

(h)  $\Omega_h = \mathbb{C}$

(i)  $\Omega_i = \mathbb{C} \setminus [-1, 1] = \{z \in \mathbb{C} \mid |\Re(z)| > 1 \text{ or } \Im(z) \neq 0\}$

18. It is known that two annuli  $A_{r,R} = B_R(0) \setminus \overline{B_r(0)}$ ,  $A_{r',R}$ , are biholomorphically equivalent if and only if  $R/r = R'/r'$ . Prove the weaker statement: “There is  $r > 0$  so that for all  $\rho$ ,  $0 < \rho < r$ , the annulus  $A_{\rho,1}$  is not biholomorphically equivalent to  $A_{1/2,1}$ .”

**Hint:** Assuming the contrary, i.e. a sequence of biholomorphic maps

$$A_{r_n,1} \xrightarrow{\phi_n} A_{1/2,1} \quad , \quad r_n \xrightarrow{n \rightarrow \infty} 0 \quad ,$$

show that there would have to exist a biholomorphic map

$$\mathbb{E} \setminus \{0\} \xrightarrow{\phi} A_{1/2,1}$$

which is impossible because of the nature of its singularity at 0.

19. Find a biholomorphic map  $\mathbb{E} \rightarrow \mathbb{E} \setminus \mathbb{R}_0^+$ .

## 2 The Residue Theorem

### 2.1 The Complex Line Integral over Continuous Curves

The complex line integral of a holomorphic function extends to continuous curves. Thus let  $G \subset \mathbb{C}$  be a domain and  $f \in \mathcal{O}(G)$ . For every  $z \in G$  let  $r(z) > 0$  be so that  $B_{r(z)}(z) \subset G$ . The Taylor series of  $f$  at  $z$  converges on  $B_{r(z)}(z)$ . For each  $z$  we choose an antiderivative  $F_z$  of  $f|_{B_{r(z)}(z)}$ .

If  $\gamma: [a, b] \rightarrow B_{r(z)}(z)$  is piecewise  $C^1$  then

$$\int_{\gamma} f(z) dz = F_z(\gamma(b)) - F_z(\gamma(a)) . \quad (2.1)$$

This gives the same value for all antiderivatives  $F_z$  of  $f|_{B_{r(z)}(z)}$ . The complex line integral depends continuously on  $\gamma$ , and is additive under concatenation.

For continuous, not necessarily piecewise  $C^1$ -curves we use (??) as definition. This can also be extended to continuous curves in  $G$  not contained in a disk  $B_{\epsilon}(z) \subset G$ . To this end, let  $\gamma: [0, 1] \rightarrow G$  be continuous. Let  $\delta$  be a **Lebesgue number** of the open covering

$$\{\gamma^{-1}(B_{r(z)}(z)) \mid z \in G\}$$

of  $[0, 1]$ . Thus if  $J \subset [0, 1]$  is a subinterval with  $|J| < \delta$ , then there is  $z(J) \in G$  so that  $\gamma(J) \subset B_{r(z(J))}(z(J))$ . We define the complex line integral of  $f \in \mathcal{O}(G)$  over  $\gamma$  by summing up the differences (??) of a subdivision of the interval  $[0, 1]$ ,

$$0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_{n-1} < \tau_n = 1 \quad , \quad \forall i : \tau_i - \tau_{i-1} < \delta ,$$

$$\int_{\gamma} f(z) dz := \sum_{i=1}^n \int_{\gamma|_{[\tau_{i-1}, \tau_i]}} f(z) dz = \sum_{i=1}^n F_{z([\tau_{i-1}, \tau_i])}(\gamma(\tau_i)) - F_{z([\tau_{i-1}, \tau_i])}(\gamma(\tau_{i-1})) \quad (2.2)$$

**Problem 2.3** Check that this does not depend on the choices involved (list these first).

**Theorem 2.4 (Homotopy invariance of the integral)** Let  $G \subset \mathbb{C}$  be a domain,  $f \in \mathcal{O}(G)$  and  $\gamma_0, \gamma_1: [0, 1] \rightarrow G$  be continuous and homotopic relative end point. Then

$$\int_{\gamma_0} f(z) dz = \int_{\gamma_1} f(z) dz .$$

**Proof:** Let  $\gamma_0 \simeq_H \gamma_1 \text{ rel } \partial$ , i.e.  $H: [0, 1] \times [0, 1] \rightarrow G$  is continuous,  $H(s, t) = \gamma_s(t)$  for all  $s \in \{0, 1\}$ ,  $t \in [0, 1]$ , and  $H(s, t) = \gamma_0(t) = \gamma_1(t)$  for all  $s \in [0, 1]$ ,  $t \in \{0, 1\}$ .

For  $s \in [0, 1]$  let  $\gamma_s: [0, 1] \rightarrow G$  be the curve with  $\gamma_s(t) = H(s, t)$ . The open covering

$$\{H^{-1}(B_{r(z)}(z)) \mid z \in G\}$$

of the compact set  $[0, 1] \times [0, 1]$  has a Lebesgue number and therefore we can choose subdivisions

$$0 = \sigma_0 < \sigma_1 < \sigma_2 < \cdots < \sigma_{m-1} < \sigma_m = 1 \quad , \quad 0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_{n-1} < \tau_n = 1 \quad ,$$

and  $z(i, j) \in G$ ,  $i = 1 \dots m$ ,  $j = 1 \dots n$  so that

$$H([\sigma_{i-1}, \sigma_i] \times [\tau_{j-1}, \tau_j]) \subset B_{r(z(i,j))}(z(i, j)) \quad .$$

We may assume that  $m = 1 = \sigma_1$ . Then

$$\begin{aligned} \int_{\gamma_0} f(z) dz &= \sum_{j=1}^n F_{z(0,j)}(\gamma_0(\tau_j)) - F_{z(0,j)}(\gamma_0(\tau_{j-1})) \\ &= \sum_{j=1}^n F_{z(1,j)}(\gamma_0(\tau_j)) - F_{z(1,j)}(\gamma_0(\tau_{j-1})) \\ &= \sum_{j=1}^n F_{z(1,j)}(\gamma_0(\tau_j)) - F_{z(1,j)}(\gamma_1(\tau_j)) + F_{z(1,j)}(\gamma_1(\tau_j)) - F_{z(1,j)}(\gamma_1(\tau_{j-1})) + F_{z(1,j)}(\gamma_1(\tau_{j-1})) - F_{z(1,j)}(\gamma_0(\tau_{j-1})) \\ &= \sum_{j=1}^n \underbrace{F_{z(1,j)}(\gamma_0(\tau_j)) - F_{z(1,j)}(\gamma_1(\tau_j))}_{\text{cancel}} + \underbrace{F_{z(1,j)}(\gamma_1(\tau_j)) - F_{z(1,j)}(\gamma_1(\tau_{j-1}))}_{\text{cancel}} + \underbrace{F_{z(1,j-1)}(\gamma_1(\tau_{j-1})) - F_{z(1,j-1)}(\gamma_0(\tau_{j-1}))}_{\text{cancel}} \\ &= \sum_{j=1}^n F_{z(1,j)}(\gamma_1(\tau_j)) - F_{z(1,j)}(\gamma_1(\tau_{j-1})) \\ &= \int_{\gamma_1} f(z) dz \quad . \end{aligned}$$

The underlined terms cancel in the sum because they vanish at the ends. •

## 2.2 The Winding Number

Crucial for this definition is the following:

**Theorem 2.5** *For every continuous curve  $\gamma: [0, 1] \rightarrow S^1$  and  $\vartheta_0 \in \mathbb{R}$  so that  $\gamma(0) = e^{i\vartheta_0}$ , there is a unique continuous function*

$$\vartheta: [0, 1] \rightarrow \mathbb{R} \quad \text{so that} \quad \forall t \in [0, 1] : \gamma(t) = e^{i\vartheta(t)} \quad \text{and} \quad \vartheta(0) = \vartheta_0 \quad .$$

**Corollary 2.6** *Let  $a \in \mathbb{C}$  and  $\gamma: [0, 1] \rightarrow \mathbb{C} \setminus \{a\}$  be continuous. Let  $\vartheta_0 \in \mathbb{R}$  be so that*

$$\gamma(0) = a + |\gamma(0) - a| e^{i\vartheta_0} \quad .$$

*Then there is a unique continuous function  $\vartheta_{\gamma, \vartheta_0}: [0, 1] \rightarrow \mathbb{R}$  so that*

$$\vartheta_{\gamma, \vartheta_0}(0) = \vartheta_0 \quad \text{and} \quad \gamma(t) = a + |\gamma(t) - a| e^{i\vartheta_{\gamma, \vartheta_0}(t)} \quad .$$

*Angle functions of the same curve differ by a constant,*

$$\vartheta_{\nu, \gamma}(t) - \vartheta_{\mu, \gamma}(t) = \nu - \mu \in 2\pi\mathbb{Z} \quad \text{for all} \quad t \in \mathbb{R} \quad .$$

**Definition 2.7** A curve  $\gamma: [a, b] \rightarrow X$  is **closed** if  $\gamma(a) = \gamma(b)$ .

The **winding number** of a closed continuous curve  $\gamma: [0, 1] \rightarrow \mathbb{C} \setminus \{a\}$  is

$$w(\gamma, a) = \frac{\vartheta(1) - \vartheta(0)}{2\pi} \in \mathbb{Z}$$

where  $\vartheta: [0, 1] \rightarrow \mathbb{R}$  is any function so that  $\gamma(t) = a + |\gamma(t) - a| e^{i\vartheta(t)}$  for all  $t \in [0, 1]$ .

The closed curve  $\gamma$  **surrounds**  $a \in \mathbb{C} \setminus \gamma([0, 1])$  if  $w(\gamma, a) \neq 0$ .

**Theorem 2.8**

$$w(\gamma, a) = \frac{1}{2\pi i} \oint_{\gamma} \frac{1}{z - a} dz$$

for every continuous closed curve  $\gamma: [0, 1] \rightarrow \mathbb{C} \setminus \{a\}$ .

**Proof:** We may assume  $a = 0$  and  $\gamma(0) = \gamma(1) = 1$ ,

$$\gamma(t) = |\gamma(t)| e^{i\vartheta(t)}, \vartheta(0) = 0.$$

This curve is homotopic relative end points to one with constant modulus, via the homotopy  $H: [0, 1] \times [0, 1] \rightarrow \mathbb{C} \setminus \{0\}$  with

$$H(s, t) = (1 - s + s |\gamma(t)|) e^{i\vartheta(t)}.$$

We may therefore also assume that

$$\gamma(t) = e^{i\vartheta(t)}.$$

Let

$$0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_{n-1} < \tau_n = 1$$

be a subdivision of  $[0, 1]$  as in the definition of the complex line integral, so that we have a common antiderivative  $F_j(z)$  of  $\frac{1}{z}$  on  $\gamma([\tau_{j-1}, \tau_j])$ , hence

$$F_j(\gamma(\tau_j)) - F_j(\gamma(\tau_{j-1})) = F_j(e^{i\vartheta(\tau_j)}) - F_j(e^{i\vartheta(\tau_{j-1})}) = i\vartheta(\tau_j) - i\vartheta(\tau_{j-1})$$

because

$$\left. \frac{d}{dt} \right|_{t=u} F_j(e^{it}) = \frac{1}{e^{iu}} i e^{iu} = i.$$

Hence from the definition (??)

$$\oint_0^1 \frac{1}{z} dz = \sum_{i=1}^n i\vartheta(\tau_i) - i\vartheta(\tau_{i-1}) = i\vartheta(1) - i\vartheta(0) = 2\pi i w(\gamma, 0)$$

•

**Theorem 2.9 (Homotopy Invariance of the Winding Number)** Let  $H: [0, 1] \times [0, 1] \rightarrow \mathbb{C} \setminus \{p\}$  be continuous and so that  $H(s, 0) = H(s, 1)$  for all  $s \in [0, 1]$ . Let  $\gamma_s$ ,  $s \in [0, 1]$ , be the curves give by  $\gamma_s(t) = H(s, t)$ . Then  $w(\gamma_0, p) = w(\gamma_1, p)$ .

Note that the curves  $\gamma_s$  need not have the same endpoints for different  $s$ ,  $\gamma_0$  is not necessarily homotopic to  $\gamma_1$  relative endpoints.  $H$  is a homotopy of closed curves only.

**Proof:** Check that function  $w: [0, 1] \rightarrow \mathbb{Z}$ ,  $s \mapsto w(\gamma_s, p)$  is continuous, for instance by Theorem ???. But continuous functions from connected to discrete topological spaces must be constant. •

## 2.3 The Cauchy Integral Theorem

**Theorem 2.10** *Let  $G \subset \mathbb{C}$  be open and  $\gamma: [0, 1] \rightarrow G$  be a continuous closed curve. Then  $\gamma$  surrounds a point outside of  $G$  if and only if there is  $f \in \mathcal{O}(G)$  with  $\oint_{\gamma} f(z) dz \neq 0$ .*

**Proof:** If  $\gamma$  surrounds  $a \in \mathbb{C} \setminus G$  then

$$\oint_{\gamma} \frac{1}{z-a} dz = 2\pi i w(\gamma, a) \neq 0$$

and  $\frac{1}{z-a}$  is holomorphic on  $G$ .

Assume now that  $\gamma$  does not surround any point outside  $G$ , and let  $f \in \mathcal{O}(G)$ . We need to show that  $\oint_{\gamma} f(z) dz = 0$ . We will use some notation for segments. For  $a, b \in \mathbb{C}$  we will denote (the parametrization of) the line segment from  $a$  to  $b$  by  $[a, b]$  respectively  $\ell_{a,b}: [0, 1] \rightarrow [a, b] \subset \mathbb{C}$ ,

$$[a, b] = \{a + t(b-a) \mid t \in [0, 1]\} \quad , \quad \ell_{a,b}(t) = a + t(b-a) = (1-t)a + tb .$$

1. We may assume that  $\gamma$  is an edge path of a mesh: Let  $r: G \rightarrow \mathbb{R}^+$  be so that for every  $z \in G$  we have  $B_{r(z)}(z) \subset G$ . As in the definition of the integral, using a Lebesgue number for the covering

$$\{\gamma^{-1}(B_{r(z)}(z)) \mid z \in G\}$$

of  $[0, 1]$ , we can find  $z_1, \dots, z_n \in G$  and  $0 = t_{-1} = t_0 < t_1 < \dots < t_n = 1$  so that

$$\gamma([t_{i-1}, t_i]) \subset B_i = B_{r(z_i)}(z_i) \quad , \quad \gamma(t_i) \in B_i \cap B_{i+1} \quad \text{for } i = 0 \dots n \quad , \quad B_0 := B_1 \quad , \quad B_{n+1} := B_n .$$

Now we choose  $\delta > 0$  but so small that for each  $i$ ,  $i = 0 \dots n$ , we have

$$B_i \cap B_{i+1} \cap \delta(\mathbb{Z} + i\mathbb{Z}) \neq \emptyset .$$

2. Wlog  $\gamma$  does not surround any point not on the mesh:
3. Wlog  $\gamma$  runs through each edge an equal number of times in both directions:

•

## 2.4 The Residue Theorem

Let  $G \subset \mathbb{C}$  be a domain and  $S \subset G$  be discrete, i.e. for every  $s \in S$  there is  $r_s > 0$  so that  $B_{r(s)}(s) \cap G = \{s\}$ . Let  $f \in \mathcal{O}(G \setminus S)$  and let  $\gamma: [0, 1] \rightarrow G \setminus S$  be a closed continuous curve. Let

$$S_0 = \{s_1, s_2, \dots, s_k\} = \{s \in S \mid w(\gamma, s) \neq 0\} .$$

This set is finite.

For each  $s \in S$  choose a curve  $\beta_s: [0, 1] \rightarrow G \setminus S$  so that  $\beta(0) = \gamma(0) = \gamma(1)$ ,  $\beta(1) = s + r(s)/2$  and let  $\alpha_s(t) = s + r(s)/2e^{2\pi it}$ . Let  $\gamma_s$  be the concatenation

$$\gamma_s = \beta_s * \underbrace{\alpha_s * \dots * \alpha_s}_{w(\gamma, s) \text{ times}} * \beta_s^{-1} \quad \text{hence} \quad w(\gamma_s, s) = w(\gamma, s) . \quad (2.11)$$

The curve  $\gamma_s$  does not surround any point outside  $G \setminus S$  except  $s$ . By additivity, the concatenation

$$\gamma * \gamma_{s_1}^{-1} * \gamma_{s_2}^{-1} * \dots * \gamma_{s_k}^{-1}$$

does not surround any point outside  $G \setminus S$ . By the Cauchy Integral Theorem (and additivity of the integral)

$$\oint_{\gamma} f(z) dz = \sum_{i=1}^k \oint_{\gamma_{s_i}} f(z) dz \quad (2.12)$$

The curves  $\beta_s$  do not contribute to this integral because they are involved twice in reverse direction in the concatenation (??). Thus the integral becomes (??)

$$= \sum_{i=1}^k w(\gamma, s_i) \oint_{\alpha_{s_i}} f(z) dz = \sum_{i=1}^k w(\gamma, s_i) \underbrace{\lim_{r \rightarrow 0} \oint_{|z-s_i|=r} f(z) dz}_{=:\text{Res}(f; s_i)} .$$

We therefore define

**Definition 2.13** Let  $U \subset \mathbb{C}$ ,  $p \in \overset{\circ}{U}$  and  $f \in \mathcal{O}(U \setminus \{p\})$ , “ $p$  is an isolated singularity of  $f$ ”. The **residue of  $f$  at  $p$**  is

$$\text{Res}(f; p) = \text{Res}(f(z); z = p) = \frac{1}{2\pi i} \lim_{r \rightarrow 0} \oint_{|z-p|=r} f(z) dz = \frac{1}{2\pi i} \oint_{|z-p|=R} f(z) dz$$

where  $R$  is so that  $B_R(p) \subset U$ .

In the notation of this definition, if the Laurent series of  $f$  at  $p$  is

$$f(z) = \sum_{n \in \mathbb{Z}} c_n (z - p)^n$$

then the residue is

$$\text{Res}(f; p) = \frac{1}{2\pi i} \oint_{|z-p|=R} \sum_{n \in \mathbb{Z}} c_n (z - p)^n dz = \frac{1}{2\pi i} \sum_{n \in \mathbb{Z}} \oint_{|z-p|=R} c_n (z - p)^n dz = c_{-1}$$

since the Laurent series converges uniformly on compact sets, and

$$\oint_{|z-p|=R} (z-p)^n dz = \oint_{|z|=1} z^n dz = \begin{cases} 1 & \text{if } n = -1 \\ 0 & \text{if } n \neq -1 \end{cases} .$$

**Example 2.14** *Residues at poles of higher order can be computed by differentiation. As an example, we compute*

$$\text{Res} \left( \frac{e^z}{\sin(z)^2}; z = 0 \right) .$$

Since  $\frac{e^z}{\sin(z)^2}$  has a pole of order 2 at  $z = 0$ , we have

$$\frac{e^z}{\sin(z)^2} = \frac{c_{-2}}{z^2} + \frac{c_{-1}}{z} + c_0 + \dots ,$$

hence

$$z^2 \frac{e^z}{\sin(z)^2} = c_{-2} + c_{-1}z + c_0z^2 + \dots ,$$

$$\begin{aligned} \text{Res} \left( \frac{e^z}{\sin(z)^2}; z = 0 \right) &= c_{-1} = \left. \frac{d}{dz} \frac{z^2 e^z}{\sin(z)^2} \right|_{z=0} = \left. \frac{2ze^z + z^2 e^z}{\sin(z)^2} - \frac{2z^2 e^z \cos(z)}{\sin(z)^3} \right|_{z=0} \\ &= \left. \frac{2e^z}{\sin(z)} + e^z - \frac{2e^z \cos(z)}{\sin(z)} \right|_{z=0} \\ &= 1 + e^z \left. \frac{2 - 2 \cos(z)}{\sin(z)} \right|_{z=0} = 1 \end{aligned}$$

We compute the same residue by manipulating power series. Recall that

$$\frac{1}{1-q} = \sum_{n=0}^{\infty} q^n \quad \text{for } |q| < 1$$

and denote by  $h_i$  any holomorphic functions.

$$\begin{aligned} \frac{e^z}{\sin(z)^2} &= \frac{e^z}{\left(z - \frac{z^3}{6} + z^5 h_1(z)\right)^2} = \frac{e^z}{z^2 - \frac{z^4}{3} + z^6 h_2(z)} \\ &= \frac{e^z}{z^2} \frac{1}{1 - \frac{z^2}{3} + z^4 h_2(z)} = \frac{e^z}{z^2} \left( 1 + \frac{z^2}{3} - z^4 h_3(z) \right) \\ &= \frac{1}{z^2} \left( 1 + z + \frac{z^2}{2} + z^3 h_4(z) \right) \left( 1 + \frac{z^2}{3} - z^4 h_3(z) \right) \\ &= \frac{1}{z^2} \left( 1 + z + \frac{5z^2}{6} + z^3 h_5(z) \right) = \frac{1}{z^2} + \frac{1}{z} + \frac{5}{6} + zh_5(z) . \end{aligned}$$

**Theorem 2.15 (Residue Theorem)** *Let  $G \subset \mathbb{C}$  be a domain and  $S \subset G$  a discrete subset. Let  $\gamma: [0, 1] \rightarrow G \setminus S$  be closed, continuous and not surround any point outside  $G$ . Let  $f \in \mathcal{O}(G \setminus S)$ . Then*

$$\oint_{\gamma} f(z) dz = 2\pi i \sum_{s \in S} w(\gamma, s) \text{Res}(f; s) .$$

### 2.4.1 Improper Integrals

A typical case is  $\int_{-\infty}^{\infty} R(x) dx$  where  $R \in \mathcal{O}(\overline{\mathcal{H}} \setminus S)$ ,  $S \subset \mathcal{H}$  finite, such that

$$\lim_{z \rightarrow \infty} |z| R(z) = 0 .$$

Then

$$\left| \int_{|z|=T, \Im(z) \geq 0} R(z) dz \right| = \left| \int_0^\pi R(Te^{it}) iT e^{it} dt \right| \leq \max \{ |R(z)| \mid |z| = T, \Im(z) \geq 0 \} \pi T \xrightarrow{T \rightarrow \infty} 0 .$$

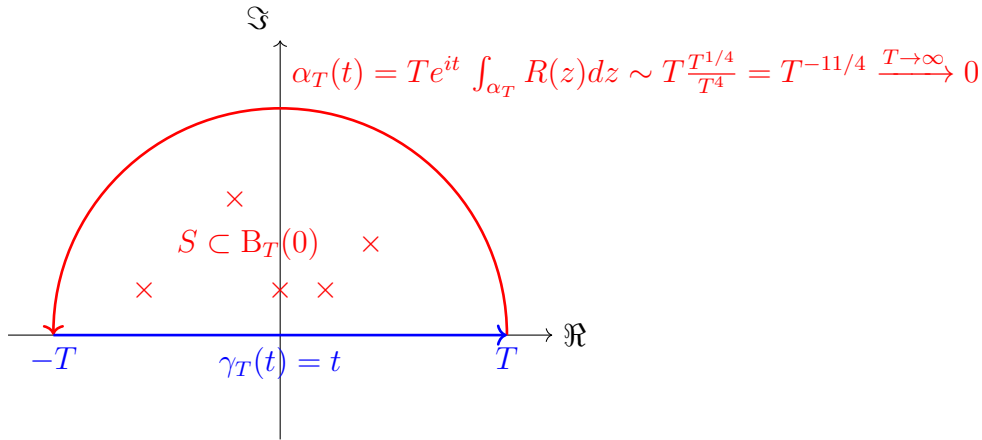
For  $T \in \mathbb{R}^+$ , let  $\gamma_T$  and  $\alpha_T$  be the paths

$$\gamma_T(t) = t, t \in [-T, T] \quad \text{and} \quad \alpha_T(t) = T e^{it}, t \in [0, \pi] .$$

We can now compute the improper integral with the residue theorem,

$$\begin{aligned} \int_{-\infty}^{+\infty} R(x) dx &= \lim_{T \rightarrow \infty} \int_{-T}^T R(x) dx \\ &= \lim_{T \rightarrow \infty} \left( \int_{-T}^T R(x) dx + \int_{|z|=T, \Im(z) \geq 0} R(z) dz \right) \\ &= \oint_{\gamma_T * \alpha_T} R(z) dz \quad \text{if } T \text{ is so large that } S \subset B_T(0) \\ &= 2\pi i \sum_{s \in \mathcal{H}} \text{Res}(R, s) \end{aligned}$$

because  $w(\gamma_T * \alpha_T, s) = 1$  for all  $s \in S$  and sufficiently large  $T$ .



**Example 2.16** The Fourier transform of a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  with  $\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$  is the function  $\hat{f}: \mathbb{R} \rightarrow \mathbb{R}$  with

$$\hat{f}(s) = \int_{-\infty}^{\infty} e^{-ist} f(t) dt .$$



We compute the Fourier transform of  $a(t) = \frac{1}{1+t^2}$ . Since  $|e^{-iz}| = e^{\Im(z)}$ , we can use the auxilliary path  $\alpha_T$  as above in the upper half plane if  $s \leq 0$ . If  $s \geq 0$  we need to close the path of integration in the lower half plane.

$$\hat{a}(s) = \int_{-\infty}^{\infty} \frac{e^{-ist}}{1+t^2} dt = 2\pi i \begin{cases} \operatorname{Res}\left(\frac{e^{-isz}}{1+z^2}; z=i\right) & \text{if } s \leq 0 \\ -\operatorname{Res}\left(\frac{e^{-isz}}{1+z^2}; z=-i\right) & \text{if } s \geq 0 \end{cases}.$$

The winding numbers are  $+1$  in the upper half plane and  $-1$  in the lower half plane. The residues are

$$\begin{aligned} \operatorname{Res}\left(\frac{e^{-isz}}{1+z^2}; z=\pm i\right) &= e^{\pm is} \operatorname{Res}\left(\frac{1}{1+z^2}; z=\pm i\right) = e^{\pm is} \operatorname{Res}\left(\frac{-i/2}{z-i} + \frac{i/2}{z+i}; z=\pm i\right) \\ &= \frac{\mp i(\pm 1)e^{\pm is}}{2} \end{aligned}$$

Thus

$$\hat{a}(s) = 2\pi i \frac{(\pm 1) \mp i e^{\pm is}}{2} = \pi e^{-|s|}.$$

**Example 2.17** In order to compute

$$\int_0^{\infty} \frac{\sqrt[4]{x}}{1+x^4} dx$$

we exploit the symmetry of the integrand  $f(z) = \frac{\sqrt[4]{z}}{1+z^4}$ ,

$$f(iz) = i^{1/4} f(z) \tag{2.18}$$

because  $i^4 = 1$ . Of course there is a choice in the 4th root of  $i$  here. We will extend  $\sqrt[4]{\cdot}$  from  $\mathbb{R}^+$  holomorphically so that

$$\sqrt[4]{e^{it}} = e^{it/4} \quad \text{for all } t \in [-\pi/4, 3\pi/4].$$

Thus  $\sqrt[4]{i} = \sqrt[4]{e^{i\pi/2}} = e^{i\pi/8}$ , for instance. We now consider the paths

$$\begin{aligned} \gamma_T(t) &= t, \quad t \in [0, T] \\ \omega_T(t) &= it, \quad t \in [0, T] \\ \alpha_T(t) &= Te^{it}, \quad t \in [0, \pi/2] \end{aligned}$$

Because of the symmetry (??),

$$\int_{\omega_T} f(z) dz = \int_0^T f(it)i dt = i^{5/4} \int_0^T f(t) dt = i^{5/4} \int_{\gamma_T} f(z) dz.$$

From the Residue Theorem, for  $T > 1$ ,

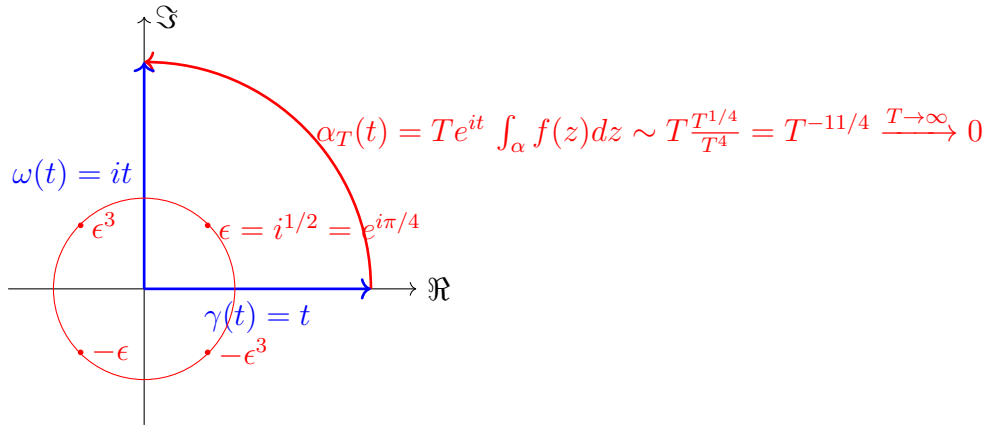
$$\begin{aligned} 2\pi i \operatorname{Res}(f, \epsilon) &= \oint_{\gamma_T * \alpha_T * \omega_T^{-1}} f(z) dz = \lim_{T \rightarrow \infty} \oint_{\gamma_T * \alpha_T * \omega_T^{-1}} f(z) dz \\ &= \lim_{T \rightarrow \infty} \int_{\gamma_T} f(z) dz + \underbrace{\lim_{T \rightarrow \infty} \int_{\alpha_T} f(z) dz}_{=0} - \underbrace{\lim_{T \rightarrow \infty} \int_{\omega_T} f(z) dz}_{=i^{5/4} \lim_{T \rightarrow \infty} \int_{\gamma_T} f(z) dz} \\ &= \int_0^{\infty} \frac{\sqrt[4]{x}}{1+x^4} dx \times (1 - i^{5/4}) \end{aligned}$$

hence

$$\int_0^\infty \frac{\sqrt[4]{x}}{1+x^4} dx = \frac{2\pi i \operatorname{Res}(f, \epsilon)}{1 - i^{5/4}}.$$

For the residue we get

$$\begin{aligned} \frac{1}{1+z^4} &= \frac{1}{(z-\epsilon)(z-\epsilon^3)(z+\epsilon)(z+\epsilon^3)}, \\ \operatorname{Res}\left(\frac{\sqrt[4]{z}}{1+z^4}, \epsilon\right) &= \frac{\epsilon^{1/4}}{(\epsilon-\epsilon^3)(\epsilon+\epsilon)(\epsilon+\epsilon^3)} = \frac{i^{1/8}}{\sqrt{2}2\epsilon i\sqrt{2}} = \frac{i^{-11/8}}{4}. \\ \int_0^\infty \frac{\sqrt[4]{x}}{1+x^4} dx &= \frac{2\pi i i^{-11/8}}{4(1-i^{5/4})} = \frac{\pi i^{-3/8}}{2(1-i^{5/4})} = \frac{\pi}{2} \frac{1}{i^{3/8} + i^{-3/8}} = \frac{\pi}{4 \cos(3\pi/16)}. \end{aligned}$$



## 2.4.2 Period Integrals of Trigonometric Rational Functions

In order to compute

$$\int_0^{2\pi} \frac{1 + \cos(t)}{2 + \sin(t)} dt$$

we first rewrite the integrand

$$\frac{1 + \cos(t)}{2 + \sin(t)} = \frac{1 + \frac{e^{it} + e^{-it}}{2}}{2 + \frac{e^{it} - e^{-it}}{2i}} = \frac{1 + \frac{z + \frac{1}{z}}{2}}{2 + \frac{z - \frac{1}{z}}{2i}} = i \frac{z^2 + 2z + 1}{z^2 + 4iz - 1} =: f(z), \quad z = e^{it}.$$

The integral then is the same as a complex line integral over the curve  $\gamma: [0, 2\pi] \rightarrow \mathbb{C}$ ,  $\gamma(t) = e^{it}$ ,

$$\int_0^{2\pi} f(e^{it}) dt = \int_0^{2\pi} f(e^{it}) \frac{1}{ie^{it}} \underbrace{ie^{it}}_{\gamma'(t)} dt = \oint_\gamma \frac{f(z)}{iz} dz = \oint_{|z|=1} \frac{f(z)}{iz} dz.$$

In the case at hand,

$$\int_0^{2\pi} \frac{1 + \cos(t)}{2 + \sin(t)} dt = \oint_{|z|=1} \frac{z^2 + 2z + 1}{z^2 + 4iz - 1} \frac{1}{z} dz = \oint_{|z|=1} \underbrace{\frac{(z+1)^2}{(z-i(-2-\sqrt{3}))(z-i(-2+\sqrt{3}))z}}_{J(z)} dz.$$

The integrand  $J(z)$  has no singularities  $z$  with  $|z| = 1$ . The winding number of  $\gamma$  is 1 around points in  $B_1(0)$  and 0 else. By the residue theorem, the integral is therefore equal to

$$2\pi i \sum_{|q| < 1} \text{Res}(J(z); z = p) .$$

There are two singularities of  $J(z)$  in  $B_1(0)$ , with residues

$$\text{Res}(J(z); z = 0) = -1$$

$$\begin{aligned} \text{Res}\left(J(z); z = i(-2 + \sqrt{3})\right) &= \frac{(z+1)^2}{(z - i(-2 - \sqrt{3}))z} \Big|_{z=i(-2+\sqrt{3})} = \frac{(1 - 2i + \sqrt{3}i)^2}{(i(-2 + \sqrt{3}) - i(-2 - \sqrt{3}))i(-2 + \sqrt{3})} \\ &= -\frac{-6 + 4\sqrt{3} - 4i + 2\sqrt{3}i}{2\sqrt{3}(-2 + \sqrt{3})} = \frac{6 - 4\sqrt{3} + 4i - 2\sqrt{3}i}{6 - 4\sqrt{3}} . \end{aligned}$$

The integral is

$$2\pi i \left( -1 + \frac{6 - 4\sqrt{3} + 4i - 2\sqrt{3}i}{6 - 4\sqrt{3}} \right) = 2\pi \frac{4 - 2\sqrt{3}}{4\sqrt{3} - 6} = \frac{8\sqrt{3}}{12} \pi = \frac{2\pi}{\sqrt{3}} .$$

## 2.5 Logarithmic Derivative

Let  $U \subset \mathbb{C}$  be open,  $p \in U$  and  $f: U \rightarrow S$  be holomorphic. If  $f$  has a pole of order  $-k$  or a zero of order  $k$  at  $p$ , then

$$f(z) = (z - p)^k g(z)$$

where  $g(p) \neq 0, \infty$ . Hence

$$\begin{aligned} f'(z) &= k(z - p)^{k-1} g(z) + (z - p)^k g'(z) , \\ \frac{f'}{f}(z) &= \frac{k}{z - p} + \frac{g'}{g} . \end{aligned}$$

Since  $g(p) \neq 0, \infty$ ,

$$\text{ord}(f; p) = k = \oint_{|z-p|=r} \frac{f'(z)}{f(z)} dz = \oint_{\gamma} \frac{f'(z)}{f(z)} dz$$

provided that  $r$  is so small that there are no other poles/zeros of  $f$  in  $B_r(p)$ , respectively that  $w(\gamma, p) = 1$  and  $\gamma$  does not surround any points outside  $U$  nor any other poles or zeros of  $f$ . We can further rewrite the integral

$$\oint_{\gamma} \frac{f'(z)}{f(z)} dz = \int \frac{1}{f(\gamma(t))} f'(\gamma(t)) \gamma'(t) dt = \oint_{f \circ \gamma} \frac{1}{z} dz = w(f \circ \gamma, 0) .$$

**Theorem 2.19 (Argument Principle, Pole-Zero-Counting integral, winding number)** *Let  $G \subset \mathbb{C}$  be a domain and  $f$  be meromorphic on  $G$ , i.e.  $f: G \rightarrow S$  is holomorphic as defined in . Let  $A \subset G$  be open and let  $\gamma$  be a boundary curve for  $A$ , i.e.*

$$\forall a \in \text{interior } A : w(\gamma, a) = 1 \quad , \quad \forall a \in \mathbb{C} \setminus \bar{A} : w(\gamma, a) = 0 .$$

*Then the number of zeros minus the number of poles of  $f$  in  $A$ , both counted with multiplicity, is*

$$\sum_{p \in A} \text{ord}(f; p) = \oint_{\gamma} \frac{f'(z)}{f(z)} dz = w(f \circ \gamma, 0) .$$

From the homotopy invariance of the winding number we immediately have

**Corollary 2.20 (Rouché's Theorem)** *Let  $G \subset \mathbb{C}$  be a domain and  $f, g \in \mathcal{O}(G)$ . Let  $A \subset G$  and  $\gamma = \partial A$  a boundary curve of  $A$ . If*

$$\forall z \in \partial A : |g(z)| < |f(z)| ,$$

*then  $f$  and  $f + g$  have the same number of zeros, counted with multiplicity, in  $A$ , i.e.*

$$\sum_{a \in A} \text{ord}(f, a) = \sum_{a \in A} \text{ord}(f + g, a) .$$

**Proof:** By the Pole-Zero counting winding number theorem, this follows once we have that

$$w(f \circ \gamma, 0) = w((f + g) \circ \gamma, 0) .$$

But this is a consequence of the homotopy invariance of the winding number. The curves

$$f \circ \gamma, (f + g) \circ \gamma : [0, 1] \rightarrow \mathbb{C} \setminus \{0\}$$

are homotopic. A homotopy is

$$H : [0, 1] \times [0, 1] \rightarrow \mathbb{C} \setminus \{0\} \quad , \quad H(s, t) = f(\gamma(t)) + sg(\gamma(t)) .$$

This is well-defined, because for  $s, t \in [0, 1]$ , we have

$$\begin{aligned} |H(s, t)| &= |f(\gamma(t)) + sg(\gamma(t))| \geq |f(\gamma(t))| - |sg(\gamma(t))| = |f(\gamma(t))| - s|g(\gamma(t))| \\ &\geq |f(\gamma(t))| - |g(\gamma(t))| > 0 . \end{aligned}$$

•

**Definition 2.21 (Multiplicity)** *Let  $G \subset S = \mathbb{C} \cup \{\infty\}$  be a domain and  $f \in \mathcal{O}(G, S)$ . For  $a \in S$  we define*

$$f_a(z) = \begin{cases} f(z) - a & \text{if } a \in \mathbb{C} \\ \frac{1}{f(z)} & \text{if } a = \infty \end{cases}$$

*The multiplicity of  $a$  as a value of  $f$  is*

$$m(f, a) := \sum_{u \in G} \text{ord}(f_a, u) .$$

If  $p(z) = \sum_{k=0}^n a_k z^k \in \mathbb{C}[z]$  has degree  $n = \deg p$ ,  $a_n \neq 0$ , then the only zero of  $\frac{1}{p(z)}$  is at  $\infty$ , and

$$\text{ord}\left(\frac{1}{p(z)}, z = \infty\right) = \text{ord}\left(\frac{1}{p(1/z)}, z = 0\right) = n$$

because

$$\frac{1}{p(1/z)} z^{-j} = \frac{1}{z^j \sum_{k=0}^n a_k z^{-k}} = \frac{1}{\sum_{k=0}^n a_k z^{j-k}} = \frac{1}{\sum_{k=0}^n a_k z^{j-k}} = \frac{1}{a_n z^{j-n} + a_{n-1} z^{j-n+1} + \dots + a_0 z^j}$$

is holomorphic at 0 exactly for  $j \leq n$ . Thus  $m(p, \infty) = \deg p$ . If  $a \in \mathbb{C}$  then

$$p(z) - a = c \prod_{i=1}^r (z - \lambda_i)^{m_i} \quad , \quad \sum_{i=1}^r m_i = \deg p$$

and

$$m(p, a) = \sum_{z \in S} \text{ord}(p(z) - a, z) = \sum_{i=1}^r \text{ord}(p(z) - a, z = \lambda_i) = \sum_{i=1}^r m_i = \deg p .$$

Thus a polynomial assumes all values with the same multiplicity equal to its degree.

A rational function is a quotient  $R(z) = \frac{p(z)}{q(z)}$  of two coprime polynomials (i.e. having no common zero).

**Theorem 2.22** *A rational function assumes all values with equal multiplicity.*

**Proof:** If  $\phi, \psi: S \rightarrow S$  are biholomorphic, then for all  $f \in \mathcal{O}(S, S)$ ,  $a \in S$  we have

$$m(f, a) = m(\phi \circ f \circ \psi, \phi(a)) .$$

Let  $R$  be a rational function and  $a \in S$  finite i.e.  $a \in \mathbb{C}$ . We will show that  $m(R, a) = m(R, \infty)$ . If  $R - a$  has a zero or a pole at  $\infty$  we replace  $R(z)$  with  $R(b + 1/z)$  where  $b \in \mathbb{C}$  and  $R(b) \neq a, \infty$ , without changing the multiplicity:

$$m(R, a) = m(R - a, 0) = m(R(b + 1/z) - a, 0) .$$

Since  $\tilde{R}(z) = R(b + 1/z) - a$  has only finitely many zeros and poles and none of them at  $\infty$ , there is  $T$  so that  $\tilde{R}^{-1}\{0, \infty\} \subset B_T(0)$ . Let  $\alpha_T$  be a boundary curve of  $B_T(0)$ , e.g.  $\alpha_T(t) = Te^{it}$  for  $t \in [0, 2\pi]$ . By the theorem on the counting integral,

$$m(R, a) - m(R, \infty) = m(\tilde{R}, 0) - m(\tilde{R}, \infty) = \sum_{z \in R^{-1}(\{0, \infty\})} \text{ord}(\tilde{R}, z) = \frac{1}{2\pi i} w(\tilde{R} \circ \alpha_T, 0) \xrightarrow{T \rightarrow \infty} 0$$

because  $\tilde{R}(\alpha_T(t)) \xrightarrow{T \rightarrow \infty} \tilde{R}(\infty) \in \mathbb{C}$ , i.e.  $\tilde{R} \circ \alpha_T$  converges to a constant curve, and constant curves have zero winding numbers. •

## 3 Compact Convergence

### 3.1 The basic complete functions spaces and the Arzela Ascoli Theorem

If  $E$  is a set and  $(M, d)$  is a metric space, then on  $M^E = \{f: E \rightarrow M \mid f \text{ a map}\}$  we consider the almost distance

$$d_\infty: M^E \times M^E \rightarrow \mathbb{R}_0^+ \cup \{\infty\} \quad \text{with}$$

$$d(f, g) = \sup \{d(f(e), g(e)) \mid e \in E\} \in \mathbb{R}_0^+ \cup \{\infty\} .$$

If  $(M, d)$  is complete, then  $(M^E, d_\infty)$  is complete, i.e. every Cauchy sequence converges. Convergence with respect to  $d_\infty$  is called **uniform convergence**.

If  $T$  is a topology on  $E$ , then

$$C(E, M) = \{f: E \rightarrow M \mid f \text{ continuous}\} \subset M^E$$

is closed with respect to  $d_\infty$  and therefore complete if  $(M, d)$  is complete. If  $E$  is compact then  $d_\infty$  is finite on  $C(E, M)$ . Thus if  $(M, d)$  is complete, then  $(C(E, M), d_\infty)$  is a complete metric space as well.

**Definition 3.1** Let  $(E, T)$  be a compact topological space and  $(M, d)$  be a metric space. A set  $H \subset \mathbb{C}((E, T), (M, d))$  of continuous functions is **pointwise totally bounded** if for each  $e \in E$ , the set  $H(e) = \{h(e) \mid h \in H\}$  is totally bounded. The set  $H$  is **equicontinuous** if

$$\forall e \in E, \epsilon > 0 \exists U_e \in T, e \in U_e \forall h \in H : h(U_e) \subset B_\epsilon(h(e)) .$$

**Theorem 3.2 (Arzela-Ascoli)** Let  $(E, T)$  be a compact topological space and  $(M, d)$  be a metric space. A subset  $H \subset C(E, M)$  is totally bounded if and only if

1.  $H$  is pointwise totally bounded and
2.  $H$  is equicontinuous.

If  $H$  consists of differentiable functions  $\mathbb{R}^n \supset U \subset \bar{U} \xrightarrow{h} \mathbb{R}^k$  and there is a common bound on the functions in  $H$  as well as one on their derivatives, then  $H$  is pointwise (totally) bounded and equicontinuous.

We will use the following immediate consequence of the Arzela Ascoli-Theorem in this case:

**Corollary 3.3** Let  $H \subset C^1(\overline{B_1^{\mathbb{R}^n}}(0), \mathbb{R}^k)$  be a set of continuously differentiable functions

$$h: \overline{B_1^{\mathbb{R}^n}}(0) = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\} \rightarrow \mathbb{R}^k .$$

If  $H$  and  $dH = \{dh \mid h \in H\}$  are locally bounded, then  $H$  is pointwise totally bounded and equicontinuous. In particular, any sequence  $(h_n)_{n \in \mathbb{N}} \in H^{\mathbb{N}}$  has a uniformly convergent subsequence.

Note that the subsequence need not converge to a differentiable function. Since all norms on a finitely dimensional real vector space are equivalent, boundedness of  $dH$  does not depend on the norm we choose to specify this on  $\text{Hom}(\mathbb{R}^n, \mathbb{R}^k)$ .

**Proof:** The differential of a function  $h: \overline{B_1^{\mathbb{R}^n}}(0) \rightarrow \mathbb{R}^k$  is the map

$$dh: \overline{B_1^{\mathbb{R}^n}}(0) \rightarrow \text{Hom}(\mathbb{R}^n, \mathbb{R}^k) \quad , \quad p \mapsto d_p h$$

so that for all  $p, q \in \overline{B_1^{\mathbb{R}^n}}(0)$ ,

$$h(q) = h(p) + d_p h(q - p) + R(q, p)$$

with a function  $R$  so that

$$\lim_{q \rightarrow p} \frac{R(q, p)}{|q - p|} = 0 .$$

The linear function  $d_p h$  is uniquely determined by this, even if  $p \in \partial \overline{B_1^{\mathbb{R}^n}}(0)$ .

We now assume  $H$  and  $dH$  locally bounded, and let  $e \in \overline{B_1^{\mathbb{R}^n}}(0)$ . Then there is  $U \subset^{\text{open}} \overline{B_1^{\mathbb{R}^n}}(0)$  and  $C \in \mathbb{R}$  so that

$$\forall x \in U, h \in H : \|h(x)\| < C, \|d_x h\|_{\text{op}} < C .$$

Thus  $H(e) \subset B_C^{\mathbb{R}^k}(0)$  is bounded and by the Heine-Borel theorem totally bounded. For equicontinuity, we estimate

$$\|h(x) - h(e)\| \leq \max_{u \in U} \|d_u h\|_{\text{op}} \|x - e\| \leq C \text{diam } U \leq 2C \quad \text{hence} \quad h(U) \subset B_{2C}^{\mathbb{R}^k}(h(e))$$

for all  $h \in H$  and  $x \in U$ . •

## 3.2 Compact Convergence and Normal Families

**Definition 3.4** Let  $(E, T)$  be a topological space and  $(M, d)$  be a metric space. A sequence  $(f_n)_{n \in \mathbb{N}} \in (M^E)^{\mathbb{N}}$  **converges compactly** if for every compact  $K \subset E$  the sequence  $(f_n|_K)_{n \in \mathbb{N}} \in (M^K)^{\mathbb{N}}$  of the restrictions converges uniformly: For  $(f_n)_{n \in \mathbb{N}} \in (M^E)^{\mathbb{N}}$ ,

$$f_n \xrightarrow{n \rightarrow \infty} f \text{ compact} \iff \forall K \subset E \text{ compact}, \epsilon > 0 \exists n_{K, \epsilon} \forall n > n_{K, \epsilon} \forall k \in K : d(f_n(k), f(k)) < \epsilon .$$

A subset  $H \subset M^E$  is **locally bounded** if

$$\forall e \in E \exists U \in T, e \in U, m \in M, r \in \mathbb{R}^+ : H(U) \subset B_r(m) ,$$

where  $H(U) = \{h(u) \mid u \in U, h \in H\}$ .

A subset  $H \subset M^E$  is **normal** if every sequence  $(h_n)_{n \in \mathbb{N}} \in H^{\mathbb{N}}$  contains a compactly convergent subsequence.

**Example 3.5** Power series converge compactly on their open disc of convergence: If  $r$  is the radius of convergence of a power series  $p = \sum_{k=0}^{\infty} a_n(z - p)^k$ , then the series converges uniformly on  $B_\rho(p)$  whenever  $\rho < r$ . If  $K \subset E = B_r(p)$  is compact then  $K \subset B_{\rho_K}(p)$  for some  $\rho_K < r$ . Thus the series converges compactly.

**Theorem 3.6 (Montel)** Let  $G \subset \mathbb{C}$  be a domain and let  $H \subset \mathcal{O}(G)$  be locally bounded. Then  $H$  is normal.

**Proof:** For  $e \in G$ , let  $e \in U \stackrel{\text{open}}{\subset} G$  and  $C \in \mathbb{R}$  be so that  $H(U) \subset B_C(0)$ . Choose  $r > 0$  so that  $B_{3r} \subset U$ . Then for  $q \in U$ ,  $h \in H$  we have  $h(q) \in B_C(0)$ , hence  $H$  is pointwise bounded, and by the Cauchy Integral Formula, for  $q \in B_r(e)$ , we have

$$|h'(q)| = \left| \frac{1}{2\pi i} \oint_{|z-p|=2r} \frac{h(z)}{(z-q)^2} dz \right| \leq \frac{1}{2\pi} 2\pi 2r C \frac{1}{r^2} = \frac{2C}{r}$$

because  $|q-p| < r$ ,  $|z-p| = 2r$  imply  $|z-q| > r$ , hence  $\frac{1}{|z-q|^2} < r^2$ . This shows that the set of the derivatives of the functions in  $H$  is also locally bounded. By the corollary ?? to the Arzela-Ascoli Theorem,  $H$  is normal. •

**Problem 3.7** Prove that compact convergence of holomorphic functions propagates to derivatives.

**Theorem 3.8 (Compact Convergence of the Derivative)** Let  $G \subset \mathbb{C}$  be a domain and let  $(h_n)_{n \in \mathbb{N}} \in \mathcal{O}(G)^{\mathbb{N}}$  converge locally uniformly to  $h$ . Then  $h \in \mathcal{O}(G)$ . Show that the sequence of derivatives  $(h'_n)_{n \in \mathbb{N}} \in \mathcal{O}(G)^{\mathbb{N}}$  converges locally uniformly to  $h'$ .

**Proof:** This follows from the Cauchy integral theorem. Since  $(h_n)_{n \in \mathbb{N}}$  converges locally uniformly to  $h$ , every point  $e \in G$  has a neighbourhood, wlog  $B_{3r}(e)$ , so that  $(h_n)_{n \in \mathbb{N}}$  converges uniformly on  $B_{3r}(e)$ . Thus for every  $\epsilon > 0$  there is  $N_\epsilon \in \mathbb{N}$ , so that

$$\forall n > N_\epsilon, x \in B_{3r}(e) : |h_n(x) - h(x)| < \epsilon .$$

The Cauchy integral formula now gives an estimate for the derivatives at  $q$  with  $|q-e| < r$ ,

$$\begin{aligned} |h'_n(q) - h'(q)| &= \left| \frac{1}{2\pi i} \oint_{|z-e|=2r} \frac{h_n(z) - h(z)}{(z-q)^2} dz \right| \\ &< \frac{1}{2\pi} 2\pi r \frac{\epsilon}{r^2} = \frac{\epsilon}{r} \end{aligned}$$

Thus  $(h'_n)_{n \in \mathbb{N}}$  converges uniformly on  $B_r(e)$ . •

## 4 The Riemann Mapping Theorem

### 4.1 Holomorphic maps of the unit disc

**Theorem 4.1 (Schwarz Lemma)** Let  $f: B_1(0) \rightarrow B_1(0)$  be holomorphic with  $f(0) = 0$ . Then  $|f'(0)| \leq 1$  and for all  $z \in B_1(0)$  we have  $|f(z)| \leq |z|$ .

If  $|f'(0)| = 1$  or if there is  $p \in B_1(0) \setminus \{0\}$  with  $|f(p)| = |p|$ , then there is  $m \in \mathcal{S}^1$  ( $m = e^{i\theta}$  for some  $\theta \in \mathbb{R}$ ) so that for all  $z$  we have  $f(z) = mz$ . Thus  $f$  is rotation by the angle  $\theta$ .



**Proof:** By Theorem ?? on the Taylor series of a holomorphic function,  $f$  is given by a power series converging on all of  $B_0(0)$  and starting with  $z$ , because  $f(0) = 0$ . Therefore  $f(z) = zg(z)$  with  $g \in \mathcal{O}(B_1(0))$ . In particular we have

$$|g(z)| < \frac{1}{|z|} \quad \text{for all } z \in B_1(0) .$$

By the Maximum Principle ??, for any  $r$ ,  $0 < r < 1$  the function  $|g|$  assumes its maximum on  $\overline{B}_r(0)$  on the boundary, hence

$$\forall 0 < r < 1, |z| < r : |g(z)| \leq \max_{|u|=r} |g(u)| \leq \frac{1}{r} .$$

But this implies

$$\forall z \in B_1(0) : |g(z)| \leq 1 \text{ hence } |f(z)| \leq |z| .$$

If  $|f(p)| = |p|$  for some  $p \in B_1(0) \setminus \{0\}$  or if  $p = 0$  and  $1 = |f'(0)| = |g(0)|$ , then  $|g(p)| = 1$  and  $p$  is a maximum of  $|g|$ , which must be on the boundary of  $B_1(0)$  if  $g$  is not constant. Hence  $g$  is constant. •

**Corollary 4.2** *If  $f: \mathcal{H} \rightarrow \mathcal{H}$  is biholomorphic, then*

$$f(z) = \frac{az + b}{cz + d}$$

for some  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}^+(2, \mathbb{R})$ .

**Proof:** The **Cayley transform** is the biholomorphic map

$$c: \mathcal{H} \rightarrow B_1(0) \quad , \quad c(z) = \frac{z - i}{z + i} \quad , \quad c(i) = 0, \quad c(0) = -1, \quad c(\infty) = 1 . \quad (4.3)$$

Its inverse is given by

$$c^{-1}(y) = \frac{iy + i}{-y + 1} .$$

By the Schwarz Lemma, every biholomorphic map of the disc fixing 0 is a rotation,  $R_t: z \mapsto e^{it}z$ ,  $t \in \mathbb{R}$ . Therefore a biholomorphic map of the upper half plane fixing  $i$  must be of the form

$$\begin{aligned} c^{-1} \circ R_t \circ c &= \mu \left[ \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{it} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \right] = \mu \left[ \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{it} & -ie^{it} \\ 1 & i \end{pmatrix} \right] \\ &= \mu \begin{pmatrix} i(e^{it} + 1) & e^{it} - 1 \\ -e^{it} + 1 & i(e^{it} + 1) \end{pmatrix} = \mu \left[ \frac{1}{2ie^{it/2}} \begin{pmatrix} i(e^{it} + 1) & e^{it} - 1 \\ -e^{it} + 1 & i(e^{it} + 1) \end{pmatrix} \right] \\ &= \mu \underbrace{\begin{pmatrix} \cos(t/2) & \sin(t/2) \\ -\sin(t/2) & \cos(t/2) \end{pmatrix}}_{\in \text{GL}^+(2, \mathbb{R})} \end{aligned}$$

where  $\mu A$  denotes the Möbius transform of the matrix  $A$ . •

**Lemma 4.4 (Transitivity of Möbius transformations)** For  $p \in \mathbb{E}$  let  $w_p \in \mathcal{O}(\mathbb{E})$  be the function with

$$w_p(z) = \frac{z - p}{-\bar{p}z + 1} .$$

Then  $w_p$  is a biholomorphic map  $\mathbb{E} \rightarrow \mathbb{E}$ ,  $w_p(p) = 0$ ,  $w_p(0) = -p$ .

**Proof:** The function  $w_p$  defines a biholomorphic map of the Riemann sphere, its inverse is given by

$$w_p^{-1} = w_{-p} \quad , \quad w_p^{-1}(y) = \frac{y + p}{\bar{p}y + 1} .$$

We only need to check  $w_p(\mathbb{E}) \subset \mathbb{E}$ , i.e.

$$\begin{aligned} |z - p| &< |1 - \bar{p}z| \\ |z|^2 + |p|^2 - z\bar{p} - \bar{z}p &< 1 + |p|^2 |z|^2 - \bar{p}z - p\bar{z} . \end{aligned}$$

•

Some simple domains can be mapped to  $\mathbb{E}$  via Möbius transforms and powers. For instance the quarter disc is biholomorphically equivalent to the disc,

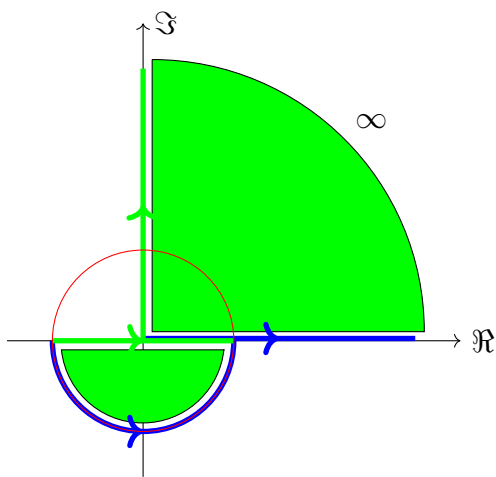
$$Q = \{z \in \mathbb{C} \mid |z| < 1, \Re(z) > 0, \Im(z) > 0\} = \left\{ r e^{it} \mid 0 < r < 1, 0 < t < \frac{\pi}{2} \right\} \cong \mathbb{E} .$$

To construct such a map, recall the Cayley transform (??)

$$c: \mathcal{H} \rightarrow \mathbb{E} \quad , \quad z \mapsto \frac{z - i}{z + i} .$$

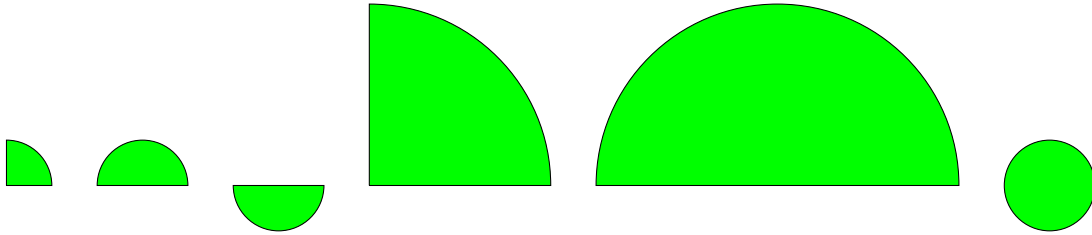
This maps

$$i \mapsto 0 \mapsto -1 \quad , \quad 1 \mapsto -i \quad , \quad \infty \mapsto 1$$



An explicit biholomorphic map  $Q \rightarrow \mathbb{E}$  is the composition

$$Q \xrightarrow{\text{square}} \mathcal{H} \cap \mathbb{E} \xrightarrow{z \mapsto -z} -\mathcal{H} \cap \mathbb{E} \xrightarrow{c^{-1}} \mathcal{H} \cap -i\mathcal{H} \xrightarrow{\text{square}} \mathcal{H} \xrightarrow{c} \mathbb{E} .$$



## 4.2 The Riemann Mapping Theorem

**Theorem 4.5** Let  $\emptyset \neq G \subsetneq \mathbb{C}$  be a simply connected domain different from  $\mathbb{C}$ . Then there is a biholomorphic map

$$\phi: G \rightarrow \mathbb{E} = \mathbb{B}_1(0) .$$

**Proof:** The proof proceeds in three steps.

1. There is an injective holomorphic map  $\phi: G \rightarrow \mathbb{E}$ . Such a map  $\phi$  is then biholomorphic onto its image.

$$\phi \xrightarrow{\cong} \phi(G) \subset \mathbb{E} .$$

Let  $q \in \mathbb{C} \setminus G$ . Since  $G$  is simply connected, there is a function  $g \in \mathcal{O}(G)$  so that  $g(z)^2 = z - p$ , thus “ $z - q$  has a square root on  $G$ ”. The function  $g$  is injective because  $g$  has a left inverse.  $g$  is not constant, hence open. Also if  $p \neq 0$ ,  $p \in g(G)$ , then  $-p \notin g(G)$ . Let  $0 \neq p \in g(G)$ . Then for some  $r > 0$ ,  $\mathbb{B}_r(p) \subset g(G)$ , because  $g$  is open. But then

$$\mathbb{B}_r(-p) \cap g(G) = \emptyset .$$

By inversion we can then map all of  $g(G)$  into  $\mathbb{E}$ . We define

$$\phi: G \rightarrow \mathbb{E} \quad , \quad \phi(z) = \frac{r}{g(z) + p} .$$

We may thus assume  $0 \in G \subset \mathbb{E}$ .

2. Surjectivity and maximal derivative at 0: Let  $0 \in G \subset \mathbb{E}$ ,  $f: G \rightarrow \mathbb{E}$  holomorphic,  $f(0) = 0$ , injective, **not surjective**. Then there is an injective holomorphic function  $F: G \rightarrow \mathbb{E}$  with  $F(0) = 0$  and

$$|F'(0)| > |f'(0)| .$$

Let  $q \in \mathbb{E} \setminus f(G)$  and  $w_1 = w_q$  as in Lemma ???. Then  $0 \notin w_1(f(G))$ . Since  $w_1(f(G))$  is simply connected, we have a holomorphic square root

$$\text{sqrt}: w_1(f(G)) \rightarrow \mathbb{E} \quad , \quad (\text{sqrt}(w_1(f(z))))^2 = w_1(f(z)) \quad \text{for all } z \in G .$$

Let  $w_2 = w_{\text{sqrt}(w_1(0))}$ . Then the composition

$$F = h \circ f = \underbrace{w_2 \circ \text{sqrt} \circ w_1}_{h} \circ f: \mathbb{E} \rightarrow \mathbb{E}$$

is holomorphic and maps  $0 \mapsto 0$ . The inverse of  $h$  is

$$h^{-1}(z) = w_1^{-1}((w_2^{-1}(z))^2)$$

and therefore we have

$$h^{-1}(\mathbb{E}) \subset \mathbb{E} .$$

Since  $h^{-1}$  is not injective on  $\mathbb{C}$  and therefore not a rotation, the Schwarz Lemma gives  $(h^{-1})'(0) < 1$ , hence

$$|F'(0)| = |h'(0)| |f'(0)| > |f'(0)| .$$

3. Maximizers for the derivative at 0 exist and are biholomorphic: Among the injective functions  $f: G \rightarrow E$  with  $f(0) = 0$  there is one with maximal  $|f'(0)|$  and this is biholomorphic.

Crucial for this step is the functional

$$\mathcal{O}(G, \mathbb{E}, \text{inj}) := \{f \in \mathcal{O}(G, E) \mid f(0) = 0, f \text{ injective}\} \xrightarrow[E]{f \mapsto |f'(0)|} \mathbb{R} .$$

By the Cauchy Integral formula,  $E$  is bounded, because for  $f \in \mathcal{O}(G, \mathbb{E}, \text{inj})$  we can estimate

$$|f'(0)| = \left| \frac{1}{2\pi i} \oint_{|z|=r} \frac{f(z)}{z^2} dz \right| \leq \frac{1}{r^2} \quad \text{if } B_r(0) \subset G .$$

Thus  $\sup_{f \in \mathcal{O}(G, \mathbb{E}, \text{inj})} |f'(0)| = s < \infty$ . Since  $\mathcal{O}(G, \mathbb{E}, \text{inj})$  is locally bounded, it is normal by Montel's Theorem. Thus there is a compactly convergent sequence

$$(f_n)_{n \in \mathbb{N}} \in \mathcal{O}(G, \mathbb{E}, \text{inj})^{\mathbb{N}} \quad , \quad \lim_{n \rightarrow \infty} f_n =: f \quad , \quad \lim_{n \rightarrow \infty} |f'_n(0)| = |f'(0)| = s .$$

By step 2, the function  $f$  is surjective. If  $f$  were not injective, then there would be two points  $x, y \in G$  so that  $f(x) = f(y) =: w$ . The zeros of the function  $f(z) - w$  must be isolated since  $f$  can not be constant. We can therefore find a closed continuous path  $\gamma$  in  $G$  avoiding all zeros of  $f(z) - w$  surrounding both  $x, y$  once. Since  $f_n - w$  converges uniformly on  $\gamma$  to  $f - w$  there is  $N \in \mathbb{N}$  so that for all  $t$ ,

$$|f(\gamma(t)) - f_N(\gamma(t))| < \min_t |f(\gamma(t)) - w| .$$

By Rouché's Theorem,  $f$  and  $f_N$  have the same number of zeros surrounded by  $\gamma$ , a contradiction.

•

**Problem 4.6** *At the end of the proof of the Riemann Mapping Theorem we used lower semicontinuity of the number of leaves, i.e. the multiplicity. Prove the more general*

**Theorem 4.7** *Let  $G \subset \mathbb{C}$  be a domain and  $(f_n)_{n \in \mathbb{N}} \in \mathcal{O}(G)^{\mathbb{N}}$ ,  $f_n \xrightarrow[\text{compact}]{n \rightarrow \infty} f$ . Let  $a \in \mathbb{C}$ ,  $M \in \mathbb{N}$  be so that*

$$\forall n \in \mathbb{N} : m(f_n, a) < M .$$

*Then*

$$m(f, a) \leq M \text{ or } f = a .$$

**Proof:** Assume that  $f$  is not constant and  $m(f, a) > M$ . Then  $f^{-1}(a) \subset G$  is discrete. There is a finite subset  $P \subset f^{-1}(a)$  and  $r > 0$  so that

$$\forall p \in P : \{p\} = B_r(p) \cap f^{-1}(a), P \cap \mathcal{S}_r(p) = \emptyset \quad \text{and} \quad \sum_{p \in P} \text{ord}(f(z) - a; z = p) > M$$

The union  $\mathcal{S} = \bigcup_{p \in P} \mathcal{S}_r(p)$  of the boundary circles is compact and  $f(z) - a$  is nowhere zero there. Thus

$$\min_{z \in \mathcal{S}} |f(z) - a| > 0$$

exists. Since  $f_n \xrightarrow[\text{compact}]{n \rightarrow \infty} f$  there is  $N \in \mathbb{N}$  so that

$$\forall z \in \mathcal{S} : |f_N(z) - f(z)| < \min_{\zeta \in \mathcal{S}} |f(\zeta) - a|$$

By Rouché's Theorem,  $f - a$  and  $f_N - a$  have the same number of zeros in  $B = \bigcup_{p \in P} B_r(p)$ , counting multiplicities, i.e.

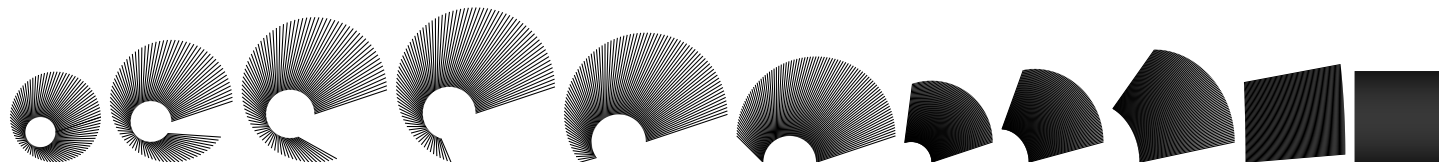
$$M = m(f_N, a) = \sum_{q \in G} \text{ord}(f_N(z) - a; z = q) \geq \sum_{q \in B} \text{ord}(f_N(z) - a; z = q) = \sum_{q \in B} \text{ord}(f(z) - a; z = q) > M$$

a contradiction. •

## References

- [1] Lars Ahlfors: *Complex Analysis: An Introduction to the Theory of Analytic Functions of One Complex Variable*
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An annulus is a rectangular domain:



# Complex Analysis I - MT333P

## 5 Homework

1. Compute real and imaginary part of all  $z \in \mathbb{C}$  with  $z^2 = -8 + 6i$ .

**Solution:** We need to compute  $a, b \in \mathbb{R}$  so that

$$(a + ib)^2 = a^2 - b^2 + 2iab = -8 + 6i ,$$

$$a^2 - b^2 = -8 \quad \text{and} \quad ab = 3 ,$$

$$a^2 - \frac{9}{a^2} = -8 ,$$

hence  $a^2 = 1$  (the other solution,  $a^2 = -9$  does not give  $a \in \mathbb{R}$ ). Thus the square roots of  $-8 + 6i$  are

$$1 + 3i \quad \text{and} \quad -1 - 3i$$

2. Compute real and imaginary part of

$$\sum_{k=0}^{123} (1 + i)^k .$$

**Solution:** The key to this is that

$$1 + i = \sqrt{2}\epsilon \quad \text{where} \quad \epsilon^4 = -1 .$$

together with the formula for the geometric series

$$\sum_{j=0}^k q^j = \frac{1 - q^{k+1}}{1 - q} \quad \text{if} \quad q \neq 1 .$$

Thus

$$\sum_{k=0}^{123} (1 + i)^k = \sum_{k=0}^{123} (\sqrt{2}\epsilon)^k = \frac{1 - (\sqrt{2}\epsilon)^{124}}{1 - \sqrt{2}\epsilon} = \frac{1 - 2^{62}(-1)^{31}}{-i} = (2^{62} + 1)i .$$

Thus the real part is 0 and the imaginary part is  $2^{62} + 1$ .

3. Recall the **standard scalar product** on  $\mathbb{R}^n$ : For  $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$  we define

$$\langle x | y \rangle = \sum_{i=1}^n x_i y_i .$$

The euclidean norm on  $\mathbb{R}^n$  is given by

$$\|x\| = \sqrt{\langle x | x \rangle}$$

and the cosine of the **angle between  $x$  and  $y$** ,  $x, y \neq 0$ , is  $\angle(x, y) \in [0, \pi]$ ,

$$\cos \angle(x, y) := \frac{\langle x | y \rangle}{\|x\| \|y\|} .$$

This is well defined because by the Cauchy-Schwarz inequality,  $|\langle x | y \rangle| \leq \|x\| \|y\|$ , and  $[0, \pi] \xrightarrow{\cos} [-1, 1]$  is bijective .

Determine all matrices  $A \in \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{R})$  preserving angles, i.e. so that for all  $x, y \in \mathbb{R}^2$  we have

$$\angle(Ax, Ay) = \angle(x, y) \quad (5.1)$$

whenever both sides are defined (i.e.  $x \neq 0 \neq y$ ,  $Ax \neq 0 \neq Ay$ ).

**Solution:** Since the cosine is injective in the range referred to in the definition of the angle, (??) means that

$$\langle Ax | Ay \rangle = \frac{\langle x | y \rangle \|Ax\| \|Ay\|}{\|x\| \|y\|} . \quad (5.2)$$

Thus if  $\{b_1, \dots, b_n\}$  is an orthonormal basis for  $\mathbb{R}^n$ , then the vectors  $Ab_1, \dots, Ab_n$  are pairwise orthogonal. Thus there is an orthonormal basis  $\{c_1, \dots, c_n\}$  and  $\lambda_i \in \mathbb{R}$  so that

$$Ab_i = \lambda_i c_i \quad , \quad i = 1, \dots, n .$$

Let  $T \in \text{O}(n)$  be the orthogonal transformation of  $\mathbb{R}^n$  with  $Tc_i = b_i$  for all  $i$ . Then

$$TA b_i = \lambda_i b_i \quad , \quad i = 1, \dots, n .$$

We may also assume that  $\lambda_i \geq 0$  because the sign of  $\lambda_i$  can also be absorbed in the isometry  $T$ . We now show that  $\lambda_i = \lambda_j$  for all  $i, j \in \{1, \dots, n\}$ . To this end we only need to look at one other angle, for instance the one between  $x = b_i + b_j$  and  $y = b_i - b_j$ . Clearly  $x, y$  are perpendicular and both have norm  $\sqrt{2}$ . Hence from (??),

$$0 = \langle Ax | Ay \rangle = \langle \lambda_i b_i + \lambda_j b_j | \lambda_i b_i - \lambda_j b_j \rangle = \lambda_i^2 - \lambda_j^2 \quad \text{hence} \quad \lambda_i = \lambda_j .$$

We thus have shown that for some  $\lambda \in \mathbb{R}_0^+$  we have

$$TA = \lambda \text{id}_{\mathbb{R}^n} \quad \text{hence} \quad A = \lambda T^{-1} .$$

In the special case  $n = 2$ , this gives

$$A = \lambda \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \quad \text{or} \quad A = \lambda \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ \sin(\alpha) & -\cos(\alpha) \end{pmatrix} .$$

because every orthogonal  $(2 \times 2)$ -matrix is of the form

$$\begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ \sin(\alpha) & -\cos(\alpha) \end{pmatrix} .$$

depending on whether it preserves or reverses orientation. Thus the conformal linear automorphisms of  $\mathbb{R}^2$  are of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \quad , \quad a, b \in \mathbb{R}, (a, b) \neq (0, 0) .$$

4. Sketch the subsets of  $\mathbb{C} = \mathbb{R}^2$  given below.

(a)  $X_a = \{z \in \mathbb{C} \mid \Re(z) + 1 = |z|\}$

**Solution:** In cartesian coordinates, this is the set  $\{(x, y) \mid y^2 = (x + 1)^2 - x^2\} = \{(x, y) \mid y^2 = 2x + 1\}$ , a parabola.

$$(b) X_b = \{z \in \mathbb{C} \mid |z|^2 = \Im(z)\}$$

**Solution:** In cartesian coordinates, this is the set  $\{(x, y) \mid y^2 - y + x^2 = 0\} = \{(x, y) \mid (y - \frac{1}{2})^2 - \frac{1}{4} + x^2 = 0\}$ , a circle.

$$(c) X_c = \{z \in \mathbb{C} \mid |z - i| + |z + i| = 4\Im(z)\}$$

5. Stereographic projection is the map

$$\mathcal{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\} \xrightarrow{\phi} \mathbb{R}^2 \cup \{\infty\} = \mathbb{C} \cup \{\infty\} =: \hat{\mathbb{C}}$$

which takes  $(0, 0, 1) \in \mathcal{S}^2$  to  $\infty$  and so that for  $p \in \mathcal{S}^2 \setminus \{(0, 0, 1)\}$  the points

$$(0, 0, 1) \quad , \quad p \quad , \quad (\phi(p), 0)$$

lie on a common line in  $\mathbb{R}^3$ . Thus  $\phi(x, y, 0) = (x, y)$ , for instance. Stereographic projection is bijective, you can compute explicit formulas for  $\phi$  and  $\phi^{-1}$ .

Let  $\text{inv}: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$  be the involution of  $\hat{\mathbb{C}}$  mapping

$$z \leftrightarrow \frac{1}{z} \quad \text{if } z \neq 0, \infty \quad \text{and} \quad 0 \leftrightarrow \infty .$$

Find the conjugate  $\phi^{-1} \circ \text{inv} \circ \phi$ , i.e. for  $x, y, z \in \mathcal{S}^2$ , what is

$$\phi^{-1}(\text{inv}(\phi(x, y, z))) ?$$

**Solution:**

$$\phi(x, y, t) = \left( \frac{x}{1-t}, \frac{y}{1-t} \right) ,$$

$$\phi(z, t) = \frac{z}{1-t} \quad \text{if we identify } \mathbb{C} \times \mathbb{R} \cong \mathbb{R}^3 , ((x + iy), t) = (x, y, t),$$

$$\text{inv}(u + iv) = \frac{u - iv}{u^2 + v^2}$$

$$\phi^{-1}(u, v) = \left( \frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1} \right)$$

$$\phi^{-1}(z) = N(2z, |z|^2 - 1) ,$$

where we denote by  $N$  the map  $\mathbb{R}^3 \setminus \{0\} \rightarrow \mathcal{S}^2$ ,  $N(v) = \frac{x}{\|v\|}$ . For  $(x, y, t) = (z, t) \in \mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ , we



get

$$\begin{aligned}
\phi^{-1}(\phi(x, y, t)^{-1}) &= \\
\phi^{-1}(\phi(z, t)^{-1}) &= \phi^{-1}\left(\frac{\bar{z}(1-t)}{|z|^2}\right) = N\left(2\frac{\bar{z}(1-t)}{|z|^2}, \left|\frac{\bar{z}(1-t)}{|z|^2}\right|^2 - 1\right) \\
&= N\left(\bar{z}, \frac{\left|\frac{\bar{z}(1-t)}{|z|^2}\right|^2 - 1}{2\frac{1-t}{|z|^2}}\right) = N\left(\bar{z}, \frac{|z|^2(1-t) - |z|^2}{2|z|^2}\right) \\
&= N\left(\bar{z}, \frac{(1-t)}{2} - \frac{|z|^2}{2(1-t)}\right) \\
&= N\left(\bar{z}, \frac{(1-t)}{2} - \frac{1-t^2}{2(1-t)}\right) \quad \text{because } 1 = |z|^2 + t^2 \\
&= N\left(\bar{z}, \frac{(1-t)}{2} - \frac{1+t}{2}\right) \\
&= N(\bar{z}, -t) = (\bar{z}, -t) \\
&= (x, -y, -t) .
\end{aligned}$$

Thus inversion corresponds to reflection at the real line,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in \text{SO}(3)$$

on the sphere.

6. Find a formula for  $\sum_{n=0}^{\infty} nq^n$ , for  $|q| < 1$ .

**Solution:** For  $|z| < 1$ , if we set

$$f(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

then

$$f'(z) = \sum_{n=1}^{\infty} nz^{n-1}$$

$$zf'(z) = \sum_{n=0}^{\infty} nz^n$$

but this can also be computed directly,

$$zf'(z) = z \left(\frac{1}{1-z}\right)' = \frac{z}{(1-z)^2} .$$

*Alternatively:*

$$\sum_{n=0}^{\infty} nq^n = \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} q^n = \sum_{k=1}^{\infty} q^k \sum_{n=0}^{\infty} q^n = \frac{q}{1-q} \frac{1}{1-q} .$$

7. Find a function  $v: \mathbb{C} \rightarrow \mathbb{R}$  so that the function  $f: \mathbb{C} \rightarrow \mathbb{C}$  given by

$$f(x + iy) = xy^2 - \frac{1}{3}x^3 + x^2 - y^2 + iv(x + iy)$$

for  $x, y \in \mathbb{R}$  is complex differentiable.

**Solution:** By the Cauchy-Riemann equations we must have

$$y^2 - x^2 + 2x = v_y \quad \text{and} \quad 2xy - 2y = -v_x ,$$

which is satisfied by

$$v(x + iy) = \frac{1}{3}y^3 - x^2y + 2xy .$$

8. What is the radius of convergence of

$$\sum_{n=0}^{\infty} \frac{z^{n!}}{n+1} ?$$

**Solution:** The  $k$ th coefficient of this series is  $\frac{1}{n+1}$  if  $k = n!$  for some  $n \in \mathbb{N}_0$  and 0 otherwise. The formula for the radius of convergence  $\rho$  gives

$$\rho = \frac{1}{\limsup_{k \rightarrow \infty} \sqrt[k!]{\frac{1}{k+1}}} = 1 ,$$

because

$$1 \geq \sqrt[k!]{\frac{1}{k+1}} \geq \sqrt[k]{\frac{1}{k+1}} \xrightarrow{k \rightarrow \infty} 1$$

9. What is the radius of convergence of the series

$$\sum_{n=0}^{\infty} 2^{n^2} z^n \quad , \quad \sum_{n=0}^{\infty} \frac{z^{n^3}}{2^{n^2}} ?$$

Prove your result!

**Hint:** Trying some values for  $z$  may be easier than using Hadamard's formula for the radius of convergence.

**Solution:** The series  $\sum_{n=0}^{\infty} 2^{n^2} z^n$  converges for no  $z \neq 0$ . To see this, assume  $2^k < z \in \mathbb{R}^+$  for some  $k \in \mathbb{Z}$ . Then

$$2^{n^2} z^n > 2^{n^2+k} \xrightarrow{n \rightarrow \infty} \infty .$$

It follows that the radius of convergence of this series is 0.

The series  $\sum_{n=0}^{\infty} \frac{z^{n^3}}{2^{n^2}}$  converges for  $z = 1$ . If  $z \in \mathbb{R}$ ,  $z > 1$  then there is  $\epsilon > 0$  so that

$$1 < 2^\epsilon < z .$$

Hence

$$\frac{z^{n^3}}{2^{n^2}} > 2^{\epsilon n^3 - n^2} \xrightarrow{n \rightarrow \infty} \infty$$

and the series does not converge. It follows that the radius of convergence of this series is 1.

10. Let  $f: \mathbb{C} \rightarrow \mathbb{C}$  be the function with  $f(z) = \bar{z} + z + z^3$  and let  $\gamma: [0, 1]$  be the curve with  $\gamma(t) = t^3 + it^2$ . Compute  $\int_{\gamma} f(z) dz$ .

**Hint:** You can compute this directly or use that the function “almost” has an antiderivative.

**Solution:** Since the function  $g(z) = z + z^3 = G'(z)$ ,  $G(z) = \frac{z^2}{2} + \frac{z^4}{4}$  is holomorphic, we can compute its integral from the endpoints,

$$\int_{\gamma} g(z) dz = G(\gamma(1)) - G(\gamma(0)) = G(1 + i) - G(0) = \frac{(1 + i)^4}{4} + \frac{(1 + i)^2}{2} = -1 + i .$$

It remains to compute

$$\begin{aligned} \int_{\gamma} \bar{z} dz &= \int_0^1 \overline{\gamma(t)} \gamma'(t) dt = \int_0^1 (t^3 - it^2)(3t^2 + 2it) dt \\ &= \int_0^1 3t^5 + 2t^3 - it^4 dt = \frac{1}{2} + \frac{1}{2} - \frac{i}{5} . \end{aligned}$$

Thus

$$\int_{\gamma} \bar{z} + z + z^3 dz = \frac{4}{5}i$$

11. Let  $\gamma: [-1, 2] \rightarrow \mathbb{C}$  be the curve with

$$\gamma(t) = 2t + i \sin(\pi t) .$$

Compute

$$f(z) = \int_{\gamma} \frac{z}{(1 - z^2)^2} dz .$$

**Solution:** The function  $f$  is the derivative of

$$F(z) = \frac{1/2}{1 - z^2} .$$

Hence

$$\int_{\gamma} f(z) dz = F(\gamma(2)) - F(\gamma(-1)) = F(4) - F(-2) = \frac{-1}{30} - \frac{-1}{6} = \frac{4}{30} = \frac{2}{15} .$$

12. Let  $f$  be complex differentiable at  $p \in \mathbb{C}$ . Show that the function  $g$  with

$$g(z) = \overline{f(\bar{z})}$$

is complex differentiable at  $\bar{p}$ .

**Hint:**  $\overline{a + ib} = a - ib$ , the complex conjugate

**Solution:**

$$\lim_{h \rightarrow 0} \frac{g(\bar{p} + h) - g(\bar{p})}{h} = \lim_{h \rightarrow 0} \frac{g(\overline{p + \bar{h}}) - g(\bar{p})}{\bar{h}} = \lim_{h \rightarrow 0} \frac{\overline{f(p + h)} - \overline{f(p)}}{\bar{h}} = \lim_{h \rightarrow 0} \frac{f(p + h) - f(p)}{h} .$$

13. Let  $R > 0$  and let  $f$  be complex differentiable on  $B_R$  with  $f(0) = 0$ ,  $f'(0) \neq 0$  and  $f(z) \neq 0$  for all  $z \in B_R \setminus \{0\}$ . Prove that

$$\oint_{|z|=r} \frac{dz}{f(z)} = \frac{2\pi i}{f'(0)} \quad \text{for all } r, 0 < r < R. \quad (5.3)$$

**Hint:** By “ $\oint_{|z|=r} \dots$ ” we mean “ $\oint_{\omega} \dots$ ” where  $\omega: [0, 1] \rightarrow \mathbb{C}$  is the curve  $\omega(t) = re^{2\pi it}$ .

Apply the Cauchy Integral Theorem to the annulus  $\{z \mid r_1 < |z| < r_2\}$  to show that the integral in (??) does not depend on  $r$ .

**Solution:** Since  $f'(0) \neq 0$  there is  $\epsilon > 0$  so that

$$f^{-1}(0) \cap B_\epsilon = \{0\}.$$

Hence  $\frac{1}{f(z)}$  is holomorphic on  $B_\epsilon \setminus B_r$  if  $0 < r < \epsilon$ , and therefore

$$\oint_{|z|=r} \frac{dz}{f(z)} = \oint_{|z|=\epsilon} \frac{dz}{f(z)}$$

for all  $r, 0 < r < \epsilon$ . Now

$$f(z) = f'(0)z + R(z) \quad \text{for } z \in B_\epsilon$$

and  $\lim_{z \rightarrow 0} R(z)z = 0$ , hence

$$\frac{1}{f(z)} = \frac{1}{f'(0)z}$$

14. Extension of  $\ln$  to  $\mathbb{C} \setminus \mathbb{R}^-$ . We define

$$\ln: \mathbb{C} \setminus \mathbb{R}^- = \{a + ib \mid a, b \in \mathbb{R}, b \neq 0 \text{ if } a < 0\} \rightarrow \mathbb{C}$$

by

$$\ln(z) = \int_{\gamma_z} \frac{1}{z} dz \quad (5.4)$$

where  $\gamma_z: [0, 1] \rightarrow \mathbb{C} \setminus \mathbb{R}^-$  is **any**  $C^1$  curve with  $\gamma_z(0) = 1$  and  $\gamma_z(1) = z$ . Show that the integral in (??) is the same for all such curves.

**Solution:** The map

$$\mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad (x, y) \mapsto (x^2 - y^2, 2xy)$$

is  $C^1$  and its restriction to the right half plane

$$\phi: \mathbb{R}^+ \times \mathbb{R} \rightarrow \{(x, y) \in \mathbb{R}^2 \mid y \neq 0 \text{ or } x > 0\}$$

is a diffeomorphism.

This is the map  $\{z \in \mathbb{C} \mid \Re(z) > 0\} \xrightarrow{z \mapsto z^2} \mathbb{C} \setminus \mathbb{R}^-$ . The inverse of  $\phi$  is

$$\phi^{-1}: \{(x, y) \in \mathbb{R}^2 \mid y \neq 0 \text{ or } x > 0\} \rightarrow \mathbb{R}^+ \times \mathbb{R}$$

$$\phi^{-1}(a, b) = \left( \sqrt{\frac{a}{2} + \sqrt{\frac{a^2 + b^2}{4}}}, \frac{b}{2\sqrt{\frac{a}{2} + \sqrt{\frac{a^2 + b^2}{4}}}} \right)$$

Let  $\gamma, \mu: [0, 1] \rightarrow \mathbb{C} \setminus \mathbb{R}^-$  be  $C^1$ -curves with  $\gamma(0) = \mu(0) = 1$  and  $\gamma(1) = \mu(1) = z$ .

Then

$$H(t, s) := \phi(s\phi^{-1}(\gamma(t)) + (1-s)\phi^{-1}(\mu(t))) \in \mathbb{C} \setminus \mathbb{R}^-$$

for all  $(t, s) \in [0, 1] \times [0, 1]$  and

$$H(0, s) = 1 \quad , \quad H(1, s) = z \quad , \quad H(t, 0) = \mu(t) \quad , \quad H(t, 1) = \gamma(t) \quad .$$

Thus  $\mu$  and  $\gamma$  are homotopic relative endpoint in  $\mathbb{C} \setminus \mathbb{R}^-$ . Since  $\frac{1}{z}$  is holomorphic in  $\mathbb{C} \setminus \mathbb{R}^-$ , the complex line integrals over  $\gamma$  and  $\mu$  coincide.

15. For  $n \in \mathbb{Z}$ , compute

$$\oint_{|z|=5} \frac{e^z - e^{-z}}{z^n} dz \quad .$$

**Hint:** Recall that  $\oint_{|z|=5} \dots$  denotes the integral  $\int_{\mu} \dots$  where  $\mu$  is the oriented boundary curve of  $B_5$ ,  $\mu: [0, 1] \rightarrow \mathbb{C}$ ,  $\mu(t) = 5e^{2\pi t}$ .

**Solution:** The power series for the numerator of the integrand,

$$e^z - e^{-z} = 2 \sinh(z) = 2 \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!}$$

hence the integrand is

$$2 \sum_{k=0}^{\infty} \frac{z^{2k+1-n}}{(2k+1)!} = 2 \sum_{k < \frac{n-1}{2}} \frac{z^{2k+1-n}}{(2k+1)!} + \underbrace{2 \sum_{k \geq \frac{n-1}{2}} \frac{z^{2k+1-n}}{(2k+1)!}}_{\text{entire}}$$

The closed line integral over the entire part vanishes by the Cauchy integral theorem and so we are left with

$$\oint_{|z|=5} 2 \sum_{k < \frac{n-1}{2}} \frac{z^{2k+1-n}}{(2k+1)!} dz = 2 \sum_{k < \frac{n-1}{2}} \frac{1}{(2k+1)!} \oint_{|z|=5} z^{2k+1-n} dz = \begin{cases} 0 & \text{if } n \text{ odd} \\ \frac{4\pi i}{(n-1)!} & \text{if } n \text{ even} \end{cases}$$

because for all  $q \in \mathbb{Z}$ ,  $r > 0$ , we have

$$\oint_{|z|=r} z^q dz = \begin{cases} 2\pi i & \text{if } q = -1 \\ 0 & \text{if else} \end{cases}$$

16. (a) Find the power series expansion of  $f(z) = \frac{e^z - 1}{z}$ .

**Hint:** Start with

$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \dots$$

This will also “define”  $f(0)$ .

**Solution:**

$$\frac{e^z - 1}{z} = 1 + \frac{z}{2} + \frac{z^2}{3!} + \dots + \frac{z^{k-1}}{k!} + \dots = \sum_{k=0}^{\infty} \frac{z^k}{(k+1)!} \quad (5.5)$$

(b) Compute the first four terms  $a_0, a_1, a_2, a_3$  of the power series for

$$g(z) = \frac{1}{f(z)} = \frac{z}{e^z - 1} = \sum_{k=0}^{\infty} a_k z^k .$$

Show that for  $n > 0$ ,

$$\sum_{k=0}^n \binom{n+1}{k} k! a_k = 0 .$$

**Solution:** We must have  $g(z) \frac{e^z - 1}{z} = 1$  for all  $z$ . By (??) this leads to

$$1 = a_0 \quad , \quad 0 = \frac{1}{2} a_0 + a_1 \quad , \quad \dots \quad ,$$

$$0 = \sum_{k=0}^n \frac{a_k}{(n-k+1)!} = \frac{1}{(n+1)!} \sum_{k=0}^n \binom{n+1}{k} k! a_k$$

17. Exploit the differential equation “ $\tan' = 1 + \tan^2$ ” to derive a recursive formula for the coefficients of the power series for  $\tan$  around 0.

**Solution:** If  $\sum_{k=0}^{\infty} a_k z^k = \tan(z)$  is the Taylor series of  $\tan$  around 0, then

$$\tan'(z) = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n .$$

$$\tan(z)^2 = \sum_{n=0}^{\infty} \left( \sum_{j=0}^n a_j a_{n-j} \right) z^n$$

“Comparing coefficients”, i.e. by uniqueness of the Taylor coefficients, this gives

$$(n+1) a_{n+1} = \sum_{j=0}^n a_j a_{n-j} .$$

18. Let  $(a_n)_{n \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}_0}$  be so that for all  $z \in B_{1/237}$  we have that

$$\sum_{k=0}^{\infty} a_k z^k = \frac{1}{\cos((1+i)z)} .$$

Compute  $\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}$ .

**Hint:** This is an attempted obfuscation. Do not compute the Taylor series of the right hand side!

**Solution:** The zeros of the denominator of the right hand side form the set

$$H = \left\{ \frac{\pi(2k+1)}{2(1+i)} \mid k \in \mathbb{Z} \right\} .$$

The points closest to 0 in  $H$  are  $\pm q := \pm \frac{\pi}{2(1+i)}$  and their distance from 0 is

$$r := \left| \frac{\pi}{2(1+i)} \right| = \frac{\pi}{2\sqrt{2}} .$$

By the corollary of the Cauchy integral theorem for the Taylor series of a holomorphic function, the series on the left hand side must be the Taylor series of the function  $f$  on the right hand side and its radius of convergence is at least  $r$ .

On the other hand, the radius of convergence can not be bigger than  $r$  because then  $f$  would extend holomorphically to  $q$ . But

$$\lim_{z \rightarrow q} f(z) = \lim_{z \rightarrow \pi/2} \frac{1}{\cos(z)} = \infty .$$

19. Let  $p \in U \stackrel{\text{open}}{\subset} \mathbb{C}$  so that

$$\forall u \in U, t \in [0, 1] : tu + (1 - t)p \in U .$$

Let  $f \in \mathcal{O}(U)$  and  $\gamma: [0, 1] \rightarrow U$  be a closed  $C^1$ -curve. Show that  $\oint_{\gamma} f(z) dz = 0$ .

**Hint:**  $\gamma$  is the main part of a rectangular domain. As in the case of the annulus, the contribution of the auxilliary curve cancels.

**Solution:** Let  $q = \gamma(0) = \gamma(1)$ . The map

$$\phi: Q \rightarrow U \quad , \quad \phi(t, s) = s\gamma(t) + (1 - s)p$$

is  $C^1$  and if  $\alpha$  denotes the boundary curve of  $Q$ , then  $\phi \circ \alpha$  is a reparametrization of the concatenation

$$\beta * \gamma^{-1} * \beta^{-1}$$

where  $\beta$  is the curve

$$\beta: [0, 1] \rightarrow U \quad , \quad \beta(t) = tq + (1 - t)p$$

lies in  $U$ . By the Cauchy Integral Theorem for rectangular domains,

$$0 = \oint_{\phi \circ \alpha} f(z) dz = \int_{\beta} f(z) dz - \int_{\gamma} f(z) dz - \int_{\beta} f(z) dz$$

20. Let  $f$  be an entire function and assume that for all  $z \in \mathbb{C}$  we have

$$|f(z)| \leq \ln(1 + |z|) .$$

Show that  $f(z) = 0$  for all  $z \in \mathbb{C}$ .

**Hint:** Look at the proof of Liouville's Theorem.

**Solution:** Since  $f$  is entire, we have

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

for all  $z \in \mathbb{C}$  and some sequence  $(c_n)_{n \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}_0}$ . By the estimate of the Taylor coefficients  $c_n$ , we have

$$|c_n| \leq \frac{\max \{|f(z)| \mid |z| = r\}}{r^n} \leq \frac{\ln(1 + r)}{r^n} \xrightarrow{r \rightarrow \infty} 0$$

if  $n > 0$ . It follows that  $f(z) = c_0$ . Since  $\ln(1) = 0$  we must have  $c_0 = 0$  as well.

21. Let  $\gamma: [-\pi, \pi] \rightarrow \mathbb{C}$  be the curve with

$$\gamma(t) = \cos(t) + i \left( \sin(t) + \frac{e^{-\sqrt{t^2+1}}}{16} \sin(43t) \right) .$$

Compute

$$\int_{\gamma} \frac{1}{\sin(z)} dz .$$

**Hint:** Sketch the curve. It is homotopic relative endpoint to a simpler one. Then use (??).

**Solution:** The integrand

$$f(z) = \frac{1}{\sin(z)}$$

is holomorphic on  $\mathbb{C} \setminus \pi\mathbb{Z}$ . The curve  $\gamma$  is homotopic to the unit circle in this domain via the homotopy

$$H(t, s) = e^{it} + si \frac{e^{-\sqrt{t^2+1}}}{16} \sin(43t) \quad , \quad (t, s) \in [-\pi, \pi] \times [0, 1] .$$

The function  $H$  is  $C^1$  and

$$H(t, 0) = e^{it} \quad , \quad H(t, 1) = \gamma(t) \quad , \quad H(-\pi, s) = H(\pi, s) = -1 ,$$

$H$  is a homotopy rel endpoint. To show that  $H$  is a homotopy in the domain where  $f$  is holomorphic, we estimate

$$1 - \frac{1}{16} \leq |H(t, s)| \leq 1 + \frac{1}{16}$$

for all  $(t, s) \in [-\pi, \pi] \times [0, 1]$  and therefore  $H(t, s) \notin \pi\mathbb{Z}$  for all such  $(t, s)$ . It thus remains to compute

$$\oint_{|z|=1} \frac{1}{\sin(z)} dz = \frac{2\pi i}{\sin'(0)} = 2\pi i$$

by (??).

22. Let  $U, V \subset \mathbb{C}$  be open and let  $f: U \rightarrow V$  and  $g: V \rightarrow \mathbb{C}$  be holomorphic. Let  $p \in U$ . Show that

$$\text{ord}(g \circ f; p) = \text{ord}(g; f(p)) \text{ord}(f; p)$$

**Hint:** Look at (??) or the local form theorem ??.

**Solution:** By (??) there are functions  $\tilde{f}$  and  $\tilde{g}$  holomorphic near  $p$  respectively  $f(p)$  with  $\tilde{f}(p) \neq 0$  and  $\tilde{g}(f(p)) \neq 0$  so that

$$\begin{aligned} f(z) &= f(p) + (z - p)^{\text{ord}(f;p)} \tilde{f}(z) \\ g(y) &= g(f(p)) + (y - f(p))^{\text{ord}(g;f(p))} \tilde{g}(y) \end{aligned}$$

for  $z, y$  in sufficiently small neighbourhoods of  $p$  and  $f(p)$  respectively. Thus

$$\begin{aligned} g(f(z)) &= g(f(p)) + \left( (z - p)^{\text{ord}(f;p)} \tilde{f}(z) \right)^{\text{ord}(g;f(p))} \tilde{g} \left( f(p) + (z - p)^{\text{ord}(f;p)} \tilde{f}(z) \right) \\ &= g(f(p)) + (z - p)^{\text{ord}(f;p) \text{ord}(g;f(p))} \underbrace{\tilde{f}(z)^{\text{ord}(g;f(p))} \tilde{g} \left( f(p) + (z - p)^{\text{ord}(f;p)} \tilde{f}(z) \right)}_{Q(z)} \end{aligned}$$

Clearly

$$Q(p) = \tilde{f}(p)^{\text{ord}(g;f(p))} \tilde{g}(f(p)) \neq 0 .$$



23. Let  $f \in \mathcal{O}(B_1)$  so that for all  $z \in B_1$  we have

$$\Re(f(z)) = \Im(f(z))^2.$$

Prove that  $f$  is constant.

**Hint:** Open Mapping Theorem

**Solution:** The assumption on  $f$  says that

$$f(B_1) \subset \{x + iy \mid x, y \in \mathbb{R}, y^2 = x\} =: P$$

But the set  $P$  is a parabola in  $\mathbb{R}^2 = \mathbb{C}$  and does not contain any nonempty open sets. Hence  $f(B_1)$  does not contain any nonempty open sets and therefore  $f$  can not be open. By the Open Mapping Theorem,  $f$  must be constant.

24. Classify the isolated singularities  $p \in S_i$  of the following functions  $f_i \in \mathcal{O}(\mathbb{C} \setminus S_i)$ ,  $S_i \subset \mathbb{C}$  discrete, as removable, pole, essential. In the case of poles determine the order.

(a)  $f_a(z) = \frac{e^z - 1 - z}{z^3(z-1)}$ ,  $S_a = \{0, 1, 2\}$

(b)  $f_b(z) = \frac{z^2 - 1}{\sin(\pi z)^2}$ ,  $S_b = \mathbb{Z}$

(c)  $f_c(z) = \frac{z^2 + 1}{1 + z + z^2 + z^3}$ ,  $S_c = \{1, i, -1, -i\}$

25. Assume that for all  $z \in B_{\frac{1}{345}}(2)$  we have

$$\sum_{n=-\infty}^{\infty} c_n z^n = \frac{\sin(z)}{z^2 + \frac{1}{4}} + \frac{e^z}{1 + 5i - z}.$$

Compute  $\limsup_{n \rightarrow \infty} \sqrt[n]{|c_n|}$  and  $\limsup_{n \rightarrow \infty} \sqrt[n]{|c_{-n}|}$ .

**Solution:** The function

$$f(z) = \frac{\sin(z)}{z^2 + \frac{1}{4}} + \frac{e^z}{1 + 5i - z}$$

has poles of order 1 at  $i/2, -i/2, 1 + 5i$  and no other singularities. Therefore the Laurent series around 0 converging at 2 converges on the largest annulus  $A_{r,R} \subset \mathbb{C} \setminus \{i/2, -i/2, 1 + 5i\}$ . Hence  $r = \frac{1}{2}$  and  $R = \sqrt{26}$ . The inner radius is the inverse of the radius of convergence of the principal part, hence

$$\limsup_{n \rightarrow \infty} \sqrt[n]{|c_{-n}|} = \frac{1}{2}.$$

The outer radius is the radius of convergence of the power series  $\sum_{n=0}^{\infty} c_n z^n$ , hence

$$\limsup_{n \rightarrow \infty} \sqrt[n]{|c_n|} = \frac{1}{\sqrt{26}}.$$

# Some problems for the study week

1. Determine the type of the singularity 0 of

$$f(z) = \frac{z(e^z - 1)^2}{(\cos(z) - 1)^2 \sin(z)^4 \tan(z)} .$$

**Solution:** In the sequel  $h_i$  will denote functions holomorphic in a neighbourhood of 0 with  $h_i(0) \neq 0$ . Inserting Taylor series, we can rewrite  $f(z)$  as

$$\begin{aligned} f(z) &= \frac{z \left( \sum_{k=1}^{\infty} \frac{z^k}{k!} \right)^2 \left( \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} \right)}{\left( \sum_{k=1}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} \right)^2 \left( \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!} \right)^5} \\ &= \frac{z(zh_1(z))^2 h_2(z)}{(z^2 h_3(z))^2 (zh_5(z))^5} = \frac{z^{1+2} h_1(z)^2 h_2(z)}{z^{2 \times 2 + 5} h_3(z)^2 h_5(z)^5} = z^{-6} h_6(z) \end{aligned}$$

hence 0 is a pole of order 6 of this function.

2. Determine the type of the singularity 0 of

$$f(z) = \frac{e^{1/z}}{\cosh(1/z)} .$$

**Solution:** Recall that  $\cosh(z) = \frac{e^z + e^{-z}}{2}$ , hence

$$f(z) = \frac{2e^{1/z}}{e^{1/z} + e^{-1/z}} = \frac{2e^{2/z}}{e^{2/z} - 1} .$$

The limit of this as  $z \rightarrow 0$  does not exist,

$$\lim_{z \rightarrow 0, z \in \mathbb{R}^+} f(z) = 2 \quad \neq \quad \lim_{z \rightarrow 0, z \in \mathbb{R}^-} f(z) = 0 ,$$

hence the singularity is essential.

3. Compute the complex line integral

$$\int_{\gamma} \frac{1}{(z-1)(z+3)} dz \quad \text{where } \gamma: [0, 1] \rightarrow \mathbb{C}, \gamma(t) = 2e^{it} + i \sin(\pi t) .$$

**Solution:** The integrand

$$f(z) := \frac{1}{(z-1)(z+3)} = \frac{1}{4} \left( \frac{1}{z-1} - \frac{1}{z+3} \right)$$

is holomorphic for  $z \in \mathbb{C} \setminus \{1, -3\}$ . Let  $H: [0, 1] \times [0, 1] \rightarrow \mathbb{C} \setminus \{1, -3\}$ ,

$$H(s, t) := 2e^{it} + is \sin(\pi t) .$$

This maps to  $\mathbb{C} \setminus \{1, -3\}$ , because  $|H(s, t)| = 2$  for all  $s, t$ . Then  $H(1, t) = \gamma(t)$  and  $H(0, t) = 2e^{it} = \mu(t)$ ,  $\mu$  the boundary curve of  $B_2$ . The second summand,  $\frac{1}{z-3}$  is holomorphic on  $B_2$ , hence contributes 0 to the closed line integral, by the Cauchy Integral Theorem. Thus

$$\begin{aligned} \int_{\gamma} \frac{1}{(z-1)(z+3)} dz &= \oint_{\gamma} \frac{1}{4} \left( \frac{1}{z-1} - \frac{1}{z+3} \right) dz = \oint_{\mu} \frac{1}{4} \left( \frac{1}{z-1} - \frac{1}{z+3} \right) dz \\ &= \oint_{\mu} \frac{1}{4} \frac{1}{z-1} dz = \oint_{|z|=2} \frac{1}{4} \frac{1}{z-1} dz = \frac{2\pi i}{4} = \frac{\pi i}{2} \end{aligned}$$

4. Let  $f \in \mathcal{O}(\mathbb{C})$  be so that  $f(1/z)$  has a removable singularity at  $z = 0$ . Show that  $f$  is constant.

**Solution:** Since 0 is a removable singularity of  $f(1/z)$ , the limit  $\lim_{z \rightarrow 0} f(1/z)$  exists. Thus there are  $L \in \mathbb{C}$  and  $\delta > 0$  so that for all  $z$  with  $|z| < \delta$  we have  $|f(1/z) - L| < 1$ , in particular,

$$|f(1/z)| < |L| + 1.$$

Since  $|z| < \delta$  if and only if  $|1/z| > 1/\delta$  we thus have

$$|f(z)| < |L| + 1 \quad \text{for all } z, |z| > \frac{1}{\delta}.$$

Since  $\overline{B_{1/\delta}}$  is compact and  $f$  entire, in particular continuous, the function  $f$  must be bounded. By Liouville's Theorem,  $f$  is constant.

5. Let  $f, g \in \mathcal{O}(\mathbb{C})$  so that  $f(g(z)) = 0$  for all  $z \in \mathbb{C}$ . Show that  $f$  or  $g$  is constant.

**Solution:** If  $f$  and  $g$  are both not constant, then both are open maps. Hence so is their composition.

6. Let  $f, g$  be entire functions so that  $f(g(z)) = p(z) \in \mathbb{C}[z]$  and assume that  $\deg p > 0$ . Show that  $f(z), g(z) \in \mathbb{C}[z]$ . What follows for the degrees of these polynomials?

**Solution:** Neither of the functions is constant since  $p$  is not constant. By Liouville's Theorem neither of the two functions can be bounded. Since  $p$  has no essential singularity at  $\infty$  neither  $f$  nor  $g$  have an essential singularity and thus must be polynomial and we have  $\deg p = \deg f \times \deg g$ .

7. Show that the function

$$f(z) := \begin{cases} e^{-1/z^4} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}.$$

satisfies the Cauchy-Riemann equations (??) on all of  $\mathbb{C}$  but is not complex differentiable at  $z = 0$ .

**Solution:** We have

$$\lim_{t \rightarrow 0, t > 0} \frac{f(tz) - f(0)}{t} = 0 \quad \text{for } z = 1, i, -1, -i$$

hence the partial derivatives in (??) are all 0. But  $f$  is not even continuous in 0. For instance,

$$\lim_{t \rightarrow 0, t > 0} f((1+i)t) = \lim_{t \rightarrow 0, t > 0} e^{+1/4t^4} = \infty$$

8. For  $n \in \mathbb{N}$ , compute  $\oint_{|z-1|=1} \left( \frac{z}{z-1} \right)^n dz$ .

**Solution:** Substituting  $z$  by  $z+1$  gives

$$\oint_{|z-1|=1} \left( \frac{z}{z-1} \right)^n dz = \oint_{|z|=1} \left( \frac{z+1}{z} \right)^n dz = \oint_{|z|=1} 1 + \frac{n}{z} + \binom{n}{2} \frac{1}{z^2} + \dots + \frac{1}{z^n} dz = 2\pi n$$

because  $\oint_{|z|=1} z^k dz = \begin{cases} 0 & \text{if } k \neq -1 \\ 2\pi i & \text{if } k = -1 \end{cases}.$

9. Determine all functions  $f \in \mathcal{O}(\mathbb{C} \setminus \{0\})$  for which 0 is not an essential singularity and which satisfy

$$f\left(\frac{1}{n+1}\right) = \frac{n+1}{n} f\left(\frac{1}{n}\right) \quad \text{for all } n \in \mathbb{N}. \quad (5.6)$$

**Solution:** If  $f$  is such a function, then for some  $k$ , 0 is a removable singularity of  $z^k f(z)$ . By the recursion formula (??), for  $n \in \mathbb{N}$ ,

$$f\left(\frac{1}{n}\right) = n f(1)$$

hence

$$z^k f(z)|_{z=1/n} = n^{1-k}$$

which is bounded only for  $k = 1$ . Thus the singularity at 0 must be a pole of order 1 and  $z f(z)$  is (extends to) an entire function with

$$z f(z) = f(1) \quad \text{for all } z \in \frac{1}{\mathbb{N}} \cup \{0\}.$$

The set  $\frac{1}{\mathbb{N}} \cup \{0\}$  contains an accumulation point and the two entire functions  $z f(z)$  and  $f(1)$  coincide on this set. By the identity theorem ?? the two functions must be equal. Thus  $z f(z) = f(1)$  for all  $z \in \mathbb{C}$ . It follows that the set of functions the problem asks for is

$$\{f_\lambda \mid \lambda \in \mathbb{C}\} \quad \text{where } f_\lambda(z) = \frac{\lambda}{z}.$$

10. Find all Laurent series centered at 2 of the function

$$f(z) = \frac{1}{z^2 + z - 12}.$$

**Solution:** The partial fraction decomposition

$$f(z) = \frac{1}{z^2 + z - 12} = \frac{1}{(z-3)(z+4)} = \frac{1/7}{z-3} - \frac{1/7}{z+4} \quad (5.7)$$

shows that  $f$  has poles of order 1 at  $z = 3$  and  $z = -4$ . The distances of these singularities from the center of the Laurent series we seek are 1 and 6. To get the Laurent series centered at 2 we rewrite (??) in terms of  $z - 2$  and use the geometric series

$$\frac{1}{1-q} = \frac{-1}{q} \frac{1}{1-\frac{1}{q}} = \begin{cases} \sum_{k=0}^{\infty} q^k & \text{if } |q| < 1 \\ -\sum_{k=1}^{\infty} q^{-k} & \text{if } |q| > 1 \end{cases}.$$

This gives

$$\begin{aligned} f(z) &= \frac{1/7}{(z-2)-1} - \frac{1/7}{(z-2)+6} \\ &= \frac{-1/7}{1-(z-2)} - \frac{1/42}{1-(-(z-2)/6)} \end{aligned}$$

$$\begin{aligned}
&= \begin{cases} \sum_{k=0}^{\infty} \left(\frac{-1}{7}\right) (z-2)^k - \sum_{k=0}^{\infty} \frac{1}{42} \left(\frac{-1}{6}\right)^k (z-2)^k & \text{if } |z-2| < 1 \\ \sum_{k=1}^{\infty} \left(\frac{1}{7}\right) (z-2)^{-k} - \sum_{k=0}^{\infty} \frac{1}{42} \left(\frac{-1}{6}\right)^k (z-2)^k & \text{if } 1 < |z-2| < 6 \\ \sum_{k=1}^{\infty} \left(\frac{1}{7}\right) (z-2)^{-k} + \sum_{k=1}^{\infty} \frac{1}{42} (-6)^k (z-2)^{-k} & \text{if } |z-2| > 6 \end{cases} \\
&= \begin{cases} \sum_{k=0}^{\infty} \left(\frac{-1}{7} - \frac{1}{42} \left(\frac{-1}{6}\right)^k\right) (z-2)^k & \text{if } |z-2| < 1 \\ \sum_{k=1}^{\infty} \left(\frac{1}{7}\right) (z-2)^{-k} - \sum_{k=0}^{\infty} \frac{1}{42} \left(\frac{-1}{6}\right)^k (z-2)^k & \text{if } 1 < |z-2| < 6 \\ \sum_{k=1}^{\infty} \left(\frac{1}{7} + \frac{1}{42} (-6)^k\right) (z-2)^{-k} & \text{if } |z-2| > 6 \end{cases}
\end{aligned}$$

# References

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## 6 Complex Numbers

We are looking for a field  $\mathbb{C}$  “containing” the field  $\mathbb{R}$  of real numbers (i.e. with an injective homomorphism of fields  $\mathbb{R} \hookrightarrow \mathbb{C}$ ), so that the equation “ $z^2 + 1 = 0$ ” has a solution. There are (at least) three constructions of such a field  $\mathbb{C}$  of complex numbers:

1. Gauss Plane:  $\mathbb{C}_1 = \mathbb{R}^2 = \{x, y \mid x, y \in \mathbb{R}\}$  with componentwise addition,  $(x, y) + (x', y') = (x + x', y + y')$  and multiplication given by

$$(x, y) \cdot (x', y') = (xx' - yy', xy + x'y) .$$

2. Field extension:  $\mathbb{C}_2 := \mathbb{R}[i]/\langle i^2 + 1 \rangle$ , i.e. the quotient of the ring of real polynomials in the variable  $i$  by the ideal generated by the polynomial  $i^2 + 1$ . Since this is irreducible, the quotient is a field.
3.  $2 \times 2$  matrices, orientation and angle preserving linear maps of  $\mathbb{R}^2$ :  $\mathbb{C}_3 = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$ .

All three  $\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3$  are fields and there are natural injective field homomorphisms  $\iota_i: \mathbb{R} \hookrightarrow \mathbb{C}_i$  given by

$$\mathbb{R} \ni x \mapsto (x, 0) \in \mathbb{R}^2 \quad , \quad \mathbb{R} \ni x \mapsto x \in \mathbb{R}[i] \text{ the constant polynomial} \quad , \quad \mathbb{R} \ni x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$$

respectively. The maps

$$(x, y) \leftrightarrow x + iy \leftrightarrow \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

are isomorphisms between the  $\mathbb{C}_i$  compatible with the embeddings  $\iota_i$ .

## Examples

$$\begin{aligned}(2 + 3i)^3 &= 8 + 4 \cdot (3i) + 2 \cdot (3i)^2 + (3i)^3 \\ &= 8 + 12i + 12i^2 + 27i^3 \\ &= 8 + 12i - 12 - 27i \\ &= -4 - 15i\end{aligned}$$

$$\begin{aligned}\frac{1}{a + bi} &= \frac{a - bi}{(a + bi)(a - bi)} \\ &= \frac{a - bi}{a^2 - b^2} \quad \text{denominator real!}\end{aligned}$$

Compare this with matrix inversion:

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}^{-1} = \frac{1}{\det \begin{pmatrix} a & -b \\ b & a \end{pmatrix}} \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

## 6.1 Real Part, Imaginary Part, Complex Conjugate, Polar Coordinates, Modulus, Argument

The cartesian coordinates of a complex number

$$z = (a, b) = a + ib = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

are called **real part** and **imaginary part** of  $z$ . If  $a, b \in \mathbb{R}$  we write

$$\Re(a + ib) = a \quad , \quad \Im(a + ib) = b .$$

The **modulus** of a complex number is its euclidean norm and the square root of its determinant,

$$|a + ib| = \sqrt{a^2 + b^2} = \sqrt{\det \begin{pmatrix} a & -b \\ b & a \end{pmatrix}} .$$

The **complex conjugate**  $\bar{z}$  of a complex number  $z$  is the its mirror image under reflection at the real axis, or its transpose, when viewed as a matrix,

$$\overline{a + ib} = a - ib .$$

From this it is immediate that for complex numbers  $u, v$  we have

$$\begin{aligned}|uv| &= |u| |v| \quad , \\ |u| &= \sqrt{u\bar{u}} \quad ,\end{aligned}$$

$$\begin{aligned}
u^{-1} &= \frac{1}{u} = \frac{\bar{u}}{|u|^2}, \\
\Re(u) &= \frac{u + \bar{u}}{2}, \\
\Im(u) &= \frac{u - \bar{u}}{2i}, \\
\Re(uv) &= \langle u \mid \bar{v} \rangle, \\
\Im(uv) &= \Re(-iuv) = \langle u \mid \overline{iv} \rangle,
\end{aligned}$$

where  $\langle \cdot \mid \cdot \rangle$  denotes the standard scalar product on  $\mathbb{R}^2$ .

We will use the following fact about the trigonometric functions  $\cos$  and  $\sin$ :

**Theorem 6.1** *The functions  $\cos, \sin: \mathbb{R} \rightarrow \mathbb{R}$  are  $2\pi$ -periodic. For every pair*

$$(a, b) \in \mathcal{S}^1 := \{z \in \mathbb{C} \mid |z| = 1\} = \{(a, b) \in \mathbb{R}^2 \mid a^2 + b^2 = 1\}$$

there is  $t \in \mathbb{R}$  so that

$$\cos(t) = a \quad \text{and} \quad \sin(t) = b$$

and  $t$  is determined by  $a, b$  up to an integer multiple of  $2\pi$ . Thus the map

$$\mathbb{R}/2\pi\mathbb{Z} \xrightarrow{t \mapsto \cos(t) + i \sin(t)} \mathcal{S}^1$$

is bijective.

We write  $e^{it} := \cos(t) + i \sin(t)$  (in [?] this is called  $\text{cis}(t)$ ). Thus every complex number  $z$  can be written as

$$z = r e^{it} \quad \text{with} \quad r = |z| \in \mathbb{R}_0^+ \quad \text{and} \quad t \in \mathbb{R}.$$

A real number  $t$  so that  $z = r e^{it}$  is called an **argument** for  $z$ , denoted by  $\arg(z) \in \mathbb{R}/2\pi\mathbb{Z}$ .

The exponential notation is justified by the functional equation: The addition theorems for  $\cos$  and  $\sin$  turn the map

$$\mathbb{R} \rightarrow \mathcal{S}^1, \quad t \mapsto e^{it} = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$$

into a group homomorphism,

$$e^{i(t+s)} = e^{it} e^{is}.$$

## 7 Series

If  $A = (a_n)_{n \in \mathbb{N}_0} \in \mathbb{C}^{\mathbb{N}_0}$  or  $\mathbb{R}^{\mathbb{N}_0}$  is a sequence of real or complex numbers, then the sequence  $S$  of its partial sums is the sequence

$$S = \left( \sum_{n=0}^k a_n \right)_{k \in \mathbb{N}_0} \quad \text{i.e. the sequence}$$



$$(a_0, a_0 + a_1, a_0 + a_1 + a_2, a_0 + a_1 + a_2 + a_3, a_0 + a_1 + a_2 + a_3 + a_4, \dots)$$

If  $S$  converges we say “the series  $\sum_{n=0}^{\infty} a_n$  converges”, and write

$$\sum_{n=0}^{\infty} a_n := \lim_{k \rightarrow \infty} \sum_{n=0}^k a_n .$$

If this limit does not exist, then we say “the series  $\sum_{n=0}^{\infty} a_n$  diverges”.

A power series  $\sum_{n=0}^{\infty} a_n$  **converges absolutely** if the power series of the moduli  $\sum_{n=0}^{\infty} |a_n|$  converges.

## Examples

The **geometric series**  $\sum_{n=1}^{\infty} q^n$  converges for  $|q| < 1$  and diverges if  $|q| \geq 1$ . To see this, first note that the series diverges if  $|q| \geq 1$  because then the sequence does not converge to 0.

For  $q \neq 1$  we have the formula

$$\sum_{n=0}^k q^n = \frac{1 - q^{k+1}}{1 - q} .$$

In case  $|q| < 1$  we get

$$\sum_{n=0}^k q^n = \frac{1 - q^{k+1}}{1 - q} \xrightarrow{k \rightarrow \infty} \frac{1}{1 - q}$$

Hence

$$\sum_{n=0}^{\infty} q^n = \lim_{k \rightarrow \infty} \sum_{n=0}^k q^n = \lim_{k \rightarrow \infty} \frac{1 - q^{k+1}}{1 - q} = \frac{1}{1 - q}$$

The **Euler number** is

$$e = \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n} \right)^n = \sum_{k=0}^{\infty} \frac{1}{k!}$$

The **harmonic series** does not converge,

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{> \frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{> \frac{1}{2}} + \dots = +\infty .$$

The **alternating harmonic series** converges but not absolutely, see below.

## 7.1 Convergence Criteria:

Let  $(a_n)_{n \in \mathbb{N}_0}$  be a sequence of real or complex numbers. Then the series  $\sum_{n=0}^{\infty} a_n$  converges if any of the following criteria holds:

1. **Comparison Test:** There is a **convergent** series  $\sum_{n=0}^{\infty} b_n$ , with

$$|a_n| \leq b_n \in \mathbb{R} \quad \text{for all } n \in \mathbb{N} .$$

**Example:** Since  $\left| \cos(n^3)e^{-n^2+in^3} \right| \leq 1$ , we have

$$\left| \frac{\cos(n^3)e^{-n^2+in^3}}{2^n} \right| \leq \frac{1}{2^n} \quad \text{for all } n \in \mathbb{N} .$$

By comparison with the geometric series  $\sum_{n=0}^{\infty} \frac{1}{2^n}$ , the series

$$\sum_{n=0}^{\infty} \frac{\cos(n^3)e^{-n^2+in^3}}{2^n} .$$

converges.

2. **Quotient Test:** There is  $q \in [0, 1)$  and  $N \in \mathbb{N}$  so that for all  $n \geq N$  we have

$$\left| \frac{a_{n+1}}{a_n} \right| \leq q .$$

**Example:** If the quotient is small the series converges fast. For example, for any  $q > 0$ , we can choose  $N \in \mathbb{N}$  so that  $Nq > 1$ . Then for all  $n > N$ ,

$$\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \frac{1}{n+1} < q .$$

By the Quotient Test, the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} =: e \quad \text{converges, giving one definition of the Euler number.}$$

3. **Root Test:** There is  $q \in [0, 1)$  and  $N \in \mathbb{N}$  so that for all  $n \geq N$  we have

$$\sqrt[n]{|a_n|} \leq q .$$

4. **Leibniz Criterion, Alternating Series Test:** We have  $\lim_{n \rightarrow \infty} a_n = 0$  and there is  $N \in \mathbb{N}$  so that for all  $n \geq N$ ,  $a_n \in \mathbb{R}$  and

$$\text{sgn}a_{n+1} = -\text{sgn}a_n .$$

The sign of a real number  $\lambda$  is

$$\operatorname{sgn}\lambda = \begin{cases} +1 & \text{if } \lambda > 0 \\ 0 & \text{if } \lambda = 0 \\ -1 & \text{if } \lambda < 0 \end{cases}$$

**Example:** Recall that the harmonic series  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges. However, since  $\frac{1}{n} \xrightarrow{n \rightarrow \infty} 0$ , the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \quad \text{converges.}$$

5. **Integral Test:** There is a decreasing Riemann integrable function  $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  with  $a_n = f(n)$  for all  $n \in \mathbb{N}$ , and for some  $x \in \mathbb{R}^+$  the integral

$$\int_x^{\infty} f = \lim_{y \rightarrow \infty} \int_x^y f \quad \text{exists.}$$

A function  $f: \mathbb{R} \rightarrow \mathbb{R}$  is decreasing if  $f(x) \geq f(y)$  for all  $x, y \in \mathbb{R}$  with  $x \leq y$ .

**Example:** Let  $s \in \mathbb{R}$ . Recall that for any positive real number  $n$ , the power  $n^s$  is defined as

$$n^s := e^{s \ln(n)}.$$

Consider the function  $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ ,  $f(x) = x^s$ , and its integral for  $0 < x < y$ ,

$$\int_x^y n^s ds = \begin{cases} \ln(y) - \ln(x) & \text{if } s = -1 \\ \frac{y^{s+1} - x^{s+1}}{s+1} & \text{if } s \neq -1 \end{cases} \xrightarrow{y \rightarrow \infty} \begin{cases} \infty & \text{if } s \geq -1 \\ \frac{-x^{s+1}}{s+1} & \text{if } s \leq -1 \end{cases}.$$

By the Integral Test, the series

$$\sum_{n=1}^{\infty} n^s \quad \text{converges for real } s < -1$$

A necessary condition for the convergence of a series  $\sum_{n=1}^{\infty} a_n$  is that the sequence  $(a_n)_{n \in \mathbb{N}}$  converges to 0.

This rules out convergence of many sequences! Thus none of the series

$$\sum_{n=1}^{\infty} \frac{n^2 + 4}{3n + 2n^2}, \quad \sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n, \quad \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{17}\right)$$

converge because

$$\lim_{n \rightarrow \infty} \frac{n^2 + 4}{3n + 2n^2} = \frac{1}{2} \neq 0,$$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e \neq 0,$$

$$\lim_{n \rightarrow \infty} \sin\left(\frac{n\pi}{17}\right) \quad \text{does not exist.}$$

## 8 Power Series

A power series is (function in  $z$  given by) a series with a parameter, say  $z$ , of the form

$$f(z) = \sum_{k=0}^{\infty} a_n z^n .$$

This concept generalizes that of polynomial

$$p(z) = \sum_{k=0}^N a_n z^n \quad , \quad N = \text{degree } p \text{ if } a_N \neq 0 .$$

**Theorem 8.1 (Convergence of Power Series)** *Let  $(a_n)_{n \in \mathbb{N}_0} \in \mathbb{C}^{\mathbb{N}_0}$ . The series  $\sum_{n=0}^{\infty} a_n z^n$  converges*

*absolutely if  $|z| < \frac{1}{\limsup \sqrt[n]{|a_n|}} =: \rho$  and diverges if  $|z| > \rho$ .*

Note that this says nothing about convergence on the circle  $|z| = \rho$ . The number  $\rho$  above is called **radius of convergence** of the series.

**Proof:** If  $\sum_{n=0}^{\infty} a_n z^n$  converges, then we must have

$$\lim_{n \rightarrow \infty} a_n z^n = 0 \quad \implies \quad \limsup_{n \rightarrow \infty} |a_n z^n| = 0 \quad \implies \quad \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n z^n|} \leq 1 \quad \implies \quad |z| \leq \frac{1}{\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}} =: \rho$$

which must be read as  $\infty$  if the denominator vanishes. Thus the series diverges if  $|z| > \rho$ .

Conversly if  $|z| < \rho$ , then

$$\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n z^n|} = |z| \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} < 1$$

and there is  $q \in [0, 1)$  and  $N \in \mathbb{N}$  so that

$$\forall n > N : |a_n z^n| < q^n .$$

Since  $0 \leq q < 1$  the geoemtric series  $\sum_{j=0}^{\infty} q^j$  converges absolutely. The comparison test yields absolute

convergence of  $\sum_{n=0}^{\infty} a_n z^n$ . •

If  $M$  is a set and  $X$  a metric space with distance function  $d: X \times X \rightarrow \mathbb{R}_0^+$ , then we have (and will usually refer to) the almost distance  $d_\infty$  on the set  $X^M$  of functions  $f: M \rightarrow X$  given by

$$d_\infty(f, g) = \sup \{d(f(m), g(m)) \mid m \in M\} \in \mathbb{R}_0^+ \cup \{\infty\} .$$

Convergence with respect to  $d_\infty$  is called **uniform convergence**.

**Definition 8.2** A sequence  $(f_n)_{n \in \mathbb{N}_0} \in (X^M)^{\mathbb{N}_0}$  of functions  $f_n: M \rightarrow X$  from a set  $M$  to a metric space  $(X, d)$  converges uniformly to  $g \in X^M$  if

$$\lim_{n \rightarrow \infty} d_\infty(f_n, g) = 0 ,$$

i.e. more explicitly

$$\forall \epsilon > 0 \exists N_\epsilon \in \mathbb{N} \forall n \in \mathbb{N}, n > N, m \in M : d(f_n(m), g(m)) < \epsilon .$$

We also recall that this construction gives rise to two basic complete metric spaces:

**Theorem 8.3** If  $(X, d)$  is complete, then  $(X^M, d_\infty)$  is complete.

If  $M$  is a topological space, then the subspace  $C(M, X) = \{f: M \rightarrow X \mid f \text{ continuous}\}$  is complete with respect to  $d_\infty$ .

In particular, a locally uniformly convergent sequence of continuous functions from a topological space to  $\mathbb{R}$  or  $\mathbb{C}$  has a continuous limit.

We apply this to power series.

**Theorem 8.4** A power series  $\sum_{n=0}^{\infty} a_n z^n$  with radius of convergence  $\rho$  converges uniformly on  $B_\tau = \{z \in \mathbb{C} \mid |z| < \tau\}$  for every  $\tau < \rho$ . In particular, the function  $f$  defined by  $f(z) := \sum_{n=0}^{\infty} a_n z^n$  is continuous.

**Proof:** For  $|z| < \tau < \rho$ , we have  $\limsup_{n \rightarrow \infty} |a_n z^n| \leq \limsup_{n \rightarrow \infty} |a_n| \tau^n < q^n$  for some  $q \in [0, 1)$ . Hence there is  $N \in \mathbb{N}$  so that for all  $n > N$  we have  $|a_n z^n| < q^n$ . If  $g$  denotes the limit and  $f_n$  the partial sums of the series,

$$g(z) = \sum_{n=0}^{\infty} a_n z^n \quad , \quad f_k(z) = \sum_{n=0}^k a_n z^n \quad \text{on } B_\tau$$

then

$$d_\infty(g, f_k) = \sup_{|z| < \tau} \left| \sum_{n=0}^{\infty} a_n z^n - \sum_{n=0}^k a_n z^n \right| \leq \sup_{|z| < \tau} \sum_{n=k+1}^{\infty} |a_n z^n| .$$

By the formula for the geometric series we can estimate this,

$$\sup_{|z| < \tau} \sum_{n=k+1}^{\infty} |a_n z^n| \leq \sum_{n=k+1}^{\infty} q^n = q^{k+1} \sum_{n=0}^{\infty} q^n = \frac{q^{k+1}}{1-q} \xrightarrow{k \rightarrow \infty} 0 .$$

•

## 9 Complex Differentiability

**Definition 9.1** Let  $U \subset \mathbb{C}$  be an open subset,  $p \in U$  and  $f: U \rightarrow \mathbb{C}$  a function. Then  $f$  is **differentiable at  $p$**  if the limit

$$f'(p) := \lim_{z \rightarrow p, z \in U} \frac{f(z) - f(p)}{z - p} \quad (9.2)$$

exists. If  $f$  is complex differentiable at all  $p \in U$ , then  $f$  is called **holomorphic on  $U$** .  $f'$  the **derivative** of  $f$  and  $f$  an **antiderivative** of  $f'$ .

If  $U$  is not open, then we will say that a function  $f$  is holomorphic on  $U$  if it has a holomorphic extension to some open neighbourhood of  $U$ .

We denote the set of all holomorphic functions  $U \rightarrow \mathbb{C}$  by  $\mathcal{O}(U)$ .

Note that the quotient in (9.2) is with respect to complex multiplication.

**Theorem 9.3 (Sums, products, quotients, compositions)** If  $f, g \in \mathcal{O}(U)$  then the sum  $f + g$ , the product  $fg$  and, if  $g$  has no zeros on  $U$ , also the quotient  $f/g$  are complex differentiable and their derivatives are given by the formulae

$$\begin{aligned} (f + g)' &= f' + g' \\ (fg)' &= f'g + fg' \\ \left(\frac{f}{g}\right)' &= \frac{f'g - fg'}{g^2} \end{aligned}$$

If  $V \subset \mathbb{C}$  is open,  $h \in \mathcal{O}(V)$  and  $f(U) \subset V$ , then the composition  $h \circ f$  is holomorphic and we have the **Chain Rule**

$$(h \circ f)' = h' \circ f \cdot f'$$

i.e. for all  $u \in U$ ,  $(h \circ f)'(u) = h'(f(u))f'(u)$ .

### Examples

Obviously, the constant functions are holomorphic, with derivative 0. The function  $f$  with  $f(z) = z$  has derivative  $f'(z) = 1$ . More generally, every polynomial  $p \in \mathbb{C}[z]$  with complex coefficients is holomorphic. If

$$p(z) = \sum_{i=0}^n a_n z^n \quad \text{for all } z \in \mathbb{C}$$

then

$$p'(z) = \sum_{i=1}^n n a_n z^{n-1} \quad \text{for all } z \in \mathbb{C}.$$

**Problem 9.4** Show that this follows from Theorem ??, you do not need to go back to the definition.

Complex conjugation, i.e. the function  $c$  with  $c(z) = \bar{z}$  is nowhere complex differentiable because for all  $p \in \mathbb{C}$  the limit

$$\lim_{z \rightarrow p} \frac{\bar{z} - \bar{p}}{z - p} = \lim_{z \rightarrow 0} \frac{\bar{z}}{z}$$

does not exist, for instance because

$$\lim_{\substack{t \rightarrow 0 \\ t \in \mathbb{R}}} \frac{\overline{tv}}{tv} = \frac{\bar{v}}{v}$$

depends on  $v$ .

## 9.1 Comparison with Real Differentiation, Cauchy-Riemann Equations

The derivative of a function  $f: \mathbb{R}^m \rightarrow \mathbb{R}^k$  at  $p \in \mathbb{R}^m$  is the linear map

$$d_p f: \mathbb{R}^m \rightarrow \mathbb{R}^k \quad \text{so that} \quad f(p+h) = f(p) + d_p f \cdot h + R_p(h)$$

holds for all  $h \in \mathbb{R}^m$  with some function  $R_p: \mathbb{R}^m \rightarrow \mathbb{R}^k$  such that

$$\lim_{h \rightarrow 0} \frac{R_p(h)}{\|h\|} = 0.$$

Recall that the matrix coefficients of the differential  $d_p f$  are the partial derivatives  $\frac{\partial f_i}{\partial x_j}$ . If the partial derivatives exist in a neighbourhood of  $p$  and are continuous at  $p$ , then  $f$  is differentiable at  $p$ . The existence of the partial derivatives at  $p$  only does not suffice.

Let  $U \subset \mathbb{C}$  be an open set,  $p \in U$  and  $f: U \rightarrow \mathbb{C}$  a function. Under the identification  $\mathbb{C}$  as  $\mathbb{R}^2$ ,

$$\mathbb{C} \ni z = x + iy \leftrightarrow (\Re z, \Im z) = (x, y) \in \mathbb{R}^2$$

we can write  $f$  as a pair of real valued functions  $u, v: \mathbb{R}^2 \rightarrow \mathbb{R}$ , defined by

$$f(x + iy) = u(x + iy) + iv(x + iy) = (u(x, y), v(x, y)) \quad \text{for } x, y \in \mathbb{R}.$$

The derivative of  $f$  at a point  $p$  is now (given by) the matrix of partial derivatives

$$d_p f = \begin{pmatrix} \frac{\partial}{\partial u} x(p) & \frac{\partial}{\partial v} x(p) \\ \frac{\partial}{\partial u} y(p) & \frac{\partial}{\partial v} y(p) \end{pmatrix} =: \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}. \quad (9.5)$$

The function is complex differentiable at  $p$  if the matrix in (9.5) represents a complex number, i.e. is of the form  $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ . Thus we have proved:

**Theorem 9.6 (Cauchy-Riemann Equations)** Let  $U \subset \mathbb{C}$  be open and  $u, v: U \rightarrow \mathbb{R}$  and let  $f: U \rightarrow \mathbb{C}$  be given by

$$f(x + iy) = u(x + iy) + iv(x + iy) \quad \text{for all } x + iy \in U.$$

Then  $f$  is complex differentiable on  $U$  if and only if the component functions  $u, v$  of  $f$  are continuously partially differentiable on  $U$  and satisfy the partial differential equations

$$u_x = v_y \quad \text{and} \quad u_y = -v_x. \quad (9.7)$$

The pointwise version of this is false, see problem ??.

## 9.2 Harmonic Functions

If the second partial derivatives are continuous, we can eliminate one of the component functions of  $f$  by inserting one of the equations in (??) in the other and interchanging the order of differentiation,

$$u_{xx} = v_{yx} = v_{xy} = -u_{yy} \quad , \quad v_{xx} = -u_{yx} = -u_{xy} = -v_{yy}$$

hence

$$\Delta u = \Delta v = 0 \quad ,$$

where  $\Delta: C^2(\mathbb{R}^2) \rightarrow C(\mathbb{R}^2)$  is the Laplace operator on  $\mathbb{R}^2$ ,

$$\Delta u = \text{trace } d^2u = u_{xx} + u_{yy} \quad .$$

Functions in the kernel of  $\Delta$  are called **harmonic**.

## 9.3 The Derivative of Power Series

Differentiation of power series is formal,

**Theorem 9.8** *Let  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  be a power series with radius of convergence  $\rho$ , i.e. convergent on  $B_\rho$ .*

*Then the derivative of  $f$  exists on  $B_\rho$  and is given there by*

$$f'(z) = \sum_{n=0}^{\infty} (n+1)a_{n+1}z^n \quad . \quad (9.9)$$

*The power series for  $f'$  has the same radius of convergence  $\rho$ .*

**Proof:** The power series (??) has the same radius of convergence because  $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$ , hence

$$\limsup_{n \rightarrow \infty} \sqrt[n]{|na_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{n} \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}$$

For  $p \in B_\rho$  and  $h \in \mathbb{C}$  so that  $|p| + |h| \in B_\rho$  we will show that

$$\lim_{h \rightarrow 0} \frac{\sum_{n=0}^{\infty} a_n (p+h)^n - a_n p^n}{h} - \sum_{n=1}^{\infty} n a_n p^{n-1} = 0 \quad .$$



To this end we estimate

$$\begin{aligned}
& \frac{\sum_{n=0}^{\infty} a_n(p+h)^n - a_n p^n}{h} - \sum_{n=1}^{\infty} n a_n p^{n-1} = \frac{\sum_{n=1}^{\infty} a_n(p+h)^n - a_n p^n}{h} - \sum_{n=1}^{\infty} n a_n p^{n-1} \\
&= \frac{\sum_{n=1}^{\infty} a_n(p+h)^n - a_n p^n - n a_n p^{n-1} h}{h} = \frac{\sum_{n=1}^{\infty} a_n(p+h)^n - a_n p^n - n a_n p^{n-1} h}{h} \\
&= \frac{\sum_{n=2}^{\infty} a_n(p+h)^n - a_n p^n - n a_n p^{n-1} h}{h} = \frac{\sum_{n=2}^{\infty} a_n \sum_{j=2}^n \binom{n}{j} p^{n-j} h^j}{h} \\
&= \sum_{n=2}^{\infty} a_n \sum_{j=2}^n \binom{n}{j} p^{n-j} h^{j-1} = h \sum_{n=2}^{\infty} a_n \sum_{j=2}^n \binom{n}{j} p^{n-j} h^{j-2} = h \sum_{n=0}^{\infty} a_{n+2} \sum_{j=2}^{n+2} \binom{n+2}{j} p^{n+2-j} h^{j-2} \\
&= h \underbrace{\sum_{n=0}^{\infty} a_{n+2} \sum_{j=0}^n \binom{n+2}{j+2} p^{n-j} h^j}_{S} = h P(h) \xrightarrow{h \rightarrow 0} 0
\end{aligned}$$

In order to show this convergence we now estimate the binomial coefficient

$$\binom{n+2}{j+2} = (n+2)(n+1) \frac{n!}{(j+2)!(n-j)!} \leq (n+2)(n+1) \frac{n!}{j!(n-j)!} = (n+2)(n+1) \binom{n}{j}$$

and thus

$$|S| \leq \sum_{j=0}^n \binom{n+2}{j+2} |p|^{n-j} |h|^j \leq \sum_{j=0}^n (n+2)(n+1) \binom{n}{j} |p|^{n-j} |h|^j = (n+2)(n+1)(|p| + |h|)^n.$$

For  $P$  above this gives

$$|P(h)| = \left| \sum_{n=0}^{\infty} a_{n+2} \sum_{j=0}^n \binom{n+2}{j+2} p^{n-j} h^j \right| \leq \sum_{n=0}^{\infty} |a_{n+2}| (n+2)(n+1)(|p| + |h|)^n = g(|p| + |h|),$$

where

$$g(z) = \sum_{n=0}^{\infty} (n+2)(n+1) |a_n| z^n.$$

This power series also has radius of convergence  $\rho$ . By Theorem ??,  $g$  is continuous on  $B_\rho$ , hence  $P$  is bounded and the limit above vanishes. •

## Examples

### 1. Exponential Function

$$\exp(z) = e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

## 2. Trigonometric Functions

$$\sin(z) := \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \quad \text{and} \quad \cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

proves immediately that

$$e^{iz} = \cos(z) + i \sin(z) \quad \text{for all } z \in \mathbb{C} .$$

$$\exp' = \exp$$

$$\cos' = -\sin \quad , \quad \sin' = \cos$$

## 3. Hyperbolic Functions

$$\sinh(z) := \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \quad \text{and} \quad \cosh(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$$

proves immediately that

$$e^z = \cosh(z) + \sinh(z) \quad \text{for all } z \in \mathbb{C} .$$

$$\cosh' = \sinh \quad , \quad \sinh' = \cosh$$

4. Logarithm: On  $\mathbb{R}^+$  this is defined by  $\ln'(z) = \frac{1}{z}$  and  $\ln(1) = 0$ . Since

$$\frac{1}{1-h} = \sum_{n=0}^{\infty} h^n \quad \text{for } |h| < 1$$

we have

$$\frac{1}{z} = \sum_{n=0}^{\infty} (1-z)^n = \sum_{n=0}^{\infty} (-1)^n (z-1)^n \quad \text{for } |z| < 1 .$$

Integration gives

$$\ln(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (z-1)^{n+1} . \tag{9.10}$$

For positive real  $z, a$  we get

$$\ln(z) - \ln(a) = \ln\left(\frac{z}{a}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \left(\frac{z}{a} - 1\right)^{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)a^{n+1}} (z-a)^{n+1}$$

which converges for all  $z$ ,  $|z| < a$ . Since any  $z \in \mathbb{C}$  with  $\Re(z) > 0$  lies in  $B_a(a)$  for  $a \in \mathbb{R}^+$  sufficiently large, this formula defines an extension of  $\ln$  from  $\mathbb{R}^+$  to the right half plane  $\{z \in \mathbb{C} \mid \Re z > 0\}$ .

5. For  $a \in \mathbb{C}$  we can define

$$z^a = e^{a \ln(z)}$$

where  $\ln$  is defined, for instance on the right half plane. Thus, for  $|z| < 1$ ,

$$(1+z)^a = \exp\left(-a \sum_{k=1}^{\infty} \frac{(-z)^k}{k}\right) = \sum_{n=0}^{\infty} \frac{(-a)^n}{n!} \left(\sum_{k=1}^{\infty} \frac{(-z)^k}{k}\right)^n$$

# 10 The Cauchy Integral Theorem

**Definition 10.1** A curve or path (in  $\mathbb{C}$ ) is a continuous map  $\gamma: [a, b] \rightarrow \mathbb{C}$  from an interval  $[a, b] \subset \mathbb{R}$  to  $\mathbb{C}$ ,  $a, b \in \mathbb{R} \cup \{\pm\infty\}$ ,  $a \leq b$ .

A loop or closed path/curve is a curve  $\gamma: [a, b] \rightarrow \mathbb{C}$ ,  $a, b \in \mathbb{R}$ , with  $\gamma(a) = \gamma(b)$ .

The loop  $\gamma$  is called a **simple closed curve** if  $\gamma$  is injective except at the endpoints, i.e. if  $\gamma(t) = \gamma(t')$  only for  $t = t'$  or  $\{t, t'\} = \{a, b\}$ .

A curve  $\gamma: [a, b] \rightarrow \mathbb{C}$ ,  $a, b \in \mathbb{R}$ , is **piecewise continuously differentiable** if there are  $t_i \in [a, b]$ ,  $i = 1, \dots, n$ ,

$$a = t_0 < t_1 < t_2 < \dots < t_{n-1} < t_n = b \quad \text{so that}$$

1. the restrictions  $\gamma|_{[t_{i-1}, t_i]}$  are continuously differentiable, and
2.  $\gamma$  is continuous.

If  $\gamma([a, b]) \subset U \subset \mathbb{C}$  and  $f \in C(U, \mathbb{C})$ , then the **complex line integral of  $f$  over  $\gamma$**  is the Riemann integral

$$\int_{\gamma} f(z) dz := \int_a^b f(\gamma(t)) \gamma'(t) dt .$$

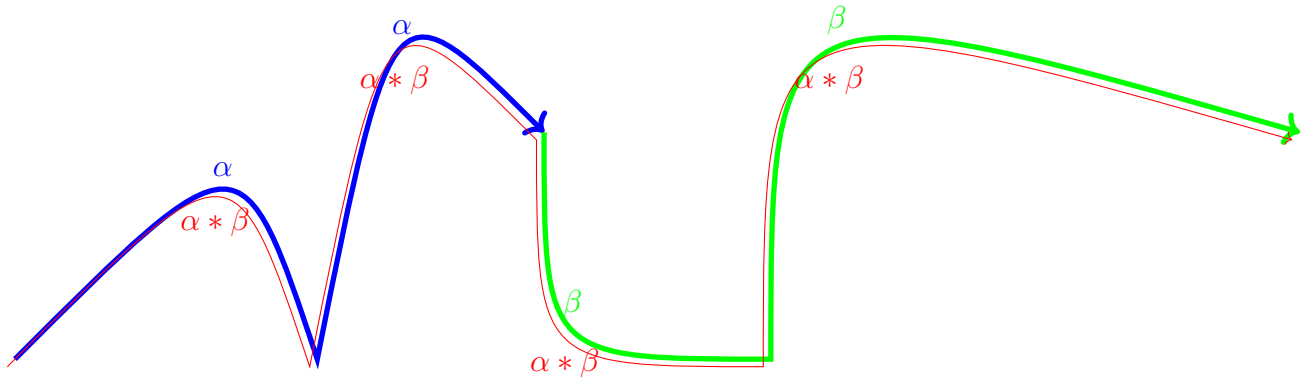
By the transformation formula the complex line integral is invariant under orientation preserving reparametrisation of the curve. Thus, if  $\phi: [u, v] \rightarrow [a, b]$  is a diffeomorphism, then

$$\begin{aligned} \int_{\gamma \circ \phi} f(z) dz &= \int_u^v f(\gamma(\phi(s))) (\gamma \circ \phi)'(s) ds = \int_u^v f(\gamma(\phi(s))) \gamma'(\phi(s)) \phi'(s) ds \\ &= \int_{\phi(u)}^{\phi(v)} f(\gamma(t)) \gamma'(t) dt = \pm \int_a^b f(\gamma(t)) \gamma'(t) dt = \pm \int_{\gamma} f(z) dz \end{aligned}$$

depending on whether  $\phi$  preserves the orientation, i.e.  $\phi' > 0$  and  $\phi(u) = a, \phi(v) = b$ , or else  $\phi$  reverses the orientation,  $\phi' < 0$  and  $\phi(u) = b, \phi(v) = a$ .

The complex line integral is additive. Thus if  $\alpha: [a, b] \rightarrow \mathbb{C}$ ,  $\beta: [c, d] \rightarrow \mathbb{C}$  are piecewise  $C^1$  curves so that  $\alpha(b) = \beta(c)$ , then the **concatenation** of the two curves is the curve

$$\alpha * \beta: [a, b + d - c] \rightarrow \mathbb{C} \quad \text{with} \quad \alpha * \beta(t) = \begin{cases} \alpha(t) & \text{if } t \in [a, b] \\ \beta(c + t - b) & \text{if } t \in [b, b + d - c] \end{cases}$$



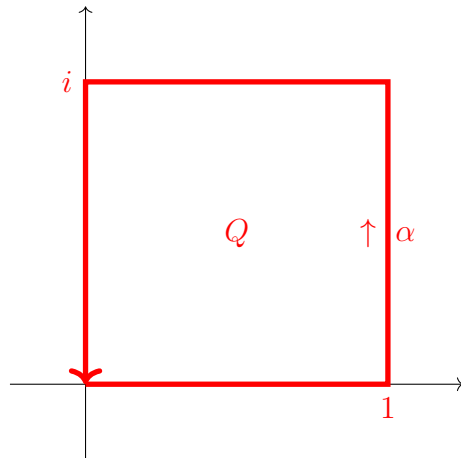
Concatenation of Curves

If  $f: \mathbb{C} \rightarrow \mathbb{C}$  continuous, then

$$\int_{\alpha * \beta} f(z) dz = \int_{\alpha} f(z) dz + \int_{\beta} f(z) dz .$$

We will often need the boundary curve of a rectangle, here the unit square:

$$\alpha: [0, 1] \rightarrow \mathbb{C} \quad , \quad \alpha(t) = \begin{cases} 4t & \text{if } 0 \leq t \leq \frac{1}{4} \\ 1 + i(4t - 1) & \text{if } \frac{1}{4} \leq t \leq \frac{1}{2} \\ 3 - 4t + i & \text{if } \frac{1}{2} \leq t \leq \frac{3}{4} \\ i(4 - 4t) & \text{if } \frac{3}{4} \leq t \leq 1 \end{cases} \quad (10.2)$$



Standard Square

We will first prove the

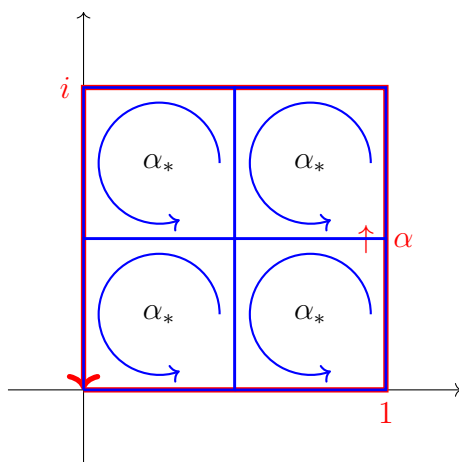
**Theorem 10.3 (Cauchy Integral Theorem for Rectangles)** Let  $\alpha$  be the curve given in (??) and let  $U \subset \mathbb{C}$  be an open set containing the square  $Q = [0, 1] + i[0, 1]$  surrounded by  $\alpha$ . Then for all  $f \in \mathcal{O}(U)$

$$\oint_{\alpha} f(z) dz = 0 .$$

**Proof:** For  $z \in \mathbb{C}$  and  $r \in \mathbb{R}_0^+$  let  $\alpha_{z,r}(t) = z + r\alpha(t)$ . Subdividing the square  $Q$  in four subsquares

$$\frac{1}{2}Q \quad , \quad \frac{1}{2} + \frac{1}{2}Q \quad , \quad \frac{i}{2} + \frac{1}{2}Q \quad , \quad \frac{1+i}{2} + \frac{1}{2}Q \quad (10.4)$$

with anti clockwise boundary curves  $\alpha_{0,1/2}$ ,  $\alpha_{1/2,1/2}$ ,  $\alpha_{(1+i)/2,1/2}$  and  $\alpha_{i/2,1/2}$



Interior arcs cancel

we see that the inner arcs of the  $\alpha_*$  cancel. Thus

$$\oint_{\alpha} f(z) dz = \oint_{\alpha_{0,1/2}} f(z) dz + \oint_{\alpha_{1/2,1/2}} f(z) dz + \oint_{\alpha_{i/2,1/2}} f(z) dz + \oint_{\alpha_{(1+i)/2,1/2}} f(z) dz .$$

Among the four subsquares in (??) let  $Q'$  be the one with maximal absolute value of the boundary integral.

Iterating this subdivision/choice gives a sequence  $(Q_n)_{n \in \mathbb{N}}$  of subsquares of  $Q$  starting with  $Q_0 = Q$  and so that

$$Q_n = Q'_{n-1} \quad \text{for all } n \in \mathbb{N}$$

is a subsquare with maximal boundary integral.

We then have

$$\left| \oint_{\partial Q_{n-1}} f(z) dz \right| \leq 4 \left| \oint_{\partial Q_n} f(z) dz \right|$$

hence

$$\left| \oint_{\alpha} f(z) dz \right| = \left| \oint_{\partial Q} f(z) dz \right| \leq 4^n \left| \oint_{\partial Q_n} f(z) dz \right| \quad \text{for all } n \in \mathbb{N} .$$

For each  $n \in \mathbb{N}$  let  $z_n$  be the lower left corner of  $Q_n$ . Then the sequence  $(z_n)_{n \in \mathbb{N}}$  is a Cauchy sequence because

$$|z_n - z_{n-1}| \leq \frac{1}{2} \text{diam } Q_{n-1} = \text{diam } Q_n = \frac{\sqrt{2}}{2^n}.$$

Let  $p = \lim_{n \rightarrow \infty} z_n$  be the limit.

Then  $p \in Q_n$  for all  $n$  and

$$\forall x \in Q_n : |x - p| \leq \text{diam } Q_n .$$

Since  $f$  is complex differentiable, we have

$$f(p + h) = f(p) + f'(p)h + R(h)$$

with a function  $R$  so that  $\lim_{h \rightarrow 0} \frac{R(h)}{h} = 0$ .

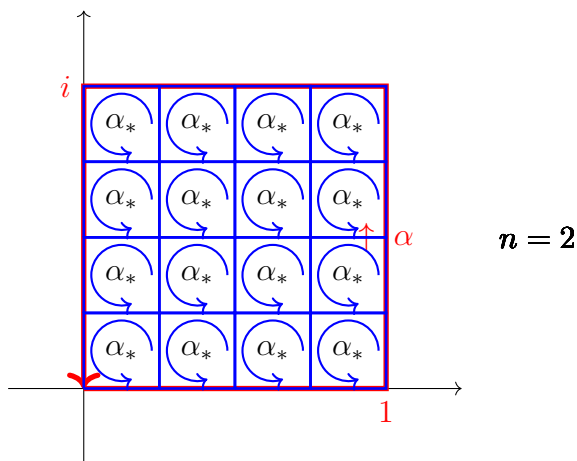
For any  $\epsilon > 0$  there is therefore  $\delta_\epsilon > 0$  so that for all  $h$  with  $|h| < \delta_\epsilon$  we have  $|R(h)| < |h|\epsilon$ .

We now choose  $n$  so large that

$$\text{diam } Q_n < \delta_\epsilon .$$

Then

$$\begin{aligned} \left| \oint_\alpha f(z) dz \right| &= \left| \oint_{\partial Q} f(z) dz \right| \leq 4^n \left| \oint_{\partial Q_n} f(z) dz \right| \\ &= \left| \oint_{\partial Q_n} R(z) dz \right| \leq 4^n \times \mathbf{length} \partial Q_n \times \epsilon \times \text{diam } Q_n \\ &= 4^n \times 4 \times 2^{-n} \times \epsilon \times \sqrt{2} \times 2^{-n} = 4\sqrt{2}\epsilon . \end{aligned} \tag{10.5}$$



Subdivision

We will now prove the version we will mainly use for now.

**Theorem 10.6 (Cauchy Integral Theorem for Rectangular Domains)** *Let  $U \subset \mathbb{C}$  be open and let  $\phi: Q \rightarrow U$  be a  $C^1$ -map. Let  $\alpha$  be the boundary curve of  $Q$  as given in (??). Then for all  $f \in \mathcal{O}(U)$*

$$\oint_{\phi \circ \alpha} f(z) dz = 0 .$$

**Proof:** As in the previous proof we subdivide the square  $Q$  to obtain a sequence of subsquares  $Q_n$  with boundary curves  $\alpha_n$  so that

$$\left| \oint_{\phi \circ \alpha} f(z) dz \right| \leq 4^n \left| \oint_{\phi \circ \alpha_n} f(z) dz \right|$$

and so that the lower left corners  $z_n$  of the squares  $Q_n$  converge to some  $p$ ,  $p \in Q_n$  for all  $n$ . As before,

$$f(\phi(p+h)) = f(\phi(p)) + d_p(f \circ \phi)h + R(h) = f(\phi(p)) + d_{\phi(p)}f d_p \phi h + R(h)$$

and for any  $\epsilon > 0$  we have  $\delta_\epsilon > 0$  so that

$$|R(h)| \leq \epsilon |h| \quad \text{if} \quad |h| < \delta_\epsilon .$$

The assumption that  $\phi$  is continuously differentiable means that the map

$$d\phi: Q \rightarrow \text{Hom}(\mathbb{R}^2, \mathbb{R}^2) \quad , \quad z \mapsto d_z \phi$$

is continuous, with respect to any norm on  $\text{Hom}(\mathbb{R}^2, \mathbb{R}^2)$  because this is a finite dimensional real vector space and all norms on finite dimensional vector spaces are equivalent, i.e. induce the same topology. We choose the operator norm on  $\text{Hom}(\mathbb{R}^2, \mathbb{R}^2)$ . Recall that the operator norm of a linear map  $T: V \rightarrow W$  of normed vector spaces is

$$\begin{aligned} \|T\|_{\text{op}} &= \sup \left\{ \frac{\|Tv\|}{\|v\|} \mid v \in V \setminus \{0\} \right\} \\ &= \sup \{ \|Tv\| \mid v \in V, \|v\| = 1 \} \\ &= \inf \{ \lambda \in \mathbb{R} \mid \forall v \in V : \|Tv\| \leq \lambda \|v\| \} \in \mathbb{R}_0^+ \cup \{\infty\} \end{aligned}$$

where  $\inf \emptyset = \infty$ . Since continuous maps map compact sets to compact sets, the function  $Q \rightarrow \mathbb{R}$ ,  $z \mapsto \|d_z \phi\|_{\text{op}}$ , has a maximum  $M$ . Hence

$$\forall z \in Q : \|d_z \phi\|_{\text{op}} \leq M \quad \text{i.e.} \quad \forall z \in Q, h \in \mathbb{R}^2 = \mathbb{C} : |d_z \phi h| \leq M |h| .$$

We now can modify the estimate (??).

$$\begin{aligned} \left| \oint_{\phi \circ \alpha} f(z) dz \right| &\leq 4^n \left| \oint_{\phi \circ \alpha_n} f(z) dz \right| \\ &= 4^n \left| \oint_{\phi \circ \alpha_n} R(z) dz \right| \\ &\leq 4^n \times \mathbf{length}(\phi \circ \alpha_n) \times \max_t |R(\alpha_n(t) - p)| \\ &\leq 4^n \times M \times \mathbf{length}(\alpha_n) \times \max_t |R(\alpha_n(t) - p)| \\ &\leq 4^n \times M \times 4 \times 2^{-n} \times \epsilon \times \sqrt{2} \times 2^{-n} \\ &= 4\sqrt{2}M\epsilon \end{aligned} \tag{10.7}$$

which holds for all  $\epsilon > 0$ , hence the integral vanishes. •

There is already an abundance of rectangular domains:

1. **Triangles** For the convex hull

$$\text{conv} \{a, b, c\} = \{ \alpha a + \beta b + \gamma c \mid \alpha, \beta, \gamma \in \mathbb{R}_0^+, \alpha + \beta + \gamma = 1 \}$$

of  $\{a, b, c\} \subset \mathbb{C}$  we can use the parametrization

$$\phi: Q \rightarrow \Delta(a, b, c) \quad , \quad \phi(s, t) = sa + tb + (1 - s - t)c .$$

2. **Homotopy of curves relative endpoint** This describes a deformation  $\Gamma$  from one curve  $\gamma_0: [0, 1] \rightarrow \mathbb{C}$  to another  $\gamma_1: [0, 1] \rightarrow \mathbb{C}$ , keeping the endpoints fixed. Differentiability of this deformation is expressed as the differentiability of the map  $\Gamma$  on  $Q$ :

**Definition 10.8** Let  $U \subset \mathbb{C}$  and let  $\gamma_0, \gamma_1: [0, 1] \rightarrow U$  be  $C^1$  curves. Then the two curves are  **$C^1$ -homotopic relative endpoint**, we write  $\gamma_0 \simeq \gamma_1 \text{ rel } \{0, 1\}$ , if there is a  $C^1$  map

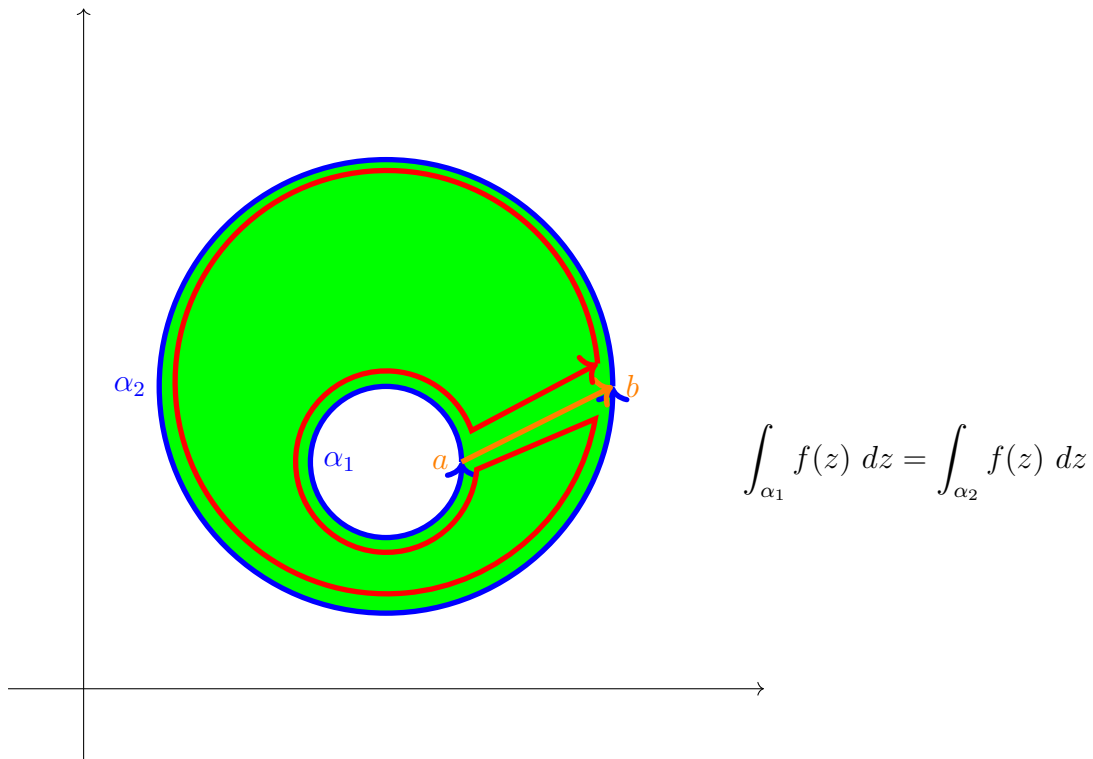
$$\Gamma: [0, 1] \times [0, 1] \rightarrow U$$

so that

$$(a) \quad \forall t \in [0, 1], s \in \{0, 1\} : \Gamma(s, t) = \gamma_s(t)$$

$$(b) \quad \forall t \in \{0, 1\}, s \in [0, 1] : \Gamma(s, t) = \gamma_0(t) = \gamma_1(t)$$

3. **Annuli**



The segment  $[a, b]$  does not contribute to the line integral over the red curve



**Problem 10.9** Find the formula for a  $C^1$ -function  $\phi: Q \rightarrow \mathbb{C}$  mapping the square  $Q$  to the annulus and the edges of  $Q$  to the circles and the segment  $[a, b]$  shown in the picture. The curve  $\phi \circ \alpha$  moves twice through the segment  $[a, b]$ , but in opposite directions.

An **annulus** is a subset of  $\mathbb{C}$  of the form  $\overline{B}_R(p) \setminus B_r(p)$ , where  $P, p \in \mathbb{C}$  and  $R > r > 0$  so that  $|P - p| + r < R$ . If  $f \in \mathcal{O}(U)$  with  $U \subset \mathbb{C}$  open and  $\overline{B}_R(p) \setminus B_r(p) \subset U$ , then  $f$  has the same complex line integral over both boundary curves of the annulus. To see this, we consider the curve  $\alpha_1 * \beta * \alpha_2^{-1} * \beta^{-1}$ . By additivity,

$$\begin{aligned} \oint_{\alpha_1 * \beta * \alpha_2^{-1} * \beta^{-1}} f(z) dz &= \int_{\alpha_1} f(z) dz + \int_{\beta} f(z) dz - \int_{\alpha_2} f(z) dz - \int_{\beta} f(z) dz \\ &= \int_{\alpha_1} f(z) dz - \int_{\alpha_2} f(z) dz \end{aligned}$$

But since the annulus is a rectangular region and  $\alpha_1 * \beta * \alpha_2^{-1} * \beta^{-1}$  its boundary curve, this vanishes by Cauchy's Integral Theorem.

A subset  $\Omega$  of a real vector space is **convex** if with any two points,  $\Omega$  also contains the segment between these points:

$$p, q \in \Omega, t \in [0, 1] \implies tp + (1 - t)q \in \Omega .$$

**Problem 10.10** Let  $\Omega \subset \mathbb{C}$  be a convex domain,  $f \in \mathcal{O}(\Omega)$  and  $\gamma: [0, 1] \rightarrow \Omega$  be a closed, piecewise continuously differentiable curve. Prove that  $\oint_{\gamma} f(z) dz = 0$ .

**Proof:** We may assume that  $\gamma(0) = \gamma(1) = 0$ . Let  $0 = t_0 < t_1 < \dots < t_n = 1$  be the only non-differentiability points of  $\gamma$ . Then  $\phi: [t_{i-1}, t_i] \times [0, 1] \rightarrow \mathbb{C}$ ,  $\phi(t, s) = s\gamma(t)$  is continuously differentiable and maps  $[t_{i-1}, t_i] \times [0, 1]$  in  $\Omega$ . The boundary curve of this rectangular region is the concatenation  $\beta_i := \ell_{\gamma(t_{i-1})} * \gamma|_{[t_{i-1}, t_i]} * \ell_{\gamma t_i}^{-1} * c_0$ , where  $c_p$  is the constant curve at  $p$  and  $\ell_x$  is the segment from 0 to  $x$ . Now  $\oint_{\gamma} f(z) dz = \sum_{i=1}^n \int_{\gamma|_{[t_{i-1}, t_i]}} f(z) dz = \sum_{i=1}^n \oint_{\beta_i} f(z) dz$ , because the integrals over the  $\ell_{\gamma_i}$  in this sum cancel (and the integrals over the constant path are 0). By Cauchy's Integral Theorem,  $\oint_{\beta_i} f(z) dz = 0$  for all  $i$ . •

## 11 Immediate Consequences of the Cauchy Integral Theorem

**Theorem 11.1 (Cauchy Integral Formula on the disc)** Let  $r > 0$  and  $p \in \mathbb{C}$ ,  $f \in \mathcal{O}(\overline{B}_r(p))$ . Then for all  $a \in B_r(p)$  we have

$$f(a) = \frac{1}{2\pi i} \oint_{|z-p|=r} \frac{f(z)}{z-a} dz . \tag{11.2}$$

**Proof:** We may assume  $p = 0$  and  $r = 1$ . Let  $a \in B_1(0)$ . For every  $\epsilon$ ,  $0 < \epsilon < 1 - |a|$ , the function

$$\frac{f(z)}{z - a} \text{ is holomorphic on the annulus } \overline{B}_1(0) \setminus B_\epsilon(a) .$$

Hence by the previous corollary of the Cauchy Integral Theorem

$$\begin{aligned} \oint_{|z|=1} \frac{f(z)}{z - a} dz &= \oint_{|z-a|=\epsilon} \frac{f(z)}{z - a} dz = \lim_{\epsilon' \rightarrow 0} \oint_{|z-a|=\epsilon'} \frac{f(z)}{z - a} dz \\ &= \lim_{\epsilon' \rightarrow 0} \oint_{|h|=\epsilon'} \frac{f(a+h)}{h} dz . \end{aligned}$$

To compute this limit we expand

$$f(a+h) = f(a) + f'(a)h + R(h) \quad , \quad \frac{R(h)}{|h|} \xrightarrow{h \rightarrow 0} 0 .$$

This leads to

$$\begin{aligned} \left| \oint_{|h|=\epsilon} \frac{R(h)}{h} dz \right| &\leq 2\pi\epsilon \sup \left\{ \left| \frac{R(h)}{h} \right| \mid |h| \leq \epsilon \right\} \xrightarrow{\epsilon \rightarrow 0} 0 , \\ \left| \oint_{|h|=\epsilon} \frac{f'(a)h}{h} dz \right| &= \left| \oint_{|h|=\epsilon} f'(a) dz \right| = 0 , \\ \oint_{|h|=\epsilon'} \frac{f(a)}{h} dz &= \int_0^{2\pi} \frac{f(a)}{e^{it}} i e^{it} dt = 2\pi i f(a) \end{aligned}$$

•

**Corollary 11.3 (Mean Value Property of Holomorphic Functions)** *If  $f \in \mathcal{O}(\overline{B}_r(p))$  then  $f(p)$  is the average of  $f$  over the boundary circle,*

$$f(p) = \frac{1}{\text{length } \mathcal{S}_r(p)} \int_{\mathcal{S}_r(p)} f = \frac{1}{2\pi r} \int_0^{2\pi} f(p + e^{it}) dt .$$

**Problem 11.4** *Deduce the mean value property for harmonic functions on  $\mathbb{R}^2$ , i.e. if  $f \in C^2(\mathbb{R}^2, \mathbb{R})$  is harmonic,  $\Delta f = 0$ , then for all  $p \in \mathbb{R}^2$  and  $r > 0$  we have*

$$f(p) = \frac{1}{\text{length } \mathcal{S}_r(p)} \int_{\mathcal{S}_r(p)} f .$$

**Proof:** We first extend  $f$  to a holomorphic function  $f + iv$ , where  $v$  needs to satisfy  $v_x = -f_y$  and  $v_y = f_x$ , i.e.  $\text{grad} v = (-f_y, f_x)$ . The vector field  $(-f_y, f_x)$  on  $\mathbb{R}^2$  has a potential  $v$  because its curl vanishes,

$$\text{curl}(-f_y, f_x) = f_{xx} + f_{yy} = \Delta f = 0 .$$

•

**Theorem 11.5 (Power Series, Taylor Series of a Holomorphic Function)** Let  $U \subset \mathbb{C}$  be open and  $f \in \mathcal{O}(U)$ . Let  $p \in U$ ,  $r > 0$  and  $\overline{B}_r(p) \subset U$ . Then

1. there is a unique power series  $\sum_{n=0}^{\infty} c_n(z-p)^n$  with positive radius of convergence  $\rho$  representing  $f$  near  $p$ , i.e.

$$\forall z \in B_\rho(p) : \sum_{n=0}^{\infty} c_n(z-p)^n = f(z) ,$$

2. The radius of convergence of this power series is the radius of the largest ball around  $p$  contained in  $U$ ,

$$\rho = \sup \{ \rho' \mid B_{\rho'}(p) \subset U \} ,$$

3. This power series is the Taylor series of  $f$  at  $p$  and its coefficients are given by

$$c_n = \frac{f^{(n)}(p)}{n!} = \frac{1}{2\pi i} \int_{|z-p|=\rho'} \frac{f(z)}{(z-p)^{n+1}} dz .$$

4. (Cauchy Estimate of the Taylor Coefficients)

$$|c_n| \leq \sup_{|z-p|=r} \frac{|f(z)|}{r^n}$$

**Problem 11.6** Compute the Taylor series of  $\arctan$  around 0.

From the definition,  $\arctan(z)$  is the antiderivative of  $f(z) = \frac{1}{1+z^2}$  with  $\arctan(0) = 0$ . We can thus integrate the geometric series “formally”,

$$\begin{aligned} \frac{1}{1+z^2} &= \sum_{k=0}^{\infty} (-z^2)^k = \sum_{k=0}^{\infty} (-1)^k z^{2k} \quad \text{for } |z| < 1 , \\ \arctan(z) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} z^{2k+1} \quad \text{for } |z| < 1 . \end{aligned}$$

**Proof:** A power series representing  $f$  near  $p$  is unique if it exists, because by Theorem ?? it must be the Taylor series of  $f$ .

For the existence of such a power series, we use the Cauchy Integral Formula and the local uniform convergence of

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \quad \text{for } |a| < 1$$

To simplify notation, we will assume  $p = 0$  and  $r = 1$ . Then for all  $q \in B_1(0)$ ,

$$\begin{aligned} f(q) &= \frac{1}{2\pi i} \oint_{|z|=1} \frac{f(z)}{z-q} dz = \frac{1}{2\pi i} \oint_{|z|=1} \frac{f(z)}{z} \frac{1}{1-\frac{q}{z}} dz \\ &= \frac{1}{2\pi i} \oint_{|z|=1} \frac{f(z)}{z} \sum_{n=0}^{\infty} \frac{q^n}{z^n} dz \\ &= \sum_{n=0}^{\infty} \underbrace{\left( \frac{1}{2\pi i} \oint_{|z|=1} \frac{f(z)}{z^{n+1}} dz \right)}_{c_n} q^n \end{aligned}$$

•

As an immediate corollaries we have

**Theorem 11.7 (Goursat's Theorem)** *A holomorphic function on domain in  $\mathbb{C}$  is infinitely often complex differentiable, in particular (as function  $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ ) of class  $C^\infty$ , smooth.*

Functions  $f \in \mathcal{O}(\mathbb{C})$  are called **entire**. If such a function is bounded, say  $|f(z)| < M$  for all  $z \in \mathbb{C}$ , then at all  $p \in \mathbb{C}$  the  $n$ th Taylor coefficient  $c_n$  satisfies

$$|c_n| \leq \sup_{|z-p|=r} \frac{|f(z)|}{r^n} \leq \frac{M}{r^n} \quad \text{for all } r > 0,$$

hence  $c_n = 0$  for all  $n > 0$ . We have proved

**Theorem 11.8 (Liouville's Theorem)** *An bounded entire function is constant.*

This also gives the probably shortest proof of the

**Theorem 11.9 (Fundamental Theorem of Algebra)** *Every complex polynomial of positive degree has at least one zero.*

**Proof:** Let  $f \in \mathbb{C}[z]$ ,  $f(z) = a_n z^n + \dots + a_0$  with  $n > 0$ ,  $a_n \neq 0$  and assume that  $f$  has no zero, i.e.  $f: \mathbb{C} \rightarrow \mathbb{C} \setminus \{0\}$ . Then  $\frac{1}{f}$  is holomorphic. But

$$|f(z)| = \left| z^n \left( a_n + \frac{a_{n-1}}{z} + \dots + \frac{a_0}{z^n} \right) \right| \geq |z|^n \left( |a_n| - \left| \frac{a_{n-1}}{z} \right| - \dots - \left| \frac{a_0}{z^n} \right| \right) \xrightarrow{z \rightarrow \infty} \infty.$$

In particular, there is  $R > 0$  so that  $|f(z)| > 1$  whenever  $|z| > R$ . On the other hand,  $\overline{B}_R(0) = \{z \mid |z| \leq R\}$  is compact, hence  $m := \min_{z \leq R} |f(z)|$  exists and is positive. Now

$$\min_{z \in \mathbb{C}} |f(z)| = \min \left\{ \min_{z \leq R} |f(z)|, \min_{z > R} |f(z)| \right\} \geq \min \{m, 1\} > 0$$

and

$$\max_{z \in \mathbb{C}} \left| \frac{1}{f(z)} \right| = \frac{1}{\min_{z \in \mathbb{C}} |f(z)|}$$

is finite,  $\frac{1}{f}$  is bounded.

By Liouville's Theorem  $\frac{1}{f}$  is constant, but this implies that  $f$  is constant and therefore not of positive degree. •

**Theorem 11.10 (Identity Theorem)** *Let  $U \subset \mathbb{C}$  be a domain (an open and connected subset),  $p \in U$ , and  $f, g \in \mathcal{O}(U)$ . Let  $A \subset U$  be a subset with an accumulation point in  $U$  and  $f|_A = g|_A$ . Then  $f = g$ .*

A topological space  $X$  is **connected** if the only open and closed subsets of  $X$  are  $\emptyset$  and  $X$ , or, equivalently, if  $X = G \cup H$  with  $G, H$  open and disjoint can only be if one of  $G, H$  is empty.

For open subsets of  $\mathbb{R}^n$  this is equivalent to being **path connected**: A topological space  $X$  is **path connected** if for every two points  $p, q \in X$  there is a continuous map  $c: [0, 1] \rightarrow X$  (a “path”), joining  $p$  to  $q$ ,  $p \xrightarrow{c} q$ , i.e.  $c(0) = p$ ,  $c(1) = q$ .

**Proof:** The Identity Theorem now follows from the power series expansion of holomorphic functions. In the setting of the theorem, consider the difference  $h = f - g \in \mathcal{O}(U)$  and the set

$$G = \{u \in U \mid \forall k \in \mathbb{N}_0 : h^{(k)}(u) = 0\} .$$

The set  $G$  is closed because all the derivatives  $h^{(k)}$  are continuous. On the other hand, if  $p \in G$ , then all coefficients of the Taylor series of  $h$  at  $p$  vanish. But then  $h$  vanishes with all its derivatives on the ball of convergence of this power series. Thus  $p$  is an interior point of  $G$ , and we have shown that all points of  $G$  are interior,  $G$  is open.

By assumption  $h$  vanishes on a sequence  $(a_n)_{n \in \mathbb{N}}$  with  $\lim_{n \rightarrow \infty} a_n = q \in U$ ,  $a_n \neq q$  for all  $n$ , and, by continuity,  $h(q) = 0$  also. Assume there were  $k \in \mathbb{N}$  with  $h^{(k)}(q) \neq 0$ . We may assume that  $k$  is minimal with this property and that  $q = 0$ . Then for some  $r > 0$ , for all  $z \in \overline{B}_r(0)$  we have,

$$h(z) = z^k \left( c_k + z \sum_{j=0}^{\infty} c_{k+1+j} z^j \right) = z^k (c_k + z g(z)) \neq 0$$

if  $z \neq 0$  and  $|z| \max_{|\zeta| \leq r} |g(\zeta)| < |c_k|$ . Therefore there is a neighbourhood of  $q$  where  $h$  has no zero other than  $q$ . But this contradicts the assumption that  $q$  is an accumulation point of the set of zeros of  $h$ . •

**Definition 11.11** *Let  $U \subset \mathbb{C}$  be open,  $p \in U$  and  $f \in \mathcal{O}(U)$ . Then the order of  $f$  at  $p$  is*

$$\text{ord}(f, p) = \min \{k \in \mathbb{N} \mid f^{(k)}(p) \neq 0\}$$

(with the convention that  $\min \emptyset = +\infty$ ).

If the order is finite, then there is  $r > 0$  so that  $B_r(p) \subset U$  and for all  $z \in B_r(p)$  we have

$$f(z) = f(p) + (z - p)^{\text{ord}(f,p)} g(z) \quad (11.12)$$

with some  $g \in \mathcal{O}(U)$  with  $g(p) \neq 0$ . Equivalently, the order is

$$\begin{aligned} \text{ord}(f, p) &= \max \left\{ k \in \mathbb{N} \mid \frac{f(z) - f(p)}{(z - p)^k} \text{ has holomorphic extension to } p \right\} \\ &= \max \left\{ k \in \mathbb{N} \mid \exists g \in \mathcal{O}(B_r(p)) \forall z \in B_r(p) : f(z) = f(p) + (z - p)^k g(z) \right\} \end{aligned}$$

**Theorem 11.13 (Local form of a holomorphic function)** *Let  $U \subset \mathbb{C}$  be a domain and  $f \in \mathcal{O}(U)$  be not constant. Then for any  $p \in U$  there is a neighbourhood  $V \subset U$  of  $p$  and a biholomorphic map*

$$\begin{aligned} \phi: B_r(0) &\rightarrow V \quad \text{with} \quad \phi(0) = p, \quad \text{for some } r > 0 \quad \text{so that} \\ f(\phi(z)) &= f(p) + z^{\text{ord}(f;p)} \quad \text{for all } z \in B_r(0) \end{aligned}$$

**Proof:** Since  $f$  is not constant on the domain  $U$ , it must have finite order at each point. W.l.o.g.  $p = 0 = f(p)$  and  $k = \text{ord}(f, 0)$ , thus

$$f(z) = c_k z^k + \sum_{j=1}^{\infty} c_{k+j} z^{k+j} \quad \text{for all } z \in B_{r'}(0)$$

with some  $c_j, c_k \neq 0$  and  $r' > 0$ . We may further assume  $c_k = 1$  so that

$$f(z) = z^k + \sum_{j=1}^{\infty} c_{k+j} z^{k+j} = z^k g(z)$$

for all  $z \in B_r(0)$  and some  $g \in \mathcal{O}(B_{r'}(0))$  with  $g(0) = 1$ . By continuity of  $g$  there is  $r'' \leq r'$  so that  $g(B_{r''}(0)) \subset B_1(1)$ . Since  $\ln$  is given by the power series (??) on  $B_1(1)$ ,  $\ln$  is holomorphic there and we have

$$f(z) = \left( z e^{\frac{1}{k} \ln(g(z))} \right)^k = (h(z))^k \quad \text{for all } |z| < r''$$

with some function  $h \in \mathcal{O}(B_{r''}(0))$ ,  $h(0) = 0$ ,  $h'(0) = 1$ .

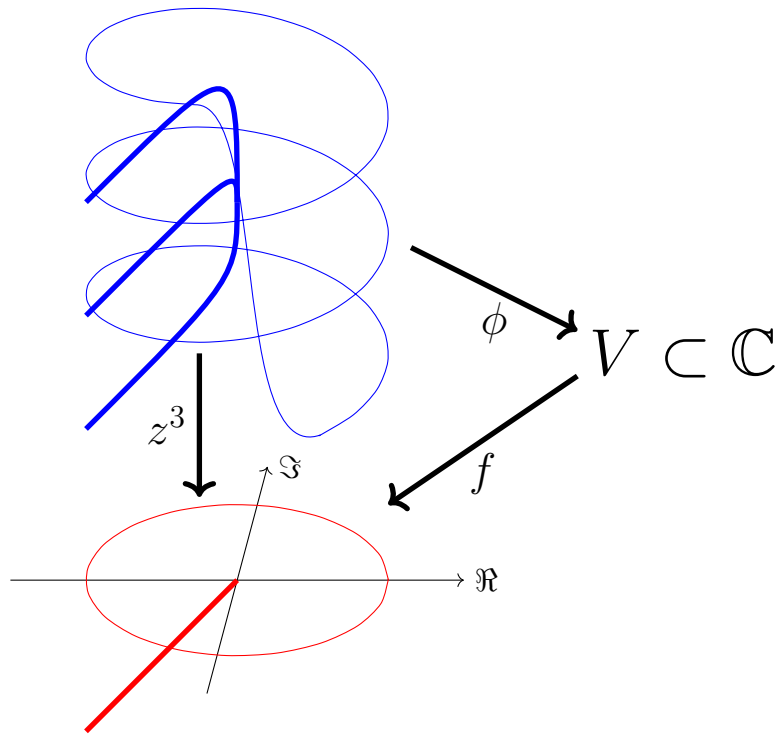
By the Inverse Function Theorem the function  $h$  is a local diffeomorphism near 0. Thus, there is an open neighbourhood  $V \subset B_{r''}(0)$ ,  $r > 0$  and a (real) differentiable map  $\phi: B_r(0) \rightarrow V$ . The derivative of  $\phi$  is the inverse of the derivative of  $h$ , hence complex multiplication

$$d_y \phi = \frac{1}{h'(\phi(y))}$$

hence  $\phi: B_r(0) \rightarrow V$  is biholomorphic and we have

$$f(\phi(z)) = z^k \quad \text{for all } z \in B_r(0) .$$

•



A holomorphic function near a point of multiplicity 3

**Corollary 11.14 (Leaves near a point of higher order)** *If  $U \subset \mathbb{C}$  is open,  $f \in \mathcal{O}(U)$  and  $p \in U$  with  $\text{ord}(f, p) = k < \infty$ . Then there is a neighbourhood  $V \subset U$  of  $p$  such that*

$$\#(f^{-1}(x) \cap V) = \begin{cases} 1 & \text{if } x = p \\ k & \text{if } x \in V \setminus \{p\} \end{cases} .$$

**Proof:** By translating the problem (i.e. replace  $f(z)$  by  $f(p+z) - f(p)$ ), we can assume that  $p = f(p) = 0$ . By Theorem ?? we may further assume that  $f(z) = z^k$ . •

**Problem 11.15** *Recall that a map  $f: X \rightarrow Y$  between topological spaces is open if  $f(U) \subset Y$  is open whenever  $U \subset X$  is open. Show that for all  $k \in \mathbb{N}$ , the  $k$ th power map  $\mathbb{C} \xrightarrow{z \mapsto z^k} \mathbb{C}$  is open.*

**Corollary 11.16 (Open Mapping Theorem)** *If  $U \subset \mathbb{C}$  is a domain and  $f \in \mathcal{O}(U)$  is nonconstant, then  $f(U)$  is also a domain. In particular  $f$  is open.*

**Proof:** By continuity,  $f(U)$  is connected. By the Theorem ?? on the local form of a holomorphic function, every point  $p \in U$  has a neighbourhood  $V \subset U$  with a biholomorphic map  $\phi: B_r(0) \xrightarrow{\cong} V$ , so that  $f(\phi(z)) = f(p) + z^{\text{ord}(f,p)}$ , hence  $f(y) = f(p) + (\phi^{-1}(y))^{\text{ord}(f,p)}$  for all  $y \in V$ . In particular

$$f(V) = f(p) + (\phi^{-1}(y))^{\text{ord}(f,p)} = f(p) + B_r(0)^{\text{ord}(f,p)} = B_{r^{\text{ord}(f,p)}}(f(p))$$

is open. •

**Corollary 11.17 (Maximum Principle)** *The absolute value of a nonconstant holomorphic function on a domain does not assume its maximum on the domain.*

**Proof:** If  $f \in \mathcal{O}(U)$  is not constant, then  $f: U \rightarrow \mathbb{C}$  is open. Since the modulus  $|\cdot|: \mathbb{C} \rightarrow \mathbb{R}_0^+$  is an open function, so is the composition  $|f|: U \rightarrow \mathbb{R}_0^+$ . Therefore  $|f(U)| \subset \mathbb{R}_0^+$  is open, hence  $\sup |f(U)| \notin |f(U)|$ .

•

This can be rephrased

**Corollary 11.18 (Maximum Principle)** *The absolute value of a nonconstant holomorphic function on a connected compact subset of  $\mathbb{C}$  assumes its maximum on the boundary of the subset.*

Since  $\Re, \Im: \mathbb{C} \rightarrow \mathbb{R}$  are all open, we have statements similar to ??, ??.

Vanishing of the complex line integral characterizes holomorphic functions:

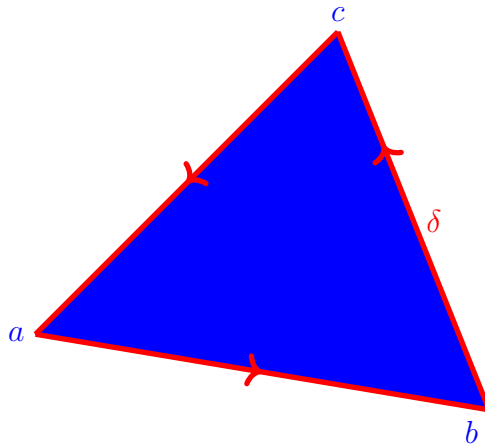
**Theorem 11.19 (Morera's Theorem)** *Let  $U \subset \mathbb{C}$  be open,  $f: U \rightarrow \mathbb{C}$  continuous. Assume that for every three points  $(a, b, c)$ ,  $a, b, c \in \mathbb{C}$ , whose convex hull  $\Delta(a, b, c)$  lies in  $U$  we have that  $\oint_{\delta} f(z) dz = 0$ , where  $\delta$  is the boundary curve of the triangle  $\Delta(a, b, c)$ . Then  $f \in \mathcal{O}(U)$ .*

A triangle is the convex hull of three points. Thus if  $a, b, c \in \mathbb{C}$ , then the triangle spanned by  $a, b, c$  is

$$\Delta(a, b, c) = \{ \alpha a + \beta b + \gamma c \mid \alpha, \beta, \gamma \in \mathbb{R}_0^+, \alpha + \beta + \gamma = 1 \} .$$

The boundary curve of  $\Delta = \Delta(a, b, c)$  is the concatenation of the edges of the triangle,

$$\delta(a, b, c)(t) = \begin{cases} a + t(b - a) & \text{if } 0 \leq t \leq 1 \\ b + (t - 1)(c - b) & \text{if } 1 \leq t \leq 2 \\ c + (t - 2)(a - c) & \text{if } 2 \leq t \leq 3 \end{cases} . \quad (11.20)$$





**Problem 11.21** *There is some ambiguity here. First the triangle might be degenerate, for instance  $b$  might be on the segment  $[a, c]$ . Even if the triangle is not degenerate, one might reverse the orientation. Check that none of these matter here. Find a formula  $X(a, b, c)$  determining the orientation of the curve (??) from the coordinates of  $a, b, c$ , if the points do not lie on a common line.*

**Proof:** Since holomorphicity is local, we may assume that  $U = B_r(p)$  for some  $p \in \mathbb{C}$ ,  $r > 0$ , and we may also assume  $p = 0$ ,  $r = 1$ , i.e.  $U = B = B_1(0)$ . We will define an antiderivative  $F$  of  $f$ . Then the theorem follows from Goursat's Theorem.

Up to a constant, an antiderivative of  $F$  must be given by line integrals over segments emanating in 0,

$$F(q) := \int_{\ell_q} f(z) dz = \int_0^1 f(tq) q dt$$

with  $\ell_q(t) = tq$  for  $t \in [0, 1]$ . We need to show that  $F' = f$ . To this end, for  $h$  so small that  $p + h \in B$ ,

$$\begin{aligned} \frac{F(p+h) - F(p)}{h} &= \frac{1}{h} \left( \int_{\ell_{q+h}} f(z) dz - \int_{\ell_q} f(z) dz \right) \\ &= \frac{1}{h} \int_{q \rightsquigarrow q+h} f(z) dz \quad \text{since } \ell_q * (q \rightsquigarrow q+h) * \ell_{q+h}^{-1} \text{ bounds a triangle in } B \\ &= \frac{1}{h} \int_0^1 f(q+th) h dt \\ &= \int_0^1 f(q+th) dt \xrightarrow{h \rightarrow 0} f(q) \end{aligned}$$

because  $f$  is continuous. •

**Notation:**  $\mathcal{H} = \{z \in \mathbb{C} \mid \Im z > 0\}$ , the upper half plane.

**Theorem 11.22 (Schwarz Reflection Principle)** *Let  $U \subset \mathbb{C}$  be open and invariant under complex conjugation, i.e.  $z \in U \implies \bar{z} \in U$ . Let  $f$  be a continuous function on  $U$  which is holomorphic on  $U \cap \mathcal{H} = \{z \in U \mid \Im z > 0\}$  and  $f(U \cap \mathbb{R}) \subset \mathbb{R}$ . Then the function  $F$  on  $U$  defined by*

$$F(z) := \begin{cases} \overline{f(\bar{z})} & \text{if } \Im z \leq 0 \\ f(z) & \text{if } \Im z \geq 0 \end{cases} \quad (11.23)$$

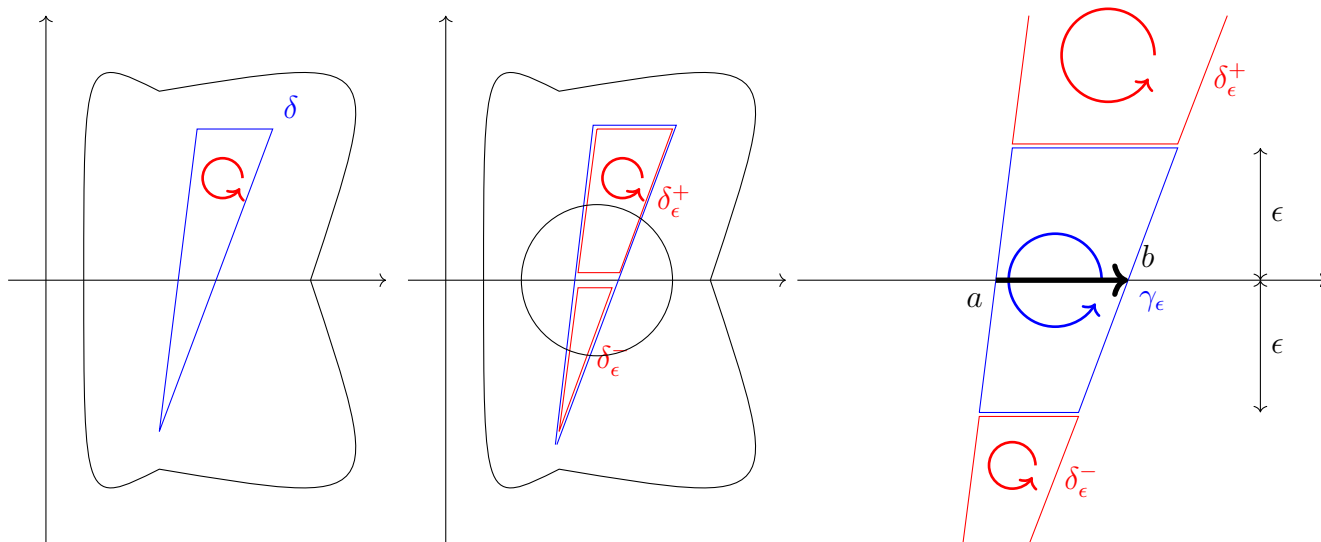
*is holomorphic on all of  $U$ .*

**Problem 11.24** *Let  $U \subset \mathbb{C}$  be open,  $f: U \rightarrow \mathbb{C}$  be a holomorphic function,  $cU = \{\bar{u} \mid u \in U\}$  the complex conjugate of the set  $U$  and  $cfc: cU \rightarrow \mathbb{C}$  be the conjugate of  $f$  with complex conjugation, i.e.*

$$cfc(x) = \overline{f(\bar{x})}.$$

*Prove that  $cU$  is open and that  $cfc \in \mathcal{O}(cU)$ .*

**Proof:** You have just shown that the restriction  $F|_{U \cap \mathcal{H}}$  is holomorphic. Since the two definitions in (??) coincide on  $\mathbb{R} \cap U$ ,  $F$  is continuous. In order to apply Morera's Theorem we need to prove that complex line integrals of  $F$  over triangles in  $U$  vanish. For triangles lying entirely in the upper or in the lower half plane this follows from Cauchy's Integral Theorem, because  $F$  is holomorphic there. If the triangle does not lie in the upper nor in the lower half plane, we split it in a triangle and a quadrilateral,



Then

$$\oint_{\delta} f(z) dz = \oint_{\delta_{\epsilon}^{+}} f(z) dz + \oint_{\delta_{\epsilon}^{-}} f(z) dz + \oint_{\gamma_{\epsilon}} f(z) dz \xrightarrow{\epsilon \rightarrow 0} 0$$

The integrals over  $\delta_{\epsilon}^{+}$  and  $\delta_{\epsilon}^{-}$  vanish for all positive  $\epsilon$  by the Cauchy Integral Theorem. The limit of the integral over  $\gamma_{\epsilon}$  converges,

$$\lim_{\epsilon \rightarrow 0} \oint_{\gamma_{\epsilon}} f(z) dz = \int_a^b f(x) dx + \int_b^a f(x) dx = 0 .$$

## 12 Isolated Singularities

If  $p \in U^{\text{open}} \subset \mathbb{C}$  and  $f \in \mathcal{O}(U \setminus p)$ , then one says “ $p$  is an **isolated singularity**” of  $f$ . The isolated singularity  $p$  is **removable** if  $f$  has a holomorphic extension  $\tilde{f}$  to  $U$ , i.e.  $\tilde{f} \in \mathcal{O}(U)$ ,  $\tilde{f}|_{U \setminus \{p\}} = f$ .  $p$  is a **pole** if  $p$  is not removable and there is some  $k \in \mathbb{N}$  so that  $p$  is a removable singularity of the function  $\tilde{f}$  with  $\tilde{f}(z) = (z - p)^k f(z)$ . The smallest such  $k$  is the **order of the pole**. An **essential singularity** is an isolated singularity that is neither removable nor a pole.

**Example 12.1** The point  $p = 0$  is a removable singularity of  $f(z) = \frac{\sin(z)}{z}$ . In fact, the holomorphic extension is given by the power series

$$\tilde{f}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k+1)!} .$$

Similarly, the point 0 is a removable singularity for functions like

$$\frac{e^z - 1 - z - \frac{1}{2}z^2}{z^3}, \quad \frac{1 - \cos(z)}{z^2},$$

and the power series of the holomorphic extension is easily derived. The value of the holomorphic extension at the removable singularity of a function  $f$  is denoted by

$$f(z)|_{z=p}.$$

For example

$$\frac{\sin(z)}{z} \Big|_{z=0} = 1, \quad \frac{e^z - 1 - z - \frac{1}{2}z^2}{z^3} \Big|_{z=0} = \frac{1}{6}, \quad \frac{1 - \cos(z)}{z^2} \Big|_{z=0} = \frac{1}{2}.$$

Usually we will not distinguish between the function and its extension over a removable singularity.

**Example 12.2** *The prototypical example of a pole of order  $k \in \mathbb{N}$  at  $p$  is the function*

$$f(z) = \frac{1}{(z - p)^k}.$$

**Example 12.3** *The points  $p = k\pi$ ,  $k \in \mathbb{Z}$ , are poles of order 1 of the function  $f \in \mathcal{O}(\mathbb{C} \setminus 2\pi\mathbb{Z})$  with*

$$f(z) = \frac{1}{\sin(z)}.$$

*These singularities are not removable since for any  $k \in \mathbb{Z}$ , the limit  $\lim_{z \rightarrow 2\pi k} f(z)$  does not exist. On the other hand, since  $\sin(\pi k + z) = (-1)^k \sin(z)$ , we have*

$$(z - \pi k)f(z) = (-1)^k \frac{z - \pi k}{\sin z - \pi k}$$

*whose singularity at  $\pi k$  is removable,*

$$(-1)^k \frac{z - \pi k}{\sin z - \pi k} \Big|_{z=\pi k} = (-1)^k.$$

**Example 12.4** *The function  $f(z) = e^{\frac{1}{z}}$  has an essential singularity at  $p = 0$ , because there is no  $k \in \mathbb{N}$  so that*

$$z^k e^{\frac{1}{z}}$$

*is bounded near  $z = 0$ . Similarly,  $z = 0$  is an essential singularity of*

$$\sin(1/z) \quad \text{and} \quad \cos(1/z).$$

## 12.1 Laurent Series

For  $r, R \in \mathbb{R}_0^+$ ,  $r \leq R$ , the concentric annulus is

$$A_{r,R}(p) = \{z \in \mathbb{C} \mid r < |z - p| < R\} .$$

The following is analogous to ?? and ??.

**Theorem 12.5** *Let  $A_{r,R}(p) \subset U \stackrel{\text{open}}{\subset} \mathbb{C}$  and  $f \in \mathcal{O}(U)$ . Then there is a unique sequence  $(c_n)_{n \in \mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$  so that*

$$f(z) = \sum_{n \in \mathbb{Z}} c_n (z - p)^n . \quad (12.6)$$

*This Laurent Series converges absolutely and uniformly on every compact subset of  $A_{r,R}(p)$ . The derivative of  $f$  is*

$$f'(z) = \sum_{n \in \mathbb{Z}} n c_n (z - p)^{n-1} .$$

*The Laurent coefficients  $c_n$  are given by the formula*

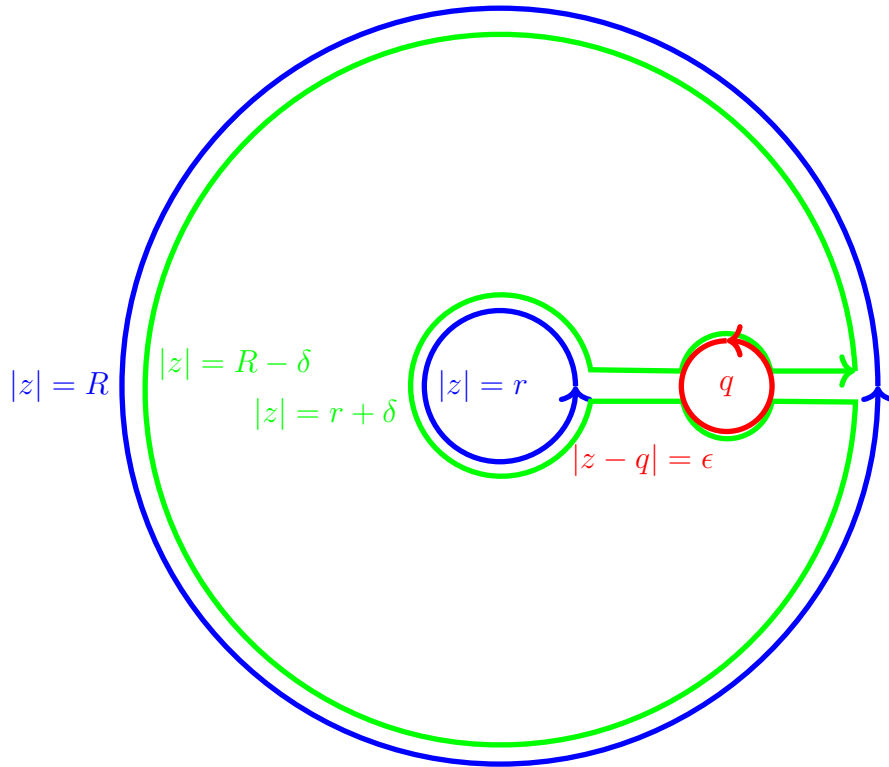
$$c_n = \frac{1}{2\pi i} \oint_{|z-p|=\rho} \frac{f(z)}{(z-p)^n} dz \quad \text{for any } \rho \in (r, R) . \quad (12.7)$$

**Proof:** We may assume  $p = 0$ . Let  $q \in A_{r,R}(0)$  and choose  $\delta$  so that

$$r < r + \delta < |q| < R - \delta < R .$$

By continuity of  $f$ , for  $\epsilon > 0$  so that  $B_\epsilon \subset A_{r,R}(0)$  we have

$$f(q) = \lim_{\rho \rightarrow 0} \frac{1}{2\pi i} \oint_{|z-q|=\rho} \frac{f(z)}{z-q} dz = \oint_{|z-q|=\epsilon} \frac{f(z)}{z-q} dz .$$



By additivity of the complex line integral and the Cauchy intergral theorem for rectangular domains,

$$0 = \oint_{\gamma} \frac{f(z)}{z - q} dz = \oint_{|z|=R-\delta} \frac{f(z)}{z - q} dz - \oint_{|z|=r+\delta} \frac{f(z)}{z - q} dz - \oint_{|z-q|=\epsilon} \frac{f(z)}{z - q} dz$$

Thus

$$2\pi i f(q) = \oint_{|z|=R-\delta} \frac{f(z)}{z - q} dz - \oint_{|z|=r+\delta} \frac{f(z)}{z - q} dz . \tag{12.8}$$

As in the proof of Theorem ??, we now use the Taylor series for  $\frac{1}{1-z}$ , this time at 0 and  $\infty$ ,

$$\frac{1}{1 - z} = \begin{cases} \sum_{k=0}^{\infty} z^k & \text{if } |z| < 1 \\ -\sum_{k=1}^{\infty} z^{-k} & \text{if } |z| > 1 \end{cases} .$$

Since  $r + \delta < |q| < R - \delta$  we can expand the integrals in (??),

$$\begin{aligned}
2\pi i f(q) &= \oint_{|z|=R-\delta} \frac{f(z)}{z-q} dz - \oint_{|z|=r+\delta} \frac{f(z)}{z-q} dz \\
&= \oint_{|z|=R-\delta} \frac{f(z)}{z} \frac{1}{1-\frac{q}{z}} dz - \oint_{|z|=r+\delta} \frac{f(z)}{z} \frac{1}{1-\frac{q}{z}} dz \\
&= \oint_{|z|=R-\delta} \frac{f(z)}{z} \sum_{k=0}^{\infty} \left(\frac{q}{z}\right)^k dz + \oint_{|z|=r+\delta} \frac{f(z)}{z} \sum_{k=1}^{\infty} \left(\frac{q}{z}\right)^{-k} dz \\
&= \sum_{k=0}^{+\infty} \left( \oint_{|z|=R-\delta} \frac{f(z)}{z^{k+1}} dz \right) q^k + \sum_{k=1}^{\infty} \left( \oint_{|z|=r+\delta} \frac{f(z)}{z^{k+1}} dz \right) q^{-k} \\
&= \sum_{k=-\infty}^{+\infty} \underbrace{\left( \oint_{|z|=\rho} \frac{f(z)}{z^{k+1}} dz \right)}_{2\pi i c_k} q^k
\end{aligned}$$

for all  $\rho$ ,  $r < \rho < R$ , since the integrands are holomorphic on  $A_{r,R}(0)$  which shows (??). •

**Corollary 12.9 (Estimate of the Laurent Coefficients)** (in the notation of Theorem ??) For all  $\rho \in (r, R)$ ,

$$|c_k| \leq \frac{\max \{ |f(z)| \mid |z-p| = \rho \}}{\rho^k}$$

**Corollary 12.10 (Principal Part)** If  $r, R \in [0, \infty]$ ,  $r < R$ ,  $p \in \mathbb{C}$  and  $f \in \mathcal{O}(A_{r,R}(p))$ , then

$$f(z) = N(z-p) + H\left(\frac{1}{z-p}\right) \quad \text{with } N \in \mathcal{O}(B_R(0)) \quad \text{and } H \in \mathcal{O}(B_{1/r}(0)) \quad \text{and } H(0) = 0.$$

$H$  is called the **principal part of  $f$  at  $p$** .

In the case  $r = 0$ , the point  $p$  is an isolated singularity. The type of this singularity is determined by the principal part  $H$ , because  $N$  is holomorphic there.

**Corollary 12.11** Let  $f \in \mathcal{O}(A_{0,R}(p))$ . Let  $H$  be the principal part of  $f$  at  $p$ . Then

1.  $p$  is a removable singularity if  $H = 0$ .
2.  $p$  is a pole of order  $k$  if  $H$  is a polynomial of degree  $k$ .
3.  $p$  is essential if  $H$  is not a polynomial.

**Theorem 12.12 (Casoratti-Weierstrass)** Let  $U \subset \mathbb{C}$  be open,  $p \in U$  and  $f \in \mathcal{O}(U \setminus \{p\})$ . Then  $p$  is an essential singularity if and only if  $f(U \setminus \{p\})$  is dense in  $\mathbb{C}$ .

By **Picard's Theorem**,  $\mathbb{C} \setminus f(U \setminus \{p\})$  contains at most one point.

**Proof:** If  $f(U \setminus \{p\})$  is not dense in  $\mathbb{C}$ , then there is  $q \in \mathbb{C}$  and  $r > 0$  so that  $f(U \setminus \{p\}) \subset \mathbb{C} \setminus B_r(q)$ . Thus  $\left| \frac{1}{f(z)-q} \right| < \frac{1}{r}$ , hence  $\frac{1}{f(z)-q}$  has a removable singularity at  $p$ . •

Thus for an isolated singularity  $p \in U \overset{\text{open}}{\subset} \mathbb{C}$  of a function  $f \in \mathcal{O}(B_r(p) \setminus \{p\})$  there are three possibilities:

1. removable:  $f$  is bounded near  $p$ ,  
 $f$  extends holomorphically to  $p$ ,  
 $\lim_{z \rightarrow p} f(z)$  exists.
2. pole: for some  $k \in \mathbb{N}$ ,  
 $p$  is a removable singularity of  $(z-p)^k f(z)$ ,  
 $\lim_{z \rightarrow p} f(z) = \infty$ ,  
 $\frac{1}{f(z)}$  is bounded near  $p$ .
3. essential: Neither  $f(z)$  nor  $\frac{1}{f(z)}$  is bounded near  $p$ ,  
Neither  $\lim_{z \rightarrow p} f(z)$  nor  $\lim_{z \rightarrow p} \frac{1}{f(z)}$  exist  
 $f(B_\rho(p) \setminus \{p\}) \subset \mathbb{C}$  is dense for all  $0 < \rho \leq r$ .

## 12.2 The Riemann Sphere, Meromorphic functions

**Definition 12.13** A meromorphic function on an open subset  $U \subset \mathbb{C}$  is a function  $f \in \mathcal{O}(U \setminus S)$  where  $S \subset U$  is a discrete subset and all  $s \in S$  are poles (of finite positive order). We denote the set of meromorphic functions on  $U$  by  $\mathcal{M}(U)$ .

Thus meromorphic functions are functions without essential singularities. If  $f \in \mathcal{M}(U)$  then for each  $p \in U$ ,  $f$  is holomorphic in a neighbourhood of  $p$  or  $p$  is a removable singularity of  $\frac{1}{f}$ .

Let  $\mathcal{S} = \mathbb{C} \cup \infty$  be a one point compactification of  $\mathbb{C}$ . Thus  $\infty \notin \mathbb{C}$  and the topology of  $\mathcal{S}$  is

$$\{U \mid U \subset \mathbb{C} \text{ open}\} \cup \{\mathcal{S} \setminus K \mid K \subset \mathbb{C} \text{ compact}\} .$$

This is the topology induced by the maps

$$z: \mathbb{C} \xrightarrow{z \mapsto z} \mathcal{S} \quad \text{and} \quad \frac{1}{z}: \mathbb{C} \xrightarrow{z \mapsto \frac{1}{z}, 0 \mapsto \infty} \mathcal{S} .$$

**Definition 12.14** A map  $f: U \rightarrow \mathcal{S}$ ,  $U \subset \mathcal{S}$ , is holomorphic if its compositions

$$z \circ f \circ z, \quad \frac{1}{z} \circ f \circ z, \quad z \circ f \circ \frac{1}{z}, \quad \frac{1}{z} \circ f \circ \frac{1}{z}$$

are holomorphic where defined. We write

$$\mathcal{O}(U, \mathcal{S}) = \{f: U \rightarrow \mathcal{S} \mid f \text{ a holomorphic map}\} .$$

We have a natural injection

$$\mathcal{M}(U) \hookrightarrow \mathcal{O}(U, \mathcal{S})$$

missing only those holomorphic maps  $f: U \rightarrow \mathcal{S}$  which are constant  $\infty$  on a component of  $U$ .

**Example 12.15** If  $f \in \mathcal{O}(U \setminus \{p\})$ ,  $p \in U \subset \mathbb{C}$ , and  $p$  is a pole of order  $k$ , then  $\frac{1}{f(z)}$  has a zero of order  $k$  at  $p$ .

A polynomial of degree  $d$  has (a pole of) order  $d$  at  $\infty$ . Since  $p(\infty) = \infty$ , we need to look at  $1/p(1/z)$ ,

$$\text{ord}(p(z); z = \infty) = \text{ord}\left(\frac{1}{p\left(\frac{1}{z}\right)}; z = 0\right) = d$$

because

$$\frac{1}{a_d \left(\frac{1}{z}\right)^d + a_{d-1} \left(\frac{1}{z}\right)^{d-1} + \dots + a_1 \left(\frac{1}{z}\right) + a_0} = \frac{z^d}{a_d + a_{d-1}z + \dots + a_1 z^{d-1} + a_0 z^d}, \quad a_d \neq 0.$$

A holomorphic map  $f: S \rightarrow S$  is a rational function, i.e. of the form  $f(z) = \frac{p(z)}{q(z)}$  with polynomials  $p$  and  $q$ .

Meromorphic functions  $U \rightarrow \mathbb{C}$  are holomorphic maps  $U \rightarrow S$ .