Homework

1. None of the following pairs $(\mathcal{P}, \mathcal{L})$ is a plane. In each case say in which way Definition 1.1 is violated.

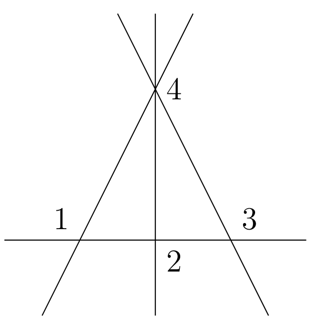
   (a) Let $\mathcal{P} = \mathbb{N} \setminus \{1\}$ be the set of natural numbers bigger than 1. Let $\mathcal{L} = \{\{a, b, ab\} \mid a, b \in \mathbb{N}, 1 < a < b\}$.
   Solution: This is not a plane, because $\{2, 3, 6\}, \{2, 6, 12\} \in \mathcal{L}$ are different but $2, 6$ are elements of both. This violates the uniqueness in Definition 1.1, part 2.

   (b) $\mathcal{P} = \mathbb{Z} \times \mathbb{Z}$, $\mathcal{L} = \{\{(ta + a_0, tb + b_0) \mid t \in \mathbb{Z}\} \mid a, b, a_0, b_0 \in \mathbb{Z}\}$.
   Solution: Denote by $l_{a,b,a_0,b_0} = \{(ta + a_0, tb + b_0) \mid t \in \mathbb{Z}\}$ the line with paramers a, b, a_0, b_0. Then $(0, 0), (2, 2) \in l_{1,1,0,0}$ and $(0, 0), (2, 2) \in l_{2,2,0,0}$ but $l_{1,1,0,0} \neq l_{2,2,0,0}$. Again uniqueness in Definition 1.1, part 2 is violated.
   Alternatively:
   The sets $l_{0,0,a_0,b_0}$, for instance $l_{0,0,0,0}$ are elements of $\mathcal{L}$, but $l_{0,0,a_0,b_0} = \{(a_0, b_0\}$ has only one element. Thus the pair $(\mathcal{P}, \mathcal{L})$ also violates Definition 1.1, part 3

2. Construct a plane so that not all lines have the same number of points.

   Hint: Be minimalistic here: On any line there must be 2 points at least. Since you want a line with a number of points different from that, try to introduce a line with 3 points.

   Solution: $(\{1, 2, 3, 4\}, \{\{1, 2, 3\}, \{1, 4\}, \{2, 4\}, \{3, 4\}\})$ is such a plane.



3. Is there a plane with 100 points and 5 lines?

   Solution: At least one of the lines, say g must have at least 6 points $\{A_1, A_2, A_3, A_4, A_5, A_6, \ldots\}$, and there must be one point X not on g. The lines (XA_i), $i = 1 \ldots 6$, are different because two different lines can intersect at most in one point. Thus we have at least 6+1=7 lines.

4. (a) How many total orders (see Definition 2) are there on $\{1, 2, 3, 4, 5\}$?
   Solution: This is the same as the number of permutations, i.e. 5!.

   (b) Define a relation $\ll$ on $\mathbb{R} \times \mathbb{R}$ so that for any $x, y, a, b \in \mathbb{R}$ we have

$$(x, y) \ll (a, b) \overset{\text{def}}{\Longleftrightarrow} (x \leq a, x \neq a) \text{ or } (x = a, y \leq b) \; .$$

   Prove that $\ll$ so defined is a total order on $\mathbb{R}^2$.
   Hint: Use that $\leq$ is a total order on $\mathbb{R}$.
   Solution: For $x \in \mathbb{R}$ we clearly have $(x, x) \ll (x, x)$.
   Now assume $(x, y) \ll (a, b)$ and $(a, b) \ll (x, y)$. There are two cases:

i. $x = a$. Then we have $y \leq b$ and $b \leq y$, hence $y = b$ because $\leq$ is a total order on $\mathbb{R}$. Thus $(x, y) = (a, b)$.

ii. $x < a$ and $a < x$, but this is impossible.

Finally assume that

$$(x, y) \ll (a, b) \quad \text{and} \quad (a, b) \ll (r, s) \ . \tag{0.1}$$

Here we need to distinguish a number of cases.

i. $x = a = r$. Then we have $y \leq b$ and $b \leq s$, hence $x = r$ and $y \leq s$ and therefore $(x, y) \ll (r, s)$.

ii. $x < a$ and $a = r$. Then we have $x < r$ and therefore $(x, y) \ll (r, s)$.

iii. $x = a$ and $a < r$. Then we have $x < r$ and therefore $(x, y) \ll (r, s)$.

iv. $x < a$ and $a < r$. Then we also have $x < r$ and therefore $(x, y) \ll (r, s)$.

In all cases we have shown that (0.1) implies $(x, y) \ll (r, s)$

5. Let $(\mathcal{P}, \mathcal{L}, \leq)$ be a plane with segments and $A, B \in \mathcal{P}$, $A \neq B$. Prove that

$$[A, B \cup [B, A = (AB) \ .$$

Solution: We may assume that $A \underset{(A,B)}{\leq} B$. Then

$$[A, B = \left\{ X \in (A, B) \,\middle|\, A \underset{(A,B)}{\leq} X \right\}$$

$$[B, A = \left\{ X \in (A, B) \,\middle|\, X \underset{(A,B)}{\leq} B \right\} \ .$$

By definition,

$$[A, B \cup [B, A \subset (A, B) \ .$$

Since $\underset{(A,B)}{\leq}$ is a total order on (A, B),

$$\begin{aligned}(A, B) &= \left\{ X \in (A, B) \,\middle|\, A \underset{(A,B)}{\leq} X \right\} \cup \left\{ X \in (A, B) \,\middle|\, X \underset{(A,B)}{\leq} A \right\} \\ &= [A, B \cup \left\{ X \in (A, B) \,\middle|\, X \underset{(A,B)}{\leq} A \right\}\end{aligned}$$

By transitivity of the total order $\underset{(A,B)}{\leq}$, since $A \underset{(A,B)}{\leq} B$, we have that

$$X \underset{(A,B)}{\leq} A \quad \text{implies} \quad X \underset{(A,B)}{\leq} B \ .$$

hence

$$\left\{ X \in (A, B) \,\middle|\, X \underset{(A,B)}{\leq} A \right\} \subset \left\{ X \in (A, B) \,\middle|\, X \underset{(A,B)}{\leq} B \right\} = [B, A$$

and therefore

$$(A, B) = [A, B \cap [B, A$$

as claimed.

6. Consider the plane $(\mathcal{P}, \mathcal{L})$ where

$$\mathcal{P} = \{1, 2, 3, a, b, c\}$$
$$\mathcal{L} = \{\{1, 2, 3\}, \{a, b, c\}\} \cup \{\{i, j\} \mid i = 1, 2, 3, \; j = a, b, c\}$$

(assuming that $a, b, c, 1, 2, 3$ are pairwise different). In order to define a plane with segments $(\mathcal{P}, \mathcal{L}, \leq)$ having this as underlying plane, we only need to specify total orders $\underset{\{1,2,3\}}{\leq}$ and $\underset{\{a,b,c\}}{\leq}$ because the other elements of $\mathcal{L}$ have only 2 elements and therefore a unique pair of mutually reverse total orders.

Is there a order function $\leq$ so that $(\mathcal{P}, \mathcal{L}, \leq)$ is a plane with segments?

Hint: Consider the remark after Definition 2.5. Wlog, we may assume that 2 is between $1, 3$ and b is between a, c (why?). Look at the half planes of the line $(2b)$. Show that 1 and 3 must be in the same half plane, and show that they must be in different half planes. A contradiction.

Solution: The segments $[1a]$ and $[a3]$ do not intersect the line $(2b)$, hence $1, a, 3$ lie in the same half plane of $(2b)$. On the other hand, the segment

$$[13] = \{1, 2, 3\}$$

intersects $(2b)$ in the point 2, hence $1, 3$ lie in different half planes of $(2b)$.

7. Let A, B, C, D be points of a Euclidean plane so that $B \neq D$ and A, C lie in different half planes of (BD). Assume that $|A, B| = |A, D|$ and $|C, B| = |C, D|$. Let X be the midpoint of $[B, D]$, i.e. the unique point on $[B, D]$ so that $|B, X| = |D, X|$.

   (a) Prove that all the angles at X are right angles , i.e.

   $$90 = |\angle BXA| = |\angle AXD| = |\angle DXC| = |\angle CXB| \; .$$

   Solution: The triangle (ABD) is isosceles, hence $|\angle ABX| = |\angle ABD| = |\angle ADB| = |\angle ADX|$ By SAS (angles at B resp. D), $(ABX) \cong (ADX)$, hence $|\angle AXB| = |\angle AXD|$. But $\angle BXD$ is straight, hence

   $$180 = |\angle BXD| = |\angle BXA| + |\angle AXD| = 2\,|\angle BXA|$$

   and therefore

   $$|\angle AXB| = |\angle AXD| = 90 \; .$$

   Replacing A with C in the above argument shows that also

   $$|\angle CXB| = |\angle CXD| = 90 \; .$$

   (b) Prove that $[A, C] \cap [B, D] = \{X\}$.
   Solution: By the additivity of the angle measure,

   $$|\angle CXA| = |\angle CXB| + |\angle BXA| = 90 + 90 = 180$$

   hence $\angle CXA$ is a straight angle, the points C, X, A lie on a common line and X is between A, C. In particular $X \in [AC]$.

8. Let $A, B, C \in l$ be points on a common line in a Euclidean plane. Let M be the midpoint of $[A, B]$. Prove that if C is not between A, B then

$$|C, A| + |C, B| = 2\,|C, M| \; .$$

Hint: This is a simple statement about real numbers.

Solution: If $A = B$ then $A = B = M$. Thus we may assume $A \neq B$ and choose the order $\leq \in \underset{(AB)}{\leq}$ so that $A \leq B$. Then there are three cases:

(a) $C \leq A$: Then $C \leq A \leq M \leq B$ and by additivity of the distance,

$$|C, A| + |A, M| = |C, M| \quad \text{and} \quad |C, M| + |M, B| = |C, B| \ .$$

Since M is the midpoint of $[A, B]$, hence $|A, M| = |M, B|$, we get

$$|C, A| = |C, M| - |A, M|$$

$$|C, B| = |C, M| + |M, B| = |C, M| + |A, M| \ .$$

Adding the two equations proves the claim.

(b) $A \leq C \leq B$ and $A \neq C \neq B$. This means that C is between A and B, which is excluded by assumption.

(c) $B \leq C$: Then $A \leq M \leq B \leq C$ and by additivity of the distance,

$$|C, A| = |C, M| + |A, M| \quad \text{and} \quad |C, M| = |C, B| + |M, B| \ .$$

Since M is the midpoint of $[A, B]$, hence $|A, M| = |M, B|$, we get

$$|C, A| = |C, M| + |A, M|$$

$$|C, B| = |C, M| - |M, B| = |C, M| - |A, M| \ .$$

As before, adding the two equations proves the claim.

9. In a plane with segments, can there be four points such that

(a) no three of the points lie on one line, and

(b) each of the four points lies in the interior region of any angle formed by the other three points.

Prove your answer.

Hint: Thus, if the points are named A, B, C, D, in any order, we must have $D \in IR(\angle ABC)$, $D \in IR(\angle ACB)$, $D \in IR(\angle BAC)$, $A \in IR(\angle BCD$, $A \in IR(\angle BDC$, $A \in IR(\angle CBD$ ...

Solution: This is true for the plane of 2 sets on 4 points. In this plane every point not on the rays of an angle lies in the interior region of that angle. This is because one of the two half planes of any line is empty.

10. Let $(\mathcal{P}, \mathcal{L})$ be a plane with finitely many points (i.e. $\mathcal{P}$ is finite). Additionally assume that

(a) For each $Q \in \mathcal{P}$ and $l \in \mathcal{L}$ with $Q \notin l$ there is exactly one $a \in \mathcal{L}$ with $Q \in a$ such that $a \cap l = \emptyset$ ("l and a do not intersect").

(b) For each $l, k \in \mathcal{L}$ there is $P \in \mathcal{P}$ such that $P \notin l$ and $P \notin k$.

Prove that all $l \in \mathcal{L}$ have the same number of points, i.e. that for $l, k \in \mathcal{L}$,

$$\# \{Q \in \mathcal{P} \,|\, Q \in l\} = \# \{Q \in \mathcal{P} \,|\, Q \in k\} \ .$$

Hint: Count the lines through a point.

Solution: (Sketch) Let $k, l \in \mathcal{L}$. By assumption there is $Q \in \mathcal{P}$ such that $Q \notin k$ and $Q \notin l$. Let $\mathcal{L}_Q$ be the set of lines through Q, i.e.

$$\mathcal{L}_Q := \{h \in \mathcal{L} \,|\, Q \in h\} \ .$$

By assumption, exactly one of the lines in $\mathcal{L}_Q$ does not intersect l and each of the other lines intersect l in exactly one point. Applying this to k in turn shows that

$$\#\{Q \in \mathcal{P} \mid Q \in l\} = \#\mathcal{L}_Q - 1 = \#\{Q \in \mathcal{P} \mid Q \in k\} \ .$$

11. Let A, B, C, X and A', B', C', X' be points in an Euclidean Plane so that the segments $[AC], [XC], [BC]$ are congruent to the segments $[A'C'], [X'C'], [B'C']$ respectively (i.e. $d(A,C) = d(A',C')$, $d(X,C) = d(X',C')$, $d(B,C) = d(B',C')$). Also assume that X lies between A, B and X' lies between A', B', and that the angles at X respectively X' are right angles, i.e.

$$|\angle AXC| = |\angle BXC| = |\angle A'X'C'| = |\angle' X'C'| = 90° \ .$$

Prove that the triangles ΔABC and $\Delta A'B'C'$ are congruent. Can we drop the assumption that X lies between A, B and X' between $A'B'$?

Solution: By SSA the we have congruences of the triangles

$$\Delta ACX \cong \Delta A'C'X' \quad \text{and} \quad \Delta BCX \cong \Delta B'C'X'$$

hence the congruences of segments

$$[AX] \cong [A'X'] \quad \text{and} \quad [BX] \cong [B'X']$$

i.e.

$$d(A,X) = d(A',X') \quad \text{and} \quad d(B,X) = d(B',X') \ .$$

Since X lies between A, B and X' lies between A', B' this implies that

$$d(A,B) = d(A,X) + d(X,B) = d(A',X') + d(X',B') = d(A',B') \quad \text{i.e} \quad [AB] \cong [A'B'] \ .$$

By SSS the triangles ΔABC and $\Delta A'B'C'$ are congruent.

12. Consider the metric g on $U = (0,\infty) \times \mathbb{R} \subset \mathbb{R}^2$ given by

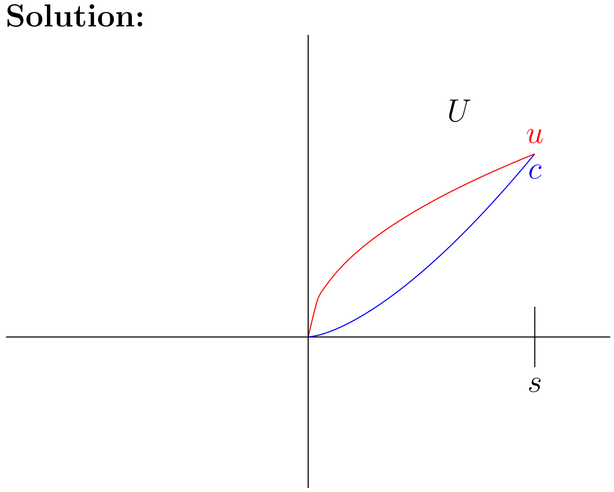
$$g_{x,y} = dx^2 + x dy^2 \ .$$

For some $s \in \mathbb{R}^+$, consider the curve $c\colon [0,s] \to \mathbb{R}^2$ with

$$c(t) = \begin{pmatrix} t \\ \frac{2}{3} t^{3/2} \end{pmatrix} \ .$$

(a) Sketch this curve.

Solution:



(b) Compute the length of c with respect to the metric g.
Solution: $c'(t) = (1, \sqrt{t})$

$$\textbf{length}^g(c) = \int_0^s \sqrt{1+t^2}\, dt = \frac{\operatorname{arcsinh}(s)}{2} + \frac{s\sqrt{1+s^2}}{2}$$

(c) Is c a geodesic? If not sketch a shorter curve with the same endpoints.
Solution: We compute the length of the curve $u\colon [0,s] \to \mathbb{R}^2$,

$$u(t) = \begin{pmatrix} t \\ \frac{2}{3}t^{1/2}s \end{pmatrix} .$$

This has the same endpoints as c. The speed of this curve is

$$u'(t) = \begin{pmatrix} 1 \\ \frac{1}{3}t^{-1/2}s \end{pmatrix} .$$

The length integral becomes

$$\textbf{length}^g(u) = \int_0^s \sqrt{1 + t\left(\frac{1}{3}st^{-1/2}\right)^2}\, dt = \int_0^s \sqrt{1+\frac{s^2}{9}}\, dt$$

$$= s\sqrt{1+\frac{s^2}{9}} = \frac{s\sqrt{9+s^2}}{3} .$$

This is less than $\frac{\operatorname{arcsinh}(s)}{2} + \frac{s\sqrt{1+s^2}}{2} = \textbf{length}^g(c)$, at least for sufficiently small s.

13. Recall the various representations of a complex number z as a pair of real numbers, a formal sum, a (2×2)-matrix:

$$z = (a,b) = a+ib = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \Re z + i\Im z = |z|\, e^{i\arg z} \text{ if } z \neq 0$$

Thus

$$\Re(z) = a \quad , \quad \Im(z) = b \quad , \quad |z| = \sqrt{\det\begin{pmatrix} a & -b \\ b & a \end{pmatrix}} = \sqrt{a^2+b^2} = \sqrt{z\overline{z}} ,$$

where $\overline{z} = (a,-b) = a - ib$ is the **complex conjugate** of z.

If $z \neq 0$, then $\arg z \in \mathbb{R}/2\pi\mathbb{Z}$ is a (equivalence class of) real number(s) so that

$$\frac{z}{|z|} = e^{i\arg z} = \cos(\arg z) + i\sin(\arg(z)) .$$

$\arg z$ is only determined up to integer multiples of 2π!

For the each of the following complex nubmers compute Real part $\Re(z)$, Imaginary part $\Im(z)$, modulus $|z|$ and argument $\arg z$.

(a) $1+i$
Solution: $\Re = 1$, $\Im = 1$, $|1+i| = \sqrt{2}$, $\arg(1+i) = \frac{\pi}{4} + 2\pi\mathbb{Z}$

(b) $1 + i\sqrt{3}$

Solution: $\Re = 1$, $\Im = \sqrt{3}$, $\left|1 + i\sqrt{3}\right| = 2$, $\arg(1 + i) = \frac{\pi}{3} + 2\pi\mathbb{Z}$

(c) $\dfrac{1}{3 + 2i}$

Hint: Try to make the denominator real.

Solution: Modulus and argument are immediately calculated:

$$\left|\frac{1}{3+2i}\right| = \frac{1}{|3+2i|} = \frac{1}{\sqrt{13}}$$

$$\arg\left(\frac{1}{3+2i}\right) = -\arg(3+2i) = -\arctan\frac{2}{3} + 2\pi\mathbb{Z}$$

In order to compute real and imaginary part, we multiply with the complex conjugate of the denominator, so that it becomes real:

$$\frac{1}{3+2i} = \frac{3-2i}{(3+2i)(3-2i)} = \frac{3-2i}{13}$$

hence $\Re = \frac{3}{13}$, $\Im = \frac{-2}{13}$.

(d) $\dfrac{1+2i}{1+3i}$

Solution: As before, modulus and argument are immediately read off:

$$\left|\frac{1+2i}{1+3i}\right| = \frac{|1+2i|}{|1+3i|} = \frac{\sqrt{5}}{\sqrt{10}} = \frac{1}{\sqrt{2}} \ .$$

$$\arg\left(\frac{1+2i}{1+3i}\right) = \arg(1+2i) - \arg(1+3i) = \arctan(2) - \arctan(3) = \arctan(2) - \arctan(3) \ ,$$

all modulo $2\pi\mathbb{Z}$.

For real and imaginary part we compute

$$\frac{1+2i}{1+3i} = \frac{(1+2i)(1-3i)}{10} = \frac{7-i}{10}$$

hence

$$\Re\frac{1+2i}{1+3i} = \frac{7}{10} \quad \text{and} \quad \Im\frac{1+2i}{1+3i} = -\frac{1}{10} \ .$$

14. For $i = a, b$, find a Möbius transformation $T_i \in \mathcal{M}$ so that

(a) $T_a(C(-1, 2)) = C(3, 4)$

Solution: This can be done by scaling and translation only. We first adjust the radius by scaling and then translate to map the correct center. Thus $T_a = \tau_{20/6}\sigma_{1/3}$, i.e.

$$T_a(z) = \frac{z}{3} + \frac{20}{6}$$

(b) $T_b(C(6, \infty)) = C(3, 4)$

Solution: Clearly the translation τ_{-6} maps $C(6, \infty)$ to $C(0, \infty)$. A Möbius transformation taking $C(3, 4)$ to $C(0, \infty)$ is $N(z) = \frac{z-4}{z-3}$. Since the inverse of the corresponding matrix is

$$\begin{pmatrix} 1 & -4 \\ 1 & -3 \end{pmatrix}^{-1} = \begin{pmatrix} -3 & 4 \\ -1 & 1 \end{pmatrix} \ ,$$

the Möbius transformation

$$N^{-1}(z) = \frac{-3z+4}{-z+1}$$

takes $C(0,\infty)$ to $C(3,4)$. We need to compose this with τ_{-6}. Thus

$$T_b(z) = \frac{-3(z-6)+4}{-(z-6)+1} = \frac{-3(z-6)+4}{-(z-6)+1} = \frac{-3z+22}{-z+7}$$

maps $C(6,\infty))$ to $C(3,4)$.

(c) $T_c(C(-5,-1)) = C(6,\infty)$

Solution: We first map to $C(0,\infty)$ and then translate. Thus

$$T_c(z) = \frac{z+1}{z+5} + 6 = \frac{7z+31}{z+5}$$

15. Recall the notation for hyperbolic lines, i.e. semicircles in the upper half plane with center on the real line or rays parallel to the imaginary axis. For a complex number $z = x+iy$ we denote by

$$\Re z = \Re(x+iy) = x \quad , \quad \Im z = \Im(x+iy) = y$$

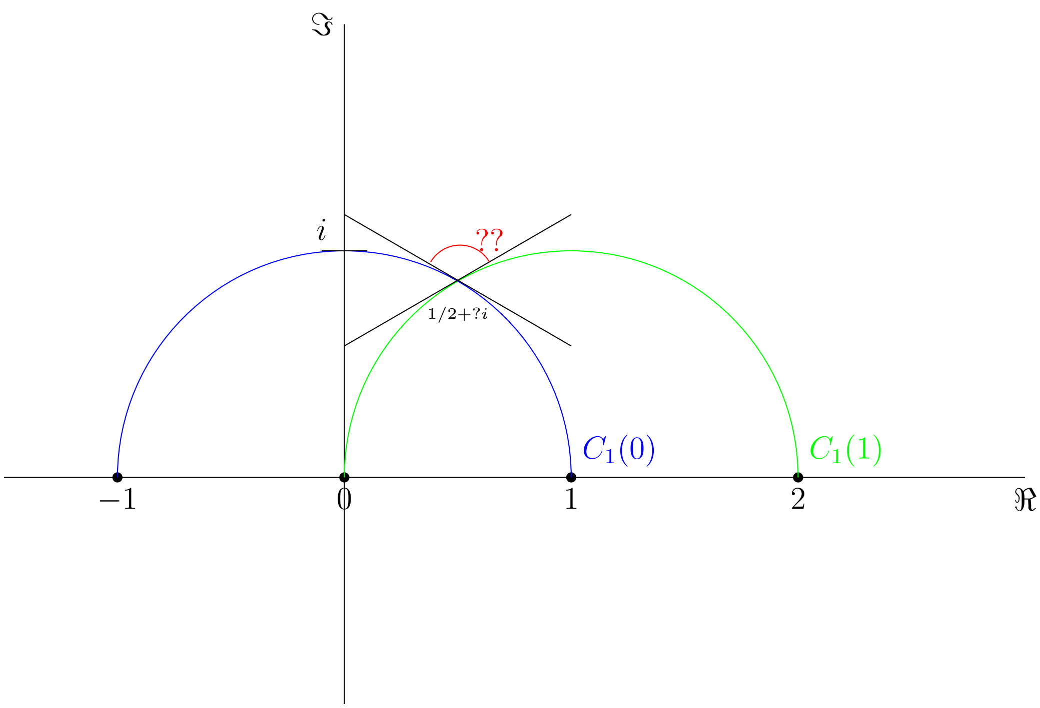
the real respectively imaginary part. Thus, for $p \in \mathbb{R}$, $r \in \mathbb{R}^+$, $A, B \in \mathbb{R}$,

$$C(a,b) = C_{\frac{|b-a|}{2}}\left(\frac{a+b}{2}\right) ,$$

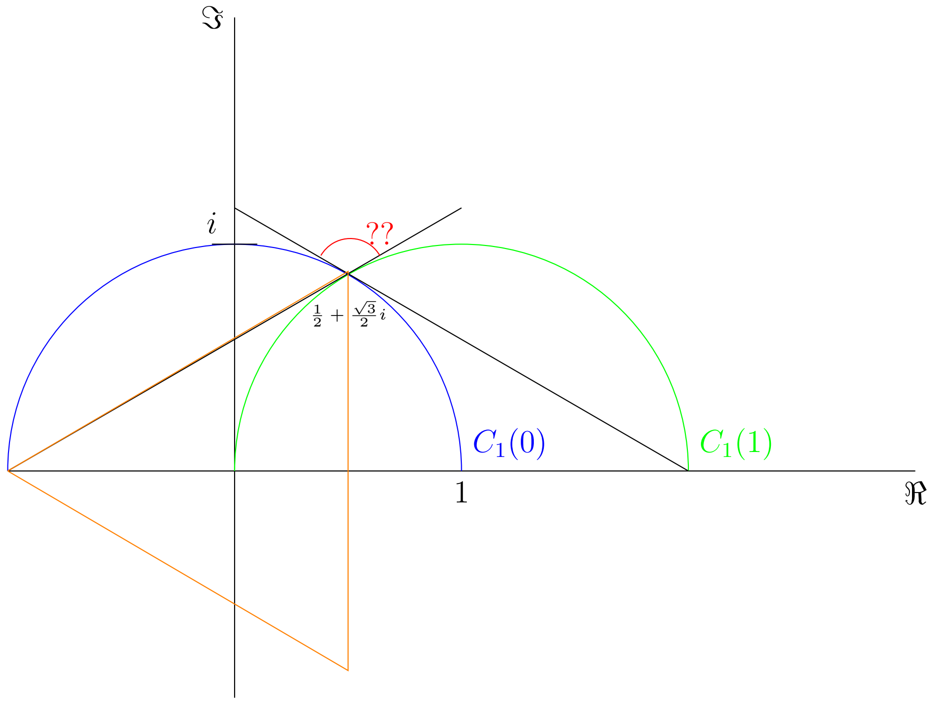
$$C_r(p) = \{z \in \mathbb{C} \,|\, \Im z > 0, |z-p| = r\} = \left\{p + re^{it} \,\middle|\, 0 < t < \pi\right\} = C(p-r, p+r) ,$$
$$C(a,\infty) = C(\infty, a) = \{z \in \mathbb{C} \,|\, \Im z > 0, \Re z = a\} = \left\{a + it \,\middle|\, t \in \mathbb{R}^+\right\} .$$

The hyperbolic angle between two hyperbolic lines intersecting at P is the standard Euclidean angle their tangents make at the point P.

Compute one of the two angles the hyperbolic lines $C_1(0)$ makes with $C_1(1)$:



Solution: The two tangents in the diagram extend,



producing the isosceles orange triangle. Hence the angles are $\frac{2\pi}{3} = 120°$ and $\frac{\pi}{3} = 60°$

Alternative solution: The two circles are:

$$C_1(0) = \left\{(x,y)\,\middle|\,x^2+y^2=1, y>0\right\} \quad , \quad C_1(1) = \left\{(x,y)\,\middle|\,(x-1)^2+y^2=1, y>0\right\} \ .$$

The circle intersect in the point (x_0, y_0) which must satisfy both equations, hence

$$x_0^2 + y_0^2 = 1 = (x_0-1)^2 + y_0^2 \quad \text{and} \quad y > 0$$

which leads to

$$(x_0, y_0) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \ .$$

Normal vectors to the circles at this point are

$$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \quad \text{and} \quad \left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$$

respectively. As tangent vectors we pick any of vectors perpendicular to these, for instance

$$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \quad \text{and} \quad \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

respectively. The angle between these is

$$\arccos\left(\frac{\left\langle\left(-\frac{\sqrt{3}}{2},\frac{1}{2}\right)\,\middle|\,\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)\right\rangle}{\left\|\left(-\frac{\sqrt{3}}{2},\frac{1}{2}\right)\right\|\left\|\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)\right\|}\right) = \arccos\left(-\frac{3}{4}+\frac{1}{4}\right) = \arccos\left(\frac{-1}{2}\right) = \frac{2\pi}{3} \ .$$

16. Let the metric g on $\mathbb{R}^2$ be given by

$$g_{(x,y)} = x^2 \, dx^2 + (1 + xy) \, dxdy + y^2 \, dy^2 \quad , \quad (x, y) \in \mathbb{R}^2 \, .$$

(a) For which (x, y) is $g_{(x,y)}$ positive definite? Sketch this region.

(b) Let $c\colon [3, 5] \to \mathbb{R}^2$ be the curve with $c(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$. Find the length integral of c with respect to this metric. You do not need to express this integral in terms of elementary functions.

17. Is it possible to make a football out with pathches so that

(a) there are 27 pentagons

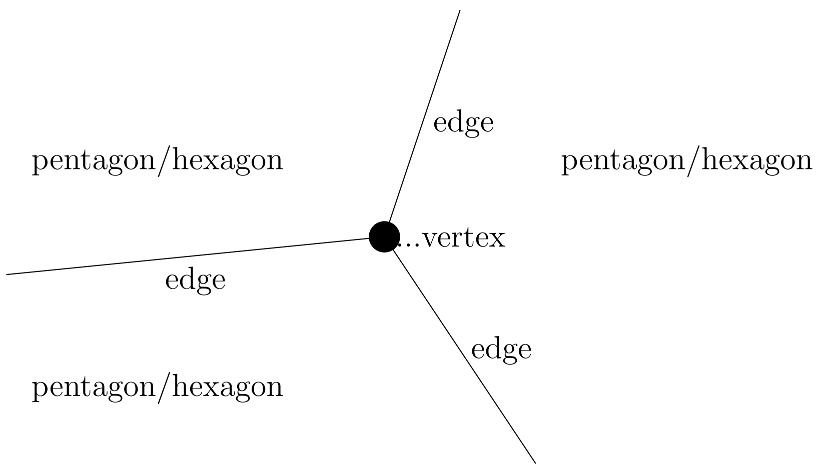
(b) the patches are hexagons or pentagons,

(c) every edge belongs to exactly two patches

(d) each vertex is met by exactly three edges

(e) each vertex belongs to exactly three patches

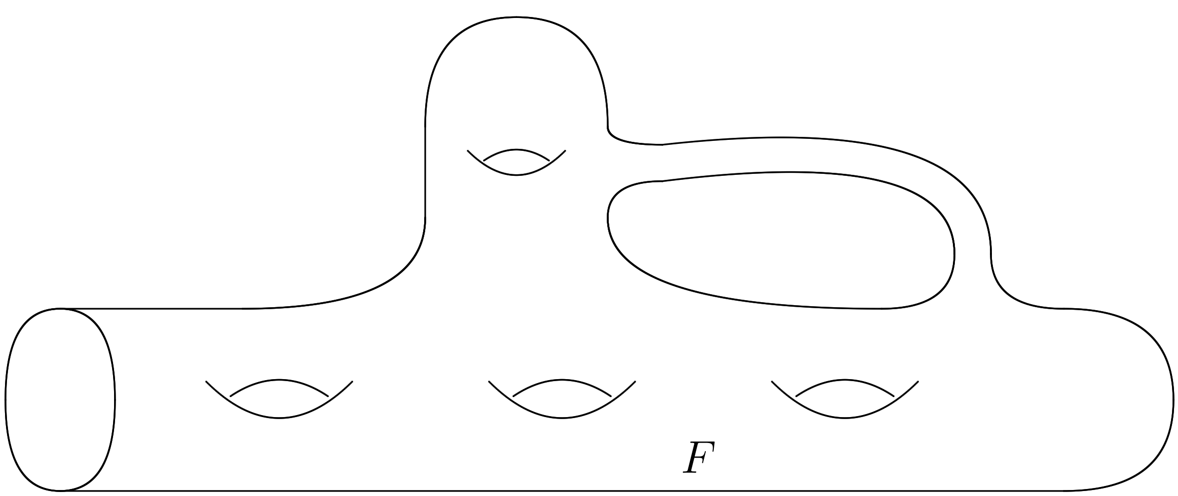
Thus the edge graph at each corner looks is a tripod, i.e. looks like



Hint: First express the Euler characteristic in terms of the numbers h, p, e, v of hexagons, pentagons, edges and vertices. From this you can eliminate the numbers e and v.

18. Compute the integral of the Gauß curvature over the surface F shown below.

Hint: Recall notions from example 5.3. You need to be careful at the boundary of this.



19. Let (A, B, C) be a triangle in an absolute geometry, i.e. in a Euclidean plane possibly not having unique parallels. Assume that the sum of the interior angles of the triangle (A, B, C) is different from 180. Show that then there are at least two parallels to (AB) through C.

Recommended Reading: Patrick D. Barry, *Geometry with Trigonometry*

1 Planes

Definition 1.1 *A* **plane** *is a pair* $(\mathcal{P}, \mathcal{L})$ *of nonempty sets* $\mathcal{P}$ *and* $\mathcal{L}$ *so that*

1. $\forall \ell \in \mathcal{L} : \ell \subset \mathcal{P}$,
2. $\forall A, B \in \mathcal{P}, A \neq B \exists! \ell \in \mathcal{L} : A \in \ell$ *and* $B \in \ell$,
3. $\forall \ell \in \mathcal{L} : \#\ell \geq 2$ *and* $\ell \neq \mathcal{P}$.

If $(\mathcal{P}, \mathcal{L})$ *is a plane and* $A, B \in \mathcal{P}$, $A \neq B$, *we denote by* (A, B) *the element of* $\mathcal{L}$ *defined by 2. Thus* $A, B \in (A, B) \in \mathcal{L}$ *and* (A, B) *is unique with this property.*

If $(\mathcal{P}, \mathcal{L})$ is a plane, we refer to the elements of $\mathcal{P}$ as points and to the elements of $\mathcal{L}$ as lines of the plane. If $A \in \ell \in \mathcal{L}$ we say that the point A lies on the line ℓ. We say that two lines ℓ and ℓ' intersect in a point P, if P lies in ℓ and P lies in ℓ'.

Example 1.2 *The* **standard plane** *is the plane* $(\mathcal{P}_{\text{std}}, \mathcal{L}_{\text{std}})$ *whose points are pairs of real numbers and whose lines are the (affine) lines. These are sets of the form*

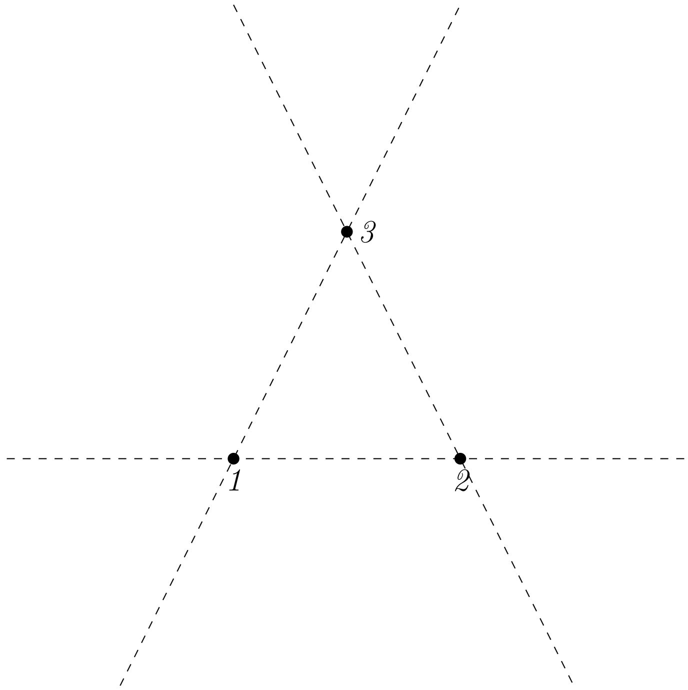
$$l_{q,v} := \{q + tv \mid t \in \mathbb{R}\}$$

for $q \in \mathbb{R}^2$ *and* $v \in \mathbb{R}^2 \setminus \{0\}$. *Thus* $\mathcal{P}_{\text{std}} = \mathbb{R}^2$ *and* $\mathcal{L}_{\text{std}} = \{l_{q,v} \mid q \in \mathbb{R}^2, v \in \mathbb{R}^2 \setminus \{0\}\}$

Example 1.3 *The smallest plane must have* 3 *points. In fact*

$$(\{1, 2, 3\} \ , \ \{\{1, 2\}, \{1, 3\}, \{2, 3\}\})$$

is a plane.



Example 1.4 *Generalizing example 1.3, for any set* M *with* $\#M > 2$ *the pair*

$$\left(M, \binom{M}{2}\right)$$

is a plane. Here

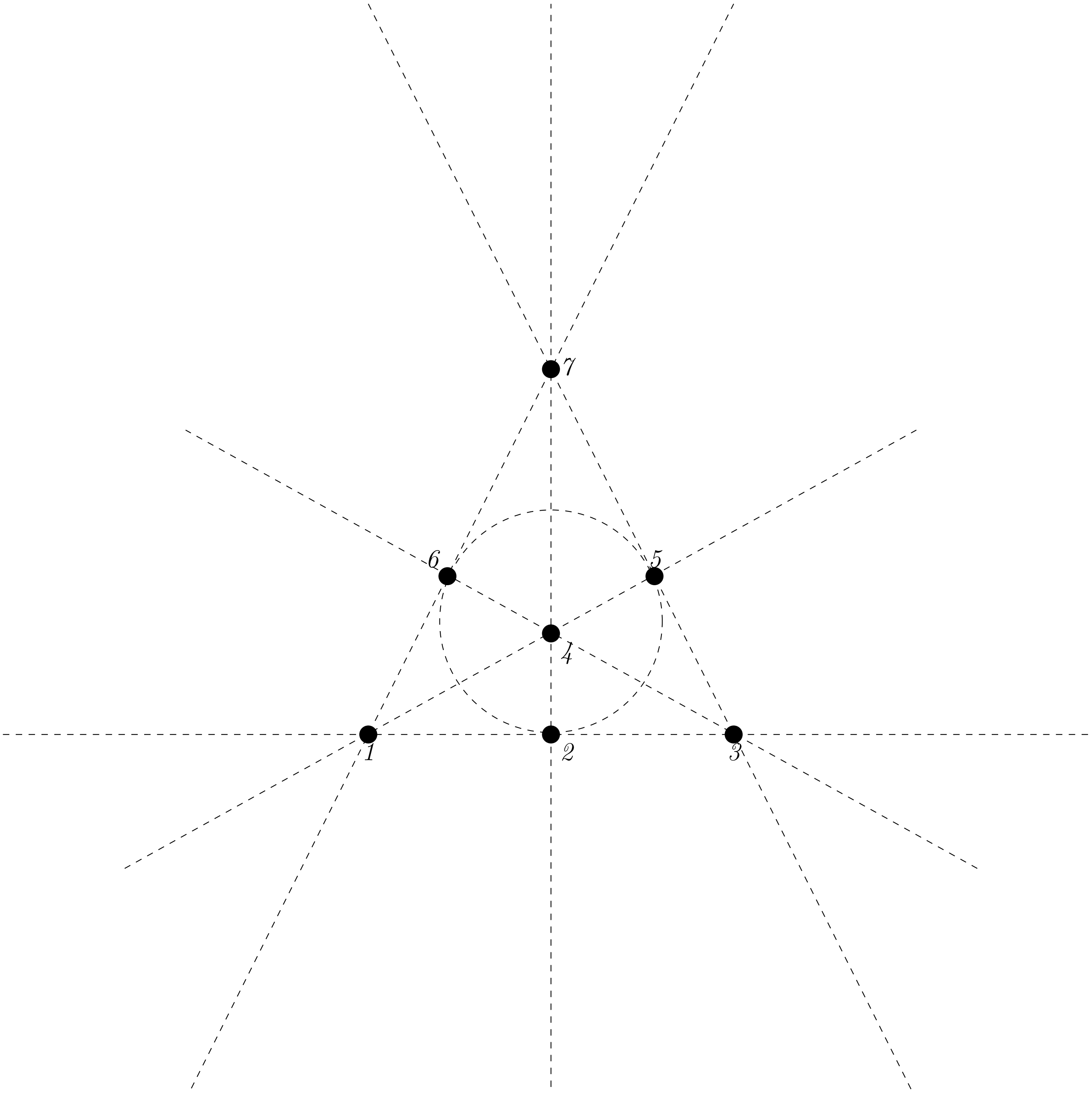
$$\binom{M}{2} = \{A \subset M \mid \#A = 2\}$$

denotes the set of 2*-sets of* M*.*

Example 1.5 *The Fano plane* $(\mathcal{P}_3, \mathcal{L}_3)$ *is a plane with 7 points and 7 lines,*

$$\mathcal{P}_3 = \{1, 2, 3, 4, 5, 6, 7\}$$

$$\mathcal{L}_3 = \{\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \{2, 4, 7\}, \{2, 5, 6\}, \{3, 4, 6\}, \{3, 5, 7\}\} \ .$$



2 Planes with segments, natural orders on lines

Definition 2.1 *A relation R on a set M is a subset of the cartesian product, $R \subset M \times M$. Usually we use some symbol like $\sim$, $=$, $<$, $<<$, $\ll$ which we write between the related elements, i.e. instead of*

$$(a, b) \in \ll \quad \text{we write} \quad a \ll b$$

Definition 2.2 *A* **total order** *on a set M is a relation $\ll$ such that for all $x, y, z \in M$ the following hold:*

1. *$x \ll y$ or $y \ll x$*
2. *$x \ll x$*
3. *if $x \ll y$ and $y \ll x$ then $x = y$*
4. *if $x \ll y$ and $y \ll z$ then $x \ll z$*

A **partial order** *on M is a relation that satsifies the last 3 of these conditions but not necessarily the first.*

If R is a relation, then its **reciprocal**, **reverse** or **opposite** is the relation $\overline{R}$ so that

$$a\,R\,b \quad \Longleftrightarrow \quad b\,\overline{R}\,a\ .$$

Problem 2.2 Check that the reciprocal of a total order is again a total order.

Solution: Let R be a total order on the set M, and let $\overline{R}$ be its reciprocal. We check the items in Definition for $\overline{R}$. By definition of the reciprocal, for $a, b \in M$, aRb is equivalent to $b\overline{R}a$, hence (for $a, b, c \in M$)

1. "$a\overline{R}b$ or $b\overline{R}a$" is equivalent to "bRa or aRb", equivalently "aRb or bRa" which holds since R is a total order.
2. $a\overline{R}a$ is equivalent to aRa.
3. If $a\overline{R}b$ and $b\overline{R}a$ then, by definition of the reciprocal, bRa and aRb. Since R is a total order this implies that $a = b$.
4. If $a\overline{R}b$ and $b\overline{R}c$ then bRa and cRb, hence cRa. Again by the definition of the reciprocal this gives $a\overline{R}c$.

If $\ll$ is a total order on a set l and $A, B, C \in l$, then we say B **lies between** A **and** C **wrt** $\ll$ equivalently B **lies between** C **and** A **wrt** $\ll$, if

$$A \ll B \ll C \quad \text{and} \quad A \neq B \neq C\ .$$

Problem 2.3 Let $\ll$ be a total order on ℓ and $A, B, C \in \ell$ so that B lies between A, C wrt $\ll$. Then $A \ll C$ and $A \neq C$. If $\overline{\ll}$ denotes the reciprocal of $\ll$, then B lies also between A, C wrt $\overline{\ll}$.

Solution: This is immediate from the definition of the reciprocal because for $A, B, C \in \ell$

$$A \ll B \ll C \quad \Longleftrightarrow \quad C \overline{\ll} B \overline{\ll} A$$

Thus a total order gives the same betweenness relation as its reciprocal.

Definition 2.5 *A* **plane with segments** *is a triple* $(\mathcal{P}, \mathcal{L}, \leq)$ *consisting of two sets* $\mathcal{P}, \mathcal{L}$ *and a order function* $\leq$ *so that*

1. $(\mathcal{P}, \mathcal{L})$ *is a plane (as defined in 1.1), and*
2. *The order function* $\leq$ *assigns to each* $\ell \in \mathcal{L}$ *a set* $\left\{ \underset{\ell}{\leq}, \underset{\ell}{\geq} \right\}$ *of two total orders on* ℓ *reciprocal to each other,* $\underset{\ell}{\geq} = \overline{\underset{\ell}{\leq}}$.
3. *For every* $\ell \in \mathcal{L}$ *there are subsets* $H_+, H_- \subset \mathcal{P} \setminus \ell$ *so that*
   (a) $\mathcal{P} = H_+ \cup \ell \cup H_-$ *and* $H_+ \cap H_- = \ell \cap H_+ = \ell \cap H_- \emptyset$
   (b) H_+ *and* H_- *are convex, i.e. if* $A, B \in H_\pm$, $X \in \mathcal{P}$, $A \underset{(A,B)}{\leq} X \underset{(A,B)}{\leq} B$ *then* $X \in H_+$.
   (c) *If* $A \in H_+$, $B \in H_-$, *then there is* $X \in (A, B) \cap \ell$ *so that* $A \underset{(A,B)}{\leq} X \underset{(A,B)}{\leq} B$ *or* $B \underset{(A,B)}{\leq} X \underset{(A,B)}{\leq} A$

The sets H_+, H_- are called the **half planes of** ℓ. Thus, given a line ℓ in a plane with segments, two points A, B lie in the same half plane of ℓ if and only if the segment $[A, B]$ does not intersect the line ℓ. Equivalently, two points $A, B \in \mathcal{P} \setminus \ell$ lie in different halp planes if and only if there is a point on ℓ that lies between A, B.

Theorem 2.6 (Pasch's "Axiom") *Let* $(\mathcal{P}, \mathcal{L}, \leq)$ *be a plane with segments. Let* $A, B, C \in \mathcal{P}$ *so that there is no* $k \in \mathcal{L}$ *with* $A, B, C \in k$ *("*A, B, C *do not lie on a common line", "*A, B, C *are not* **collinear**"*). Let* $\ell \in \mathcal{L}$, $A, B, C \notin \ell$ *and* $\ell \cap [A, B] \neq \emptyset$. *Then*

$$\ell \cap [A, C] \neq \emptyset \quad \text{or} \quad \ell \cap [B, C] \neq \emptyset \ .$$

Proof: Let H_+ and H_- be the half planes of ℓ. By assumption, the points $A, B, C \notin \ell$, hence, by 3a of Definition 2.5, $A \in H_+$, or $A \in H_-$. We assume that $A \in H_+$. If not we interchange the naming of H_+, H_-. If were in the same half plane as A, i.e. if $B \in H_+$ then by 3b $[A, B] \cap \ell = \emptyset$ contrary to the assumptions of the Theorem. As $B \notin \ell$, by 3a, we must have $B \in H_-$. Since $C \notin \ell$, by 3a, we have the two possibilities

1. <u>$C \in H_+$:</u> We have $C \in H_+$ and $B \in H_-$. By 3c, $[BC] \cap \ell \neq \emptyset$, proving the theorem in this case.
2. <u>$C \in H_-$:</u> We have $C \in H_-$ and $A \in H_+$. By 3c, $[AC] \cap \ell \neq \emptyset$, proving the theorem in this case.

Problem 2.6 There are exactly two total orders on a set with two elements and these are the opposite of each other. Thus if $\mathcal{P}$ is a set with more than 2 elements, and $(\mathcal{P}, \mathcal{L})$ is the 2-set plane, i.e.

$$\mathcal{L} = \{\{a, b\} \mid a, b \in \mathcal{P}, a \neq b\} \ ,$$

then there is a unique order function, hence there can be at most one plane with segments $(\mathcal{P}, \mathcal{L}, \leq)$. Prove that the 2-set plane (with the unique order function) always is a plane with segments.

Hint: You need to check Definition 2.5. When is $a \in \mathcal{P}$ between $p, q \in \mathcal{P}$?

As an example consider the 2-set plane of $\{1,2,3,4\}$, i.e. $(\mathcal{P}, \mathcal{L}, \leq)$ where

$$\mathcal{P} = \{1,2,3,4\}$$

$$\mathcal{L} = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$$

and the total order $\leq$ on $l = \{a,b\} \in \mathcal{L}$ is either so that $a \leq b$ or $b \leq a$. What are the half planes of $\{1,3\}$?

Solution: If $a, p, q \in \mathcal{P}$ then a is never between p and q because no l contains 3 elements. Thus one of two half planes must be empty. If $l = \{p,q\} \in \mathcal{L}$, $p, q \in \mathcal{P}$, $p \neq q$, then one of the half planes of l must be empty. Hence the half planes of l are

$$\emptyset \quad \text{and} \quad \mathcal{P} \setminus l\ .$$

These are convex because every subset of $\mathcal{P}$ is convex.

In a plane with segments we can define notions of Betweeness, Segments, Convexity, Rays, Angles.

Definition 2.8 *Let $(\mathcal{P}, \mathcal{L}, \leq)$ be a plane with segments.*

1. *Let $A, B, C \in \mathcal{P}$. We say C* **lies between** *A* **and** *B if*

$$A \neq B \quad , \quad C \in (AB) \quad , \quad C \neq A, B$$

and

$$A \underset{(A,B)}{\leq} C \underset{(A,B)}{\leq} B \quad \text{or} \quad A \underset{(A,B)}{\geq} C \underset{(A,B)}{\geq} B$$

where $\left\{ \underset{(A,B)}{\leq}, \underset{(A,B)}{\geq} \right\}$ are the natural orders on (AB).

2. *For $A, B \in \mathcal{P}$ we define the* **segment** *$[A,B]$ as*

$$[A,B] = \{A,B\} \cup \{X \in (A,B) \,|\, X \text{ between } A, B\}\ .$$

If $A \neq B$ and $\underset{(AB)}{\leq}$ is the natural order on (AB) so that $A \underset{(AB)}{\leq} B$, then

$$[AB] = \left\{ X \in (AB) \,\middle|\, A \underset{(AB)}{\leq} X \underset{(AB)}{\leq} B \right\}\ .$$

3. *A subset $V \subset \mathcal{P}$ is* **convex** *if*

$$\forall A, B \in V : [A,B] \subset V\ .$$

4. *Let $A \in \ell \in \mathcal{L}$ and $\leq$ be one of the two natural orders of ℓ. Then the sets*

$$\{X \in \ell \,|\, A \leq X\} \quad \text{and} \quad \{X \in \ell \,|\, X \leq A\}$$

are called the **rays** *of ℓ with vertex X. We usually specify a ray on a given line by a point. Thus, let $A, B \in \mathcal{P}$, $A \neq B$. The* **ray** *$[AB$ is the set of all $X \in (AB)$ so that A is not between B and X. If among the two natural orders on (AB), the order $\underset{(AB)}{\leq}$ is the one so that $A \underset{(AB)}{\leq} B$, then*

$$[AB = \left\{ X \in (AB) \,\middle|\, A \underset{(AB)}{\leq} X \right\} .$$

5. *If $\ell \in \mathcal{L}$ and $B \in \mathcal{P} \setminus \ell$, we denote by $H(\ell, B)$ the* **half plane of ℓ containing** *B,*
$$H(\ell, B) = \{X \in \mathcal{P} \,|\, [X, B] \cap \ell = \emptyset\} .$$

6. *An* **angle** *is a set of two rays with the same vertex: For $A, X, B \in \mathcal{P}$, $A \neq X \neq B$,*
$$\angle AXB := \{[X, A \ , \ [X, B\}$$
A **zero angle** *is an angle $\angle AXB$ whose rays coincide, $[XA = [XB$. A* **straight angle** *is an angle $\angle AXB$ whose rays form a line, i.e. $X \in [A, B]$.*

7. *The* **interior region of a non straight angle** *$\angle AXB$ is empty in case of the zero angle and*
$$\mathrm{IR}\,(\angle AXB) = H((AX), B) \cap H((BX), A)$$
if $[XA \neq [XB$.

Problem 2.8 Let k, ℓ be lines intersecting in one point X, $k \cap l = \{X\}$, and let H be a half plane of ℓ. Show that $(k \cap H) \cup \{X\}$ is a ray.

Solution: If $k \cap H = \emptyset$ then, since $\#k \geq 2$, there is $P \in k$, $P \neq X$, and $P \in H' = \mathcal{P} \setminus (\ell \cup H)$, the other half plane of ℓ. Let $\leq$ be the natural order on k so that $X \leq P$. We claim that
$$(k \cap H) \cup \{X\} = \{X\} = \{Y \in k \,|\, Y \leq X\} .$$
To see this, let $Y \in k$, $Y \leq X$, $Y \neq X$. Then $[Y, P] \cap \ell = \{X\}$, hence P and Y are in different half planes of ℓ. But this forces $Y \in H \cap k$ which can therefore not be empty, impossible.

We may thus assume that there is $Q \in k \cap H \neq \emptyset$, and choose the natural order $\leq$ on k so that $X \leq Q$. We claim that
$$(k \cap H) \cup \{X\} = \{R \in k \,|\, X \leq R\} .$$
In order to see this, first assume that $R \in k$, $R \neq X$ and $X \leq R$. Then X is not between R and Q. If the intersection $[RQ] \cap \ell$ were not empty, then we would have $k = \ell$, impossible. Thus $[RQ] \cap \ell = \emptyset$ and R and Q lie in the same half plane of ℓ, i.e. in H. For the reverse inclusion, assume that $R \in k \cap H$. If we had $R \leq X$, then $[R, Q] \cap \ell \ni X$, hence R and Q would have to lie in different half planes of ℓ, hence $R \notin H$.

Problem 2.9 Show that every ray r in a plane with segments is of the form
$$r = (H \cap k) \cup \{X\}$$
where $\{X\} = \ell \cap k$ is the vertex of the ray and ℓ, k are lines, H a half plane of ℓ.

Solution: Let $r = \{R \in k \,|\, X \leq R\}$ for some $X \in k$, k a line in the plane and $\leq$ one of its natural orders. Let Q be a point not in k, and let H, H' be the half planes of $\ell = (XQ)$. Then $r = (H \cap k) \cup \{X\}$ or $r = (H' \cap k) \cup \{X\}$

3 Euclidean Planes, Absolute Geometry

Definition 3.1 *A euclidean plane is a quintuple* $(\mathcal{P}, \mathcal{L}, \leq, d, |\angle\cdot|)$ *so that* $(\mathcal{P}, \mathcal{L}, \leq)$ *is a plane with segments, d ("the distance") a real valued function on* $\mathcal{P} \times \mathcal{P}$ *("pairs of points") and* $|\angle\cdot|$ *("the angle measure", "the degree measure of the angle") a function on angles (i.e. sets of two rays). We write*

$$|A, B| = d(A, B)$$

for the distance of A, B *and, if* $B \neq X \neq A$*,*

$$|\angle AXB| = |\{[X, A, [X, B\}|$$

for the measure of the angle between the rays $[X, A, [X, B$*. The distance and the degree measure of the angle must satisfy the following:*

1. **Axioms for the Distance**

   The distance must satsify $(A, B \in \mathcal{P})$

   (a) $|A, B| \geq 0$ *and* $|A, B| = 0$ *if and only if* $A = B$*.*

   (b) $|A, B| = |B, A|$

   (c) *("Additivity") If* $Q \in [A, B]$ *then* $|A, B| = |A, Q| + |Q, B|$

   (d) *("Construction") If* $A \neq B$ *then for every* $\lambda \in \mathbb{R}$*,* $\lambda \geq 0$*, there is a unique* $S \in (A, B)$ *and a unique* $R \in (A, B)$ *so that*

   $$R \underset{(A,B)}{\leq} A \underset{(A,B)}{\leq} S \quad \text{and}$$

   $$|R, A| = |A, S| = \lambda \ .$$

2. **Axioms for the Angle Measure**

   The angle measure must satisfy the properties below. $(A, X, Z, B \in \mathcal{P},\ A \neq X \neq B)$

   (a) $|\angle AXB| \in [0, 180]$

   (b) $|\angle AXB| = 0$ *if* $\angle AXB$ *is the zero angle (i.e.* $[X, A = [X, B)$ *and* $|\angle AXB| = 180$ *if* $\angle AXB$ *is a straight angle (i.e. if* $X \in [A, B]$*)*

   (c) *("Additivity") If* $Z \in IR(\angle AXB)$ *and* $\angle AXB$ *is not straight, then*

   $$|\angle AXB| = |\angle AXZ| + |\angle ZXB|$$

   (d) *("Additivity") If* $\angle AXB$ *is straight and* $Z \in \mathcal{P} \setminus (A, B)$*, then*

   $$180 = |\angle AXB| = |\angle AXZ| + |\angle ZXB|$$

   (e) *("Uniqueness") If* $P, Q \in \mathcal{P}$*,* $[P, Q] \cap (X, A) = \emptyset$ *(i.e.* P, Q *lie in the same half plane of* (XA)*), then*

   $$|\angle AXP| = |\angle AXQ| \quad \Longrightarrow \quad [X, P = [X, Q \ .$$

   (f) *("Construction") Let* H *be a half plane of* (X, A)*. Then for every* $\lambda \in (0, 180)$ *there is* $Q \in H$ *so that*

   $$|\angle QXA| = \lambda \ .$$

3. **The Congruence Axiom SAS for Triangles**

If $A, B, C, A', B', C' \in \mathcal{P}$, $A \neq B \neq C \neq A, A' \neq B' \neq C' \neq A'$ are so that

$$|A,B| = |A',B'| \quad , \quad |A,C| = |A',C'| \quad \text{and} \quad |\angle BAC| = |\angle B'A'C'|$$

then

$$|\angle ABC| = |\angle A'B'C'| \quad , \quad |\angle ACB| = |\angle A'C'B'| \quad \text{and} \quad |B,C| = |B',C'|$$

3.1 Consequences of the axioms for distane and angle measure

Proposition 3.2 *Let $(\mathcal{P}, \mathcal{L}, \leq, d, |\angle \cdot|)$ be a Euclidean plane. Let $Q \in \ell \in \mathcal{L}$ and let $\underset{\ell}{\leq}$ be one of the natural orders on ℓ. Then*

$$\ell \xrightarrow{\lambda_{\ell,Q,\underset{\ell}{\leq}}} \mathbb{R}$$
$$X \longmapsto \begin{cases} |Q,X| & \text{if } Q \underset{\ell}{\leq} X \\ -|Q,X| & \text{if } X \underset{\ell}{\leq} Q \end{cases}$$

is bijective.

Proof: To prove surjectivity, let $\lambda \in \mathbb{R}$. Then there are points R, S so that $R \underset{\ell}{\leq} Q \underset{\ell}{\leq} S$ and $|RQ| = |\lambda| = |QS|$. If $\lambda \geq 0$, then $\lambda_{\ell,Q,\underset{\ell}{\leq}}(S) = |Q,S| = |\lambda| = \lambda$. If $\lambda \leq 0$, then $\lambda_{\ell,Q,\underset{\ell}{\leq}}(R) = -|Q,R| = -|\lambda| = \lambda$.

Injectivity follows from the uniqueness of the points R, S in the last postulate for the distance. If $\lambda_{\ell,Q,\underset{\ell}{\leq}}(X) = \lambda_{\ell,Q,\underset{\ell}{\leq}}(Y) =: \lambda$ then $|Q,X| = |Q,Y|$. The only ambiguity this leaves would be that X, Y are on different sides of Q. But then the signs of $\lambda_{\ell,Q,\underset{\ell}{\leq}}(X)$ and $\lambda_{\ell,Q,\underset{\ell}{\leq}}(Y)$ would be different. •

There is a similar bijection for angles:

Proposition 3.3 *Let $(\mathcal{P}, \mathcal{L}, \leq, d, |\angle \cdot|)$ be a Euclidean plane. Let $A, Q \in \mathcal{P}$, $A \neq Q$, and let H be one of two half planes of (AQ). Then the map*

$$\{[QX \mid X \in H \cup (AQ) \setminus \{Q\}\} \xrightarrow{\alpha_{Q,A,H}} [0,180]$$
$$\angle AQX \longmapsto |\angle AQX|$$

is a bijection.

Proof: If $X \in [QA$ or $X \in (QA) \setminus [QA$ then $\angle AQX$ is zero or straight and $|\angle AQX|$ is 0 or 180. If $\alpha \in (0,180)$ then a ray $[QX$ with $\alpha_{[QA,H}([QX) = \alpha$ exists by the last axiom for the angle measure. This shows that $\alpha_{[QA,H}$ is surjective.

As before injectivity follows from the uniqueness in the last axiom for the angle measure. •

Problem 3.3 Show that if A, B, C, D are four different points on the same line of a Euclidean plane with $C \in [A, B]$, $B \in [A, D]$ and

$$|A, C|\,|D, B| = |A, D|\,|C, B| \ ,$$

then

$$\frac{1}{|A, C|} + \frac{1}{|A, D|} = \frac{2}{|A, B|} \ .$$

Hint: This is a theorem about real numbers, obfuscated by transporting it to a line in a Euclidean plane.

Solution: Let $\leq$ be the natural order on (AB) so that $A \leq B$. Then $A \leq C \leq B$ and $A \leq B \leq D$. By transitivity of the total order $\leq$, we have

$$A \leq C \leq B \leq D \ .$$

Let $x, y, z \in \mathbb{R}_0^+$ be the positive real numbers

$$x = |A, C| \quad , \quad y = |C, B| \quad , \quad z = |B, D| \ .$$

Because of the additivity of the distance, we have

$$|A, B| = x + y \quad \text{and} \quad |AD| = x + y + z \ .$$

Thus for real numbers $x, y, z > 0$ we have

$$xz = (x + y + z)y \tag{3.5}$$

and need to show that

$$\frac{1}{x} + \frac{1}{x + y + z} = \frac{2}{x + y} \ .$$

The claim is equivalent to

$$(x + y + z)(x + y) + x(x + y) = 2x(x + y + z) \ ,$$
$$2x^2 + 3xy + y^2 + zx + zy = 2x^2 + 2xy + 2xz \ ,$$
$$xy + y^2 + zy = xz \ ,$$

which is (3.5).

Theorem 3.6 (Opposite Angles) *Let A, B, X, Y, Q be points in a Euclidean plane so that Q is between A, X and also between B, Y. Then the* **opposite angles** *$\angle AQB$ and $\angle XQY$ have the same measure.*

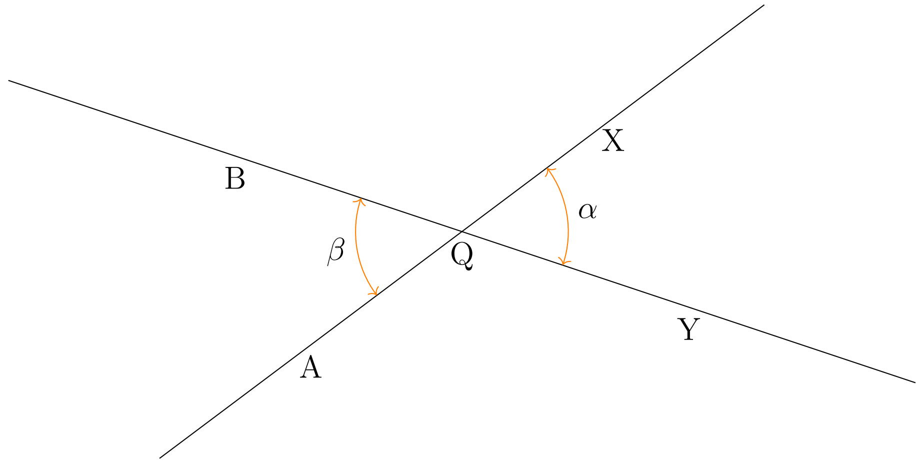
Proof: By the additivity postulates for the angle measure, in the case of straight angles, we have

$$180 = |\angle AQX| = |\angle AQB| + |\angle BQX| \ ,$$
$$180 = |\angle BQY| = |\angle BQX| + |\angle XQY| \ .$$

Subtracting the two equations gives

$$0 = |\angle AQB| - |\angle XQY| \ .$$

•



α and β are opposite angles.

3.2 Congruence

Definition 3.7 *Let $(\mathcal{P}, \mathcal{L}, \leq, |\cdot|, |\angle\cdot|)$ be a Euclidean plane. A* **polygon***, more specifically a n-***gon** *for some $n \in \mathbb{N}$, is a n-tuple $(P_1, \ldots, P_n)$ of pairwise different elements of $\mathcal{P}$, i.e. $P_i \neq P_j$ for $i \neq j$. The points P_i are referred to as the* **vertices** *of the polygon.*

Two n-gons $(P_1, \ldots, P_n)$ and $(Q_1, \ldots, Q_n)$ are said to be **congruent***, and we write*

$$(P_1, \ldots, P_n) \cong (Q_1, \ldots, Q_n)$$

if

$$\forall i, j = 1, \ldots, n : |P_i, P_j| = |Q_i, Q_j|$$

and

$$\forall i, j, k = 1, \ldots, n : |\angle P_i P_j P_k| = |\angle Q_i Q_j Q_k| \ .$$

Problem 3.7 Check that the congruence axiom is equivalent to the following: "In a Euclidean plane two triangles (A, B, C) and (A', B', C') are congruent if

$$|A, B| = |A', B'| \quad , \quad |A, C| = |A', C'| \quad \text{and} \quad |\angle BAC| = |\angle B'A'C'| \ ."$$

Note that for the congruence of two polygons, the order of the points is important. Thus, for instance, triangle (i.e. a 3-gon) (A, B, C) need not be congruent to (B, C, A).

Example 3.9 *Let (A, B, C) be a triangle in a Euclidean plane. Then the congruence*

$$(A, B, C) \cong (B, C, A)$$

means that

$$|A, B| = |B, C| = |C, A| \tag{3.10}$$

i.e. all the sides are equal, the triangle is **equilateral**, *and*

$$|\angle ABC| = |\angle BCA| = |\angle BAC| \tag{3.11}$$

i.e. all angles are also equal.

We will later see that for a triangle (A, B, C) in a Euclidean plane, (3.10) and (3.11) are equivalent.

Example 3.12 *Let (A, B, C) be a triangle in a Euclidean plane. Then the congruence*

$$(A, C, B) \cong (B, C, A)$$

yields that

$$|A, C| = |B, C| \ , \tag{3.13}$$

the sides containing C are equal. Such triangles are called **isosceles**. *For the angles this congruence gives*

$$|\angle ACB| = |\angle CBA| \tag{3.14}$$

the two angles away from C are equal.

Example 3.15 *Let (A, B, C) be an isosceles triangle, the sides containing C being equal,*

$$|A, C| = |B, C| \ .$$

Since the angle at C is

$$|\angle ACB| = |\angle BCA| \ ,$$

by the congruence axiom, we have the congruence

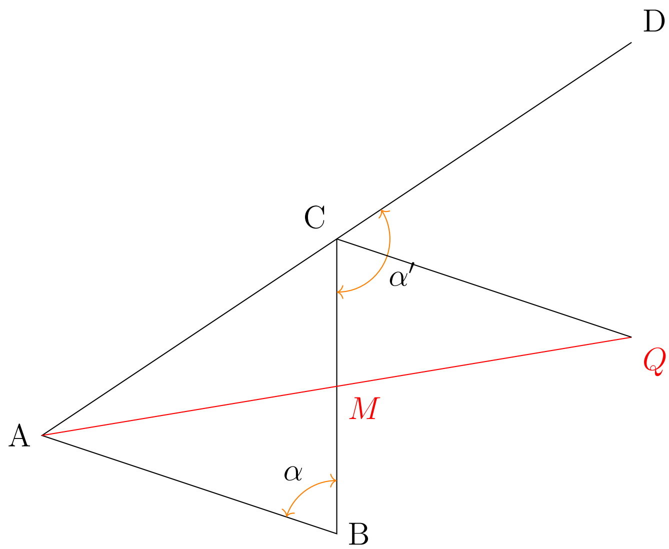
$$(A, C, B) \cong (B, C, A) \ .$$

Hence $|\angle ACB| = |\angle CBA|$. Thus we have shown that (3.13) implies (3.14). We will later show that (3.13) and (3.14) are equivalent.

Theorem 3.16 (Alternate Angles) *Let $(\mathcal{P}, \mathcal{L}, \leq, |\cdot|, |\angle\cdot|)$ be a euclidean plane, $A, B, C \in \mathcal{P}$ not on the same line, and $D \in (AC)$, $D \neq C$, so that $C \in [AD]$. Then*

$$|\angle ABC| < |\angle BCD|$$

$|\angle ABC|, |\angle BCD|$ as in the Theorem are called "alternate angles".



α and α' are alternate angles, $\alpha' > \alpha$.

Proof: Let A, B, C, D be as in the Theorem. Let M be the midpoint of the segment $[B, C]$, i.e. the point M on (BC) with $|M, B| = |M, C|$. Let $Q \in (AM)$ be so that M is the midpoint of the segment $[A, Q]$. The triangles AMB and QMC are congruent by the congruence axiom because

$$|\angle AMB| = |\angle QMC|$$

because the angles are opposite and

$$|A, M| = |Q, M| \quad \text{and} \quad |B, M| = |C, M|$$

because M is the midpoint of the two segments $[A, Q]$ and $[B, C]$. In particular, because of the congruence axiom,

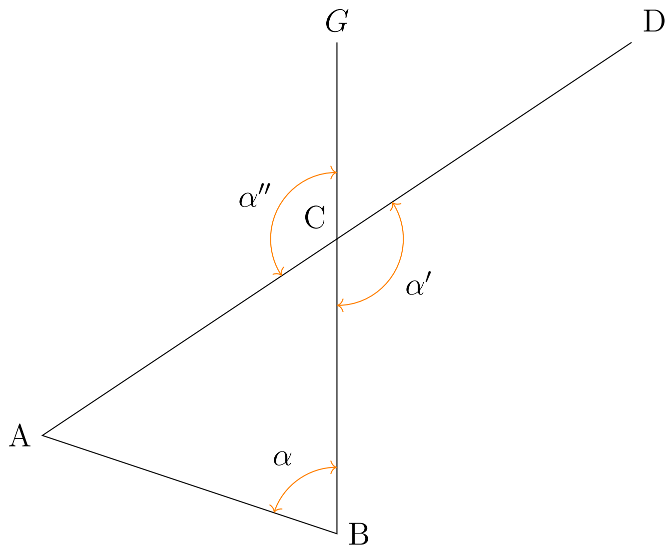
$$|\angle ABC| = |\angle ABM| = |\angle QCM| = |\angle QCB| \ .$$

Since $Q \in IR(\angle BCD)$, by the additivity axiom for the angle measure,

$$|\angle BCD| = |\angle BCQ| + |\angle QCD| > |\angle BCQ| = |\angle ABC| \ .$$

•

Corollary 3.17 *In the notation of Theorem 3.16, let* $G \in (BC)$ *so that* C *is between* B *and* G*. Then* $|\angle ACG| > |\angle ABG|$



α and α' are alternate angles, α' and α'' are opposite, hence equal. $\alpha'' = \alpha' > \alpha$.

Theorem 3.18 (The longer side lies opposite the larger angle) *If A, B, C are points in a euclidean plane, not on a common line, then*

$$|\angle ABC| < |\angle BAC| \quad \Longleftrightarrow \quad |A, C| < |B, C|$$

Proof: We know from example that $|A, C| = |B, C| \Rightarrow |\angle ABC| = |\angle BAC|$. We will now show that

$$|A, C| > |B, C| \quad \Longrightarrow \quad |\angle ABC| > |\angle BAC| \tag{3.19}$$

and

$$|A, C| < |B, C| \quad \Longrightarrow \quad |\angle ABC| < |\angle BAC| \ . \tag{3.20}$$

Then the Theorem follows, since the cases $|A, C| < |B, C|$, $|A, C| = |B, C|$, $|A, C| > |B, C|$ are mutually exclusive. It suffices to prove (3.19), the proof of (3.20) is then obtained by interchanging the roles of A and B.

Thus assume $|A, C| > |B, C|$. By the construction axiom for the distance, there is a unique point $A' \in (CB)$ so that $|CA'| = |CA|$ and B, A' lie on the same side of C on (CB), i.e. C is not between B and A'. Since

$$|C, A'| = |CA| > |CB|$$

we have that B lies between C and A', hence $B \in \mathrm{IR}\,(\angle A'AC)$. By angle additivity, we have that

$$|\angle A'AC| = |\angle A'AB| + |\angle BAC| > |\angle BAC| \ . \tag{3.21}$$

Since $|AC| = |A'C|$ we have

$$|\angle A'AC| = |\angle AA'C| \ . \tag{3.22}$$

Now $\angle AA'C$ and $\angle ABC$ are opposite-alternate angles as in Corollary 3.17, hence

$$|\angle AA'C| < |\angle ABC| \ . \tag{3.23}$$

From (3.21), (3.22), (3.23) together we now get $|\angle ABC| > |\angle BAC|$, i.e. the right hand side of (3.19). •

Corollary 3.24 (Isosceles Triangles) *Let* $(\mathcal{P}, \mathcal{L}, \leq, |\cdot|, |\angle\cdot|)$ *be a Euclidean plane and* $A, B, C \in \mathcal{P}$. *Then*

$$|A, C| = |B, C| \iff |\angle BAC| = |\angle ABC| \ .$$

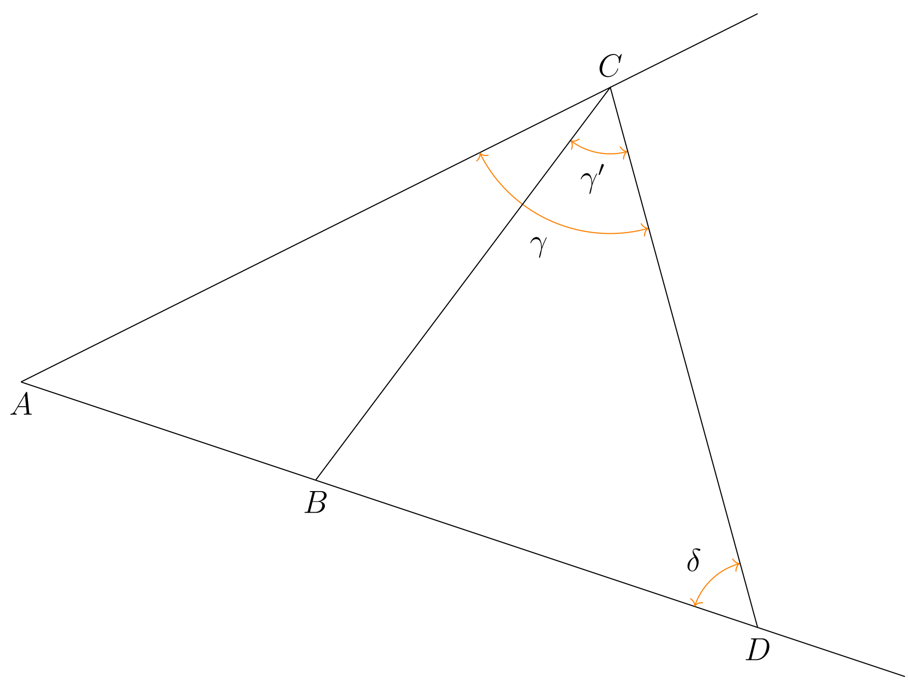
Proof: We have already shown that $|A, C| = |B, C| \implies |\angle BAC| = |\angle ABC|$. If $|A, C| \neq |B, C|$ we must have either $|A, C| > |B, C|$, which implies $|\angle BAC| < |\angle ABC|$, or else $|A, C| < |B, C|$, which implies $|\angle BAC| > |\angle ABC|$. Thus we can have equality of the angles only if the sides are equal. •

A second Corollary of the side-angle relation in Theorem 3.18 is that the distance in a Euclidean plane $(\mathcal{P}, \mathcal{L}, \leq, d, |\angle|)$ satisfies the **triangle inequality**, making $(\mathcal{P}, d)$ a **metric space**:

Problem 3.24 Show that for points A, B, C in a Euclidean plane we have

$$|A, C| \leq |A, B| + |B, C| \ . \tag{3.26}$$

Hint: If the three points are collinear, i.e. on a common line, then (3.26) becomes the triangle inequality for the absolute value on the real numbers. We may therefore assume that the points are not collinear. On the line (AB), construct a point D so that $|A, D| = |A, B| + |B, C|$. Look at the triangle (D, B, C).



$|B, D| = |B, C|$, $\delta = \gamma' \leq \gamma$.

Solution: Let D be as in the hint. Then B is between A and D, hence $B \in \mathrm{IR}(\angle ACD)$. We must have $|B, C| = |B, D|$, the triangle (C, D, B) is isosceles, $|\angle BCD| = |\angle BDC|$. By additivity of the angle measure we have

$$|\angle ACD| = |\angle ACB| + |\angle BCD| > |\angle BCD| = |\angle BDC| \ .$$

Thus in the triangle (ADC) the angle at C is larger than the angle at D. By Theorem 3.18 the side opposite C, i.e. $[A, D]$ is longer than the side opposite D, i.e. $[A, C]$. Thus

$$|A, C| < |A, D| = |A, B| + |B, C| \ .$$

We have proved a little more than claimed: Equality in (3.26) implies that the three points are collinear.

Problem 3.26 This is to establish the notion "distance of a point from a line". Let P be a point and ℓ be a line in a Euclidean plane. we then define the **distance of P from ℓ** to be

$$d(P, \ell) := \inf \{|P, Q| \mid Q \in \ell\} \ .$$

Show that there is a unique point $F(P, \ell) \in \ell$ so that

$$|F(P, \ell), P| = d(P, \ell) \ . \tag{3.28}$$

If $P \notin \ell$, show that at $F(P, \ell)$ the segment $[P, F(P, \ell)]$ makes a right angle with ℓ, i.e. for every point $Z \in \ell \setminus \{F(P, \ell)\}$ we have

$$|\angle ZF(P, \ell)P| = 90 \ . \tag{3.29}$$

We also have

$$|\angle F(P, \ell)ZP| < 90 \quad \text{for every point} \quad Z \in \ell \setminus \{F(P, \ell)\} \tag{3.30}$$

Solution: We may assume $P \notin \ell$. For the construction of $F(P, \ell)$, choose any points $A, B \in \ell$, $A \neq B$. By the construction axiom for the angle measure and the distance there is a point Q in the half plane of $\ell = (AB)$ not containing P such that

$$|A, P| = |A, Q| \quad \text{and} \quad |\angle PAB| = |\angle QAB| \ .$$

Since P and Q lie in different half planes of ℓ, there is a unique point $F(P, \ell) \in \ell$ such that

$$[P, Q] \cap \ell = \{F(P, \ell)\} \ .$$

We show that this point has the properties outlined above: If $A = F(P, \ell)$, then $A \in [P, Q]$ and hence $|\angle ZF(P, \ell)P| = 90$ by the construction above. If $A \neq F(P, \ell)$, then by SAS applies to the construction above and yields

$$(PAF(P, \ell) \cong (QAF(P, \ell)$$

hence

$$|\angle PF(P, \ell)A| = |\angle QF(P, \ell)A| \ .$$

But since these two angles add up to 180, they must both be 90 which proves (3.29). The estimate (3.30) is an imediate consequence of the alternate angles theorem. To show that $F(P, \ell)$ realizes the distance uniquely, let $Z \in \ell \setminus F(P, \ell)$ and look at the triangle $(ZPF(P, \ell))$. The angle at $F(P, \ell)$ is 90 by the alternate angles theorem, the angle at Z is < 90. By Theorem 3.18 the side opposite $F(P, \ell)$, i.e. $[Z, P]$ is strictly longer than the side opposite Z, i.e. $[P, F(P, \ell)]$. Thus $|P, Z| < |P, F(P, \ell)|$ proving (3.28).

3.3 Congruence Theorems

Theorem 3.31 (SSS) *Two triangles are congruent if and only if corresponding sides are equal: For triangles* (A,B,C), (A',B',C') *in a Euclidean Plane, we have*

$$|A,B| = |A',B'|\,,\ |A,C| = |A',C'|\,,\ |B,C| = |B',C'| \quad \Longleftrightarrow \quad (A,B,C) \cong (A',B',C')\ .$$

Proof: Let (A,B,C), (A',B',C') be triangles in a Euclidean Plane as in the theorem. Let C'' be a point in the half plane of (AB) not containing C', so that $|\angle C''AB)| = |\angle C'A'B'|$. Thus C and C'' lie in opposite half planes of (AB). By the congruence axiom SAS, $(C'',A,B) \cong (C',A',B')$. Since C and C'' lie in different half planes of (AB) but not on (A,B) thee is exactly one point X so that $(AB) \cap [CC'']$.

By the assumption of the theorem and the construction of C'', $|A,C| = |A'C'| = |A,C''|$ and $|B,C| = |B'C'| = |B,C''|$. In particular the triangles (A,C,C'') and (B,C,C'') are isosceles. By 3.24 we therefore have

$$\begin{aligned} |\angle ACX| = |\angle ACC''| = |\angle AC''C| = |\angle AC''X| \quad &\text{and} \\ |\angle BCX| = |\angle BCC''| = |\angle BC''C| = |\angle BC''X| \end{aligned}$$

Let $\leq$ be the natural order on (AB) so that $A \leq B$. As to the position of X relative to A, B we now have three cases

1. *X lies between A,B* Then $X \in \mathrm{IR}\,(\angle ACB)$ and $X \in \mathrm{IR}\,(\angle AC''B)$. By the additivity of the angle measure we have

   $$|\angle ACB| = |\angle ACX| + |\angle XCB| = |\angle AC''X| + |\angle XC''B| = |\angle AC''B|\ .$$

   By the congruence axiom SAS, applied at the angles at C respectively C'', we have $(A,C,B) \cong (A,C'',B)$.

2. $A \leq B \leq X$ If $X = B$ then

   $$|\angle ACB| = |\angle ACX| = |\angle AC''X| = |\angle AC''B|$$

   and we finish as in the first case. If $X \neq B$, then B lies between X and A, hence $B \in \mathrm{IR}\,(\angle ACX)$ and $B \in \mathrm{IR}\,(\angle AC''X)$. By additivity of the angle measure,

   $$|\angle ACB| = |\angle ACX| - |\angle BCX| = |\angle AC''X| - |\angle BC''X| = |\angle AC''B|\ .$$

   Again the congruence axiom SAS applied at the angles at C and C'' yields the congruence $(A,C,B) \cong (A,C'',B)$.

3. $X \leq A \leq B$ This case is analogous to the previous one, interchanging A with B.

•

Theorem 3.32 (ASA) *For triangles* (A,B,C), (A',B',C') *in a Euclidean Plane, we have*

$$|\angle CAB| = |\angle C'A'B'|\,,\ |A,B| = |A',B'|\,,\ |\angle ABC| = |\angle A'B'C'| \quad \Longleftrightarrow \quad (A,B,C) \cong (A',B',C')\ .$$

Proof: Given such triangles (A, B, C), (A', B', C'), let $C'' \in [AC$ be such that $|A'C'| = |A, C''|$ (construction axiom for the distance). By the congruence axiom SAS, $(A'C'B') \cong (A, C'', B)$. We may assume that $|A, C| \leq |A, C''| = |A'C'|$. If not we interchange the triangles (A, B, C), (A', B', C'). If $C \neq C''$ then additivity of the distance on the line (AC) yields that $|A, C| < |A, C| + |C, C''| = |A, C''| = |A'C'|$, hence C lies between A, C'' and therefore $C \in \mathrm{IR}\,(\angle ABC'')$. By the additivity of the angle measure, we must have

$$|\angle ABC| < |\angle ABC| + |\angle CBC''| = |\angle ABC''| = |\angle A'B'C'|$$

but this contradicts the assumption of the theorem. •

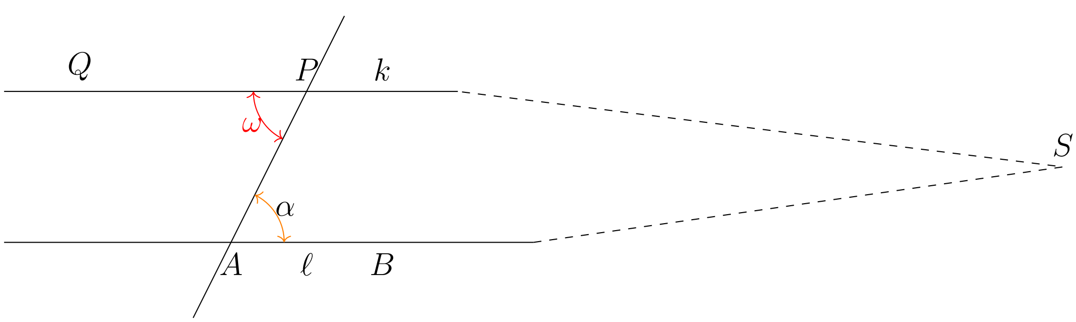
3.4 Parallels, Euclid's 5th postulate

Definition 3.33 *Let $(\mathcal{P}, \mathcal{L})$ be a plane and $g, h \in \mathcal{L}$. The we say g* **and** *h* **are parallel** *if $g = h$ or $g \cap h = \emptyset$.*

In a plane two lines g, h either intersect in exactly one point or are parallel.

Theorem 3.34 *In a Euclidean plane, for every line ℓ and every point P not on this line, there is a line k so that P lies on k and k does not intersect ℓ. Such a line k is called a parallel to ℓ through P.*

Proof: Let $A \neq B$ be two points on ℓ and $Q \neq P$ so that Q and B lie on opposite sides of (AB) and $|\angle APQ| = |\angle PAB|$. Let $k := (PQ)$. If there were a point $S \in \ell \cap k$, then $|\angle APQ|$ and $|\angle PAB|$ would be alternate angles, and by the alternate angles theorem 3.16 the one of the two lying on the opposite side of S with respect to (AP) would be bigger than the one lying on the same side as S. (Since Q and B lie on different sides of (AP) the same is true for these angles.)



If there were a point S, then $\omega > \alpha$.

•

Theorem 3.35 *(Axiom of Parallels) In a Euclidean plane $(\mathcal{P}, \mathcal{L}, \leq, |\cdot|, |\angle\cdot|)$ the following are equivalent:*

(A). *(Playfair's axiom)*

$$\forall l \in \mathcal{L}, P \in \mathcal{P} \setminus l \; \exists! \; h \in \mathcal{L} : h \cap l = \emptyset$$

Such a line h is called a parallel *to l through P.*

(B). *(Euclid's fifth postulate) Let $A, B, X, Y \in \mathcal{P}$ be pairwise different and so that B, Y lie in the same half plane of (AX). If $|\angle AXY| + |\angle BAX| < 180$, then $(AB) \cap (XY) \neq \emptyset$.*

Proof: By problem 3.34, a parallel (at least one) always exists and can be constructed by choosing the points in (B) above so that $|\angle AXY| + |\angle BAX| = 180$. Thus this theorem is about the equivalence of the uniqueness statement in (A) with (B).

To show equivalence, first assume (A) and let $A, B, X, Y \in \mathcal{P}$ be pairwise different, B, Y on the same side of (AX), with $|\angle AXY| + |\angle BAX| < 180$, hence $|\angle AXY| < 180 - |\angle BAX|$. Let $Y' \in \mathcal{P}$ be so that $|\angle AXY'| = 180 - |\angle BAX|$. Then by problem 3.34 (AB) and (XY') are parallel. Since Y is in the interior region of $\angle AXY'$, the lines (XY) and (XY') are different. Thus, if (XY) did not intersect (AB) there would be two parallels to (AB) through X contradicting (A). •

Theorem 3.36 *(Angle Sum in Triangles)*

4 Riemannian metrics on $\mathbb{R}^2$

4.1 Scalar Products on $\mathbb{R}^n$

Definition 4.1 *A scalar product s on $\mathbb{R}^n$ is a positive definite, symmetric, bilinear function $\mathbb{R}^n \times \mathbb{R}^n \xrightarrow{s} \mathbb{R}$. We often write $s(x, y) = \langle x \mid y \rangle = xy$ if a scalar product is fixed. This means that for all $x, y, z, w \in \mathbb{R}^n$ we must have*

$$\begin{aligned}
&\langle x \mid x \rangle \geq 0 \quad \textit{and} \quad \langle x \mid x \rangle = 0 \textit{ only if } x = 0 \\
&\langle x \mid y \rangle = \langle y \mid x \rangle \\
&\langle \xi x + \upsilon y \mid \zeta z + \omega w \rangle = \xi\zeta \langle x \mid z \rangle + \xi\omega \langle x \mid w \rangle + \upsilon\zeta \langle y \mid z \rangle + \upsilon\omega \langle y \mid w \rangle
\end{aligned}$$

Given a scalar product s on $\mathbb{R}^n$, we can define the **norm of a vector** and the **angle between two nonzero vectors**,

$$\begin{aligned}
\|x\|_s &= \sqrt{s(x, x)} \quad \text{for} \quad x \in \mathbb{R}^n \\
\angle_s(x, y) &= \arccos\left(\frac{s(x, y)}{\|x\|_s \|y\|_s}\right) \quad \text{for} \quad x, y \in \mathbb{R}^n \setminus \{0\}
\end{aligned}$$

A scalar product s is determined by its norm $\|\cdot\|_s$, because of the **polarization formula**

$$2s(x, y) = \|x + y\|_s^2 - \|x\|_s^2 - \|y\|_s^2 \; . \tag{4.2}$$

The **standard scalar product** on $\mathbb{R}^n$, or the **dot product**, is defined by

$$x \bullet y = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \bullet \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = x_1 y_1 + \cdots + x_n y_n \ .$$

Via matrix multiplication this can also be written as $x \bullet y = x^t y = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$. Every scalar product s on $\mathbb{R}^n$ can be expressed in terms of the standard scalar product. Let $S \in \mathrm{Mat}(\mathbb{R}, n \times n)$,

$$S = (s(e_i, e_j))_{\substack{i=1\ldots n \\ j=1\ldots n}}$$

be the matrix of the scalar products of the standard basis vectors $e_i = \begin{matrix} 1 \\ \\ \\ i \\ \\ \\ n \end{matrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $i = 1, \ldots, n$, of $\mathbb{R}^n$. Note that this is a symmetric matrix. In particular S can be diagonalized. We can rewrite the scalar product s in terms of this matrix as

$$s(v, w) = v \bullet Sw \quad \text{for all} \quad v, w \in \mathbb{R}^n \tag{4.3}$$

Problem 4.3 Check (4.3).

The norm $\|\cdot\|_s$ is given by the formula

$$\begin{aligned} \|v\|_s^2 = s(v, v) = v \bullet Sv = \sum_{1 \le i,j \le n} s(e_i, e_j) v_i v_j = \sum_{1 \le i \le n} s(e_i, e_i) v_i^2 + 2 \sum_{1 \le i < j \le n}^{n} s(e_i, e_j) v_i v_j \\ = \sum_{1 \le i \le j \le n} s_{i,j} v_i v_j \ . \end{aligned}$$

This is a **quadratic form**, i.e. a homogeneous quadratic polynomial on $\mathbb{R}^n$. Note that because of the factor 2 the $s_{i,j}$ are not exactly the matric coefficients of the matrix S. By the formula (4.2), this quadratic form determines the scalar product.

If we name the coordinates of $\mathbb{R}^n$ as $x_1, \ldots x_n$, then we write this somewhat shorter as

$$s = \sum_{1 \le i \le j \le n} s_{i,j} dx_i dx_j \tag{4.5}$$

The $\binom{n}{2} = \frac{n(n-1)}{2}$ coefficients $s_{i,j}$ are uniquley determined by the scalar product s and conversely, any family $(s_{i,j})_{i,j}$ of $\binom{n}{2}$ real numbers determines a scalar product provided (4.5) is positive definite, i.e.

$$\sum_{1 \le i \le j \le n} s_{i,j} v_i v_j > 0 \quad \text{whenever} \quad (v_1, \ldots, v_n) \neq (0, \ldots, 0) \ .$$

Example 4.6 *Consider the quadratic form*

$$q = a\ dx^2 + b\ dxdy + c\ dy^2 \quad on \quad \mathbb{R}^2\ . \tag{4.7}$$

This is positive definite if and only if

$$q((x,y)) = ax^2 + bxy + cy^2 > 0 \quad for\ all \quad (x,y) \neq (0,0)\ .$$

Evaluating this for $(x,y) = (1,0)$ and $(0,1)$ leads to $a > 0$ and $c > 0$. We may thus assume that $a, c > 0$ and rewrite (4.7) as

$$q((x,y)) = (\sqrt{a}x)^2 + \underbrace{\frac{b}{\sqrt{ac}}}_{\tilde{b}}(\sqrt{a}x)(\sqrt{c}y) + (\sqrt{c}y)^2$$

$$= (\sqrt{a}x + \sqrt{c}y)^2\left(\frac{1}{2} + \frac{b}{4\sqrt{ac}}\right) + (\sqrt{a}x - \sqrt{c}y)^2\left(\frac{1}{2} - \frac{b}{4\sqrt{ac}}\right)$$

This is positive for all $(x,y) \neq (0,0)$ if both

$$\frac{1}{2} + \frac{b}{4\sqrt{ac}} \quad and \quad \frac{1}{2} - \frac{b}{4\sqrt{ac}}$$

are positive, i.e. if and only if

$$4ac > b^2\ .$$

We have proved

Theorem 4.8 *The quadratic form*

$$a\ dx^2 + b\ dxdy + c\ dy^2 \quad , \quad a, b, c \in \mathbb{R},$$

on $\mathbb{R}^2$ is positive definite if

$$a > 0\ ,\ c > 0 \quad and \quad 4ac > b^2\ .$$

4.2 Riemannian metrics on $\mathbb{R}^n$

Definition 4.9 *A* **Riemannian metric on** *$U \subset \mathbb{R}^n$ is a function assigning to each point $p \in U$ a scalar product g_p on $\mathbb{R}^n$.*

If we name the coordinates as $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, then we write

$$g_p = \sum_{1 \leq i \leq j \leq n}^{n} g_{i,j}(p) dx_i dx_j \tag{4.10}$$

for the Riemannian metric with

$$g_p\left(\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}, \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}\right) = \sum_{1 \leq i \leq j \leq n} g_{ij}(p) v_i v_j$$

for some functions g_{ij} on $\mathbb{R}^n$.

If $c\colon [a,b] \to \mathbb{R}^n$ is a continuously differentiable curve, then the **length of c with respect to the metric** *g is defined to be the integral*

$$\mathbf{length}_g(c) = \int_a^b \|c'(t)\|_{g_{c(t)}}\ dt = \int_a^b \sqrt{g_{c(t)}(c'(t), c'(t))}\ dt = \int_a^b \sqrt{\sum_{1\le i\le j\le n} g_{ij}(c(t)c_i'(t)c_j'(t))}\ dt \tag{4.11}$$

if $c(t) = (c_1(t), \dots, c_n(t))$ and the metric is given as in (4.10).

The **(geodesic) distance** *on (U, g) is given by*

$$\mathrm{d}^g(p,q) = \inf\left\{\mathbf{length}_g(c) \,\middle|\, c\colon [a,b] \to U \text{ a continuously differentiable curve, } c(a) = p, c(b) = q\right\} \tag{4.12}$$

A **geodesic** *in (U, g) is a C^1-curve $c\colon [a, b \to U$ which locally realizes this infimum, i.e. so that*

$$\forall t \in [a,b]\ \exists \delta_t > 0 : \mathrm{d}^g(c(t-\delta_t), c(t+\delta_t)) = \mathbf{length}\left(c|[t-\delta_t, t+\delta_t]\right)\ .$$

In particular, if c is so that $\mathbf{length}(c) = \mathrm{d}^g(c(a), c(b))$, then c is a geodesic. Such geodesics are called **minimizing**.

Example 4.13 *In the* **standard metric** *on $\mathbb{R}^n$, is*

$$g_p^{std}(v,w) = v \bullet w$$

independent of p. In the style of formula (4.10) this becomes

$$g_p^{std} = \sum_i dx_i^2\ .$$

Example 4.14 *On the upper half plane*

$$\mathcal{H} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$$

we define the hyperbolic metric by

$$g_p^{hyp}(v,w) = \frac{v \bullet w}{\Im(p)}$$

i.e.

$$g_p^{hyp} = \frac{dx^2 + dy^2}{\Im(p)}$$

if we denote the coordinates $z = x + iy = (x, y)$.

Example 4.15 *On the punctured sphere*

$$S^2_p = \{(x,y,z) \in \mathbb{R}^3 \,|\, x^2+y^2+z^2 = 1 \ , \ z \neq \pm 1\}$$

we denote by r *the distance on the spere from the north pole, and by* α *the longitude. Thus we write* (r,α) *for the point*

$$(x,y,z) = (\sin(r)\cos(\alpha), \sin(r)\sin(\alpha), \cos(r)) = (r,\alpha) \ .$$

These coordinates are called **polar coordinates**. *The usual metric on the sphere thus becomes the metric*

$$g^S_{r,\alpha} = dr^2 + \sin(r)^2 d\alpha^2 \quad on \quad \{(r,\alpha) \,|\, 0 < r < \pi \ , \ \alpha \in \mathbb{R}\} \ .$$

This is positive definite only if $\sin(r) \neq 0$, *i.e. away from the poles.*

Problem 4.15 Compute the length of the curve

$$c\colon [0,2\pi] \to \{(r,\alpha) \,|\, 0 < r < \pi \ , \ \alpha \in \mathbb{R}\} \quad , \quad c(t) = (2,t) \ .$$

with respect to the spherical metric from example 4.15. Sketch this curve on the sphere and in $\mathbb{R}^2$.

Solution: By definition (4.11), we have

$$\textbf{length}\,(c) = \int_0^{2\pi} \sqrt{g^S_{c(t)}(c'(t), c'(t))}\; dt = \int_0^{2\pi} \sqrt{g^S_{(2,2t)}((0,1),(0,1))}\; dt = \int_0^{2\pi} \sqrt{\sin(2)^2}\; dt$$
$$= 2\pi \sin(2) \ .$$

4.3 Isometries

Definition 4.17 *Let* $U \subset \mathbb{R}^n$ *and* g *be a Riemannian metric on* U. *Let* $\phi\colon U \to U$ *be a differentiable map. We say that* ϕ *is an* **isometry** *of* (U,g), *if for all* $p \in U$ *and* $v,w \in \mathbb{R}^n$ *we have*

$$g_p(v,w) = g_{\phi(p)}(\mathrm{d}_p\phi v, \mathrm{d}_p\phi w) \ .$$

This is equivalent to preserving the length of curves. Thus $\phi\colon U \to U$ is an isomoetry if and only if for all differentiable curves c in U, we have

$$\textbf{length}\,(c) = \textbf{length}\,(\phi \circ c) \ .$$

Example 4.18 *Every affine linear map*

$$\alpha\colon \mathbb{R}^n \to \mathbb{R}^n \quad , \quad x \stackrel{\alpha}{\longmapsto} Ax + b \ ,$$

with $A \in \mathrm{O}\,(n)$, $b \in \mathbb{R}^n$, *is an isometry of the standard metric on* $\mathbb{R}^n$. *In fact,* $\mathrm{d}_p\alpha = A$ *and*

$$g^{std}_{\alpha(p)}(\mathrm{d}_p\alpha v, \mathrm{d}_p\alpha w) = Av \bullet Aw = v \bullet w = g^{std}_p(v,w)$$

because $A \in \mathrm{O}\,(n)$.

Example 4.19 *In the hyperbolic upper half plane $(\mathcal{H}, g^{hyp})$ we have isometries*

1. *Translations* $\tau_a(z) = z + a$ *for* $a \in \mathbb{R}$
2. *Scaling, homothety* $\sigma_\alpha(z) = \alpha z$ *for* $\alpha \in \mathbb{R}^+$
3. *Inversion* $\iota(z) = \frac{-1}{z}$

We check this for the inversion: Writing complex numbers as pairs $x + iy = (x, y)$, we have

$$\iota(x,y) = \frac{-1}{x+iy} = \frac{-x+iy}{x^2+y^2} = \left(\frac{-x}{x^2+y^2}, \frac{y}{x^2+y^2}\right) ,$$

$$\mathrm{d}_{(x,y)}\iota = \begin{pmatrix} -\frac{1}{x^2+y^2} + \frac{2x^2}{(x^2+y^2)^2} & \frac{2xy}{(x^2+y^2)^2} \\ -\frac{2xy}{(x^2+y^2)^2} & \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2} \end{pmatrix} = \frac{1}{(x^2+y^2)} \underbrace{\begin{pmatrix} \frac{x^2-y^2}{x^2+y^2} & \frac{2xy}{x^2+y^2} \\ \frac{-2xy}{x^2+y^2} & \frac{x^2-y^2}{x^2+y^2} \end{pmatrix}}_{A} .$$

The matrix on the right hand side is orthogonal. This means that $Av \bullet Aw = v \bullet w$ for all vectors $v, w \in \mathbb{R}^2$. Hence

$$g^{hyp}_{\iota(x,y)}\left(\mathrm{d}_{(x,y)}\iota v, \mathrm{d}_{(x,y)}\iota w\right) = \frac{1}{\left(\frac{y}{x^2+y^2}\right)^2} \left(\frac{1}{(x^2+y^2)}\right)^2 Av \bullet Aw$$

$$= \frac{1}{y^2} v \bullet w = g^{hyp}_{(x,y)}(v, w) .$$

Of course we could had simply computed in complex numbers:

$$\iota'(z) = \frac{1}{z^2} ,$$

$$g^{hyp}_{\iota(z)}(\iota'(z)v, \iota'(z)w) = g^{hyp}_{-1/z}\left(\frac{1}{z^2}v, \frac{1}{z^2}w\right) = \frac{\frac{v}{z^2} \bullet \frac{w}{z^2}}{\Im(-1/z)^2} = \frac{(v \bullet w)\left|\frac{1}{z^2}\right|^2}{\Im(-1/z)^2} = \frac{v \bullet w}{\Im(z)^2} = g^{hyp}_z(v, w)$$

because for a complex number $z \in \mathcal{H}$ we have $\Im(1/z) = -\Im(z)/|z|^2$, hence

$$\frac{\left|\frac{1}{z^2}\right|^2}{\Im(-1/z)^2} = \frac{1}{|z|^4\left(\Im(z)/|z|^2\right)^2} = \frac{1}{\Im(z)^2}$$

A **Möbius transformation** is a function M (on some domain, e.g. $\mathcal{H}$, of $\mathbb{C}$) of the form

$$M_A(z) = \frac{az+b}{cz+d} \quad , \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C}) .$$

$\mathrm{GL}(n, \mathbb{C})$ *is the group of complex invertible $(n \times n)$-matrices.* Let $\mathcal{M}$ denote the set of Möbius transformation.

Note that there is no unique matrix belonging to a given Möbius transform. For every $\lambda \in \mathbb{C}$, we have $M_A = M_{\lambda A}$.

Problem 4.19 Find a Möbius transformation M_A so that

$$M_A(9) = \infty \quad , \quad M_A(-1) = 0 \quad , \quad M_A(1+4i) = i \ .$$

Check that $1+4i$ lies on the hyperbolic line with ideal endpoints -1 and 9.

Solution: The first two conditions are easy to satisfy. Simply create a pole at 9 and a zero at -1, i.e. mapping

$$z \mapsto \frac{z+1}{z-9} \ . \tag{4.21}$$

The crucial observation is that all scalings fix 0 and ∞. We are therefore free to insert a factor α into (4.21), which we can use to satisfy the third condition. We thus try

$$M(z) = \alpha \frac{z+1}{z-9} \ .$$

This maps -1 to 0 and 9 to ∞ for all α. The third condition,

$$M(1+4i) = \alpha \frac{1+4i+1}{1+4i-9} = \alpha \frac{2+4i}{-8+4i} \overset{!}{=} i \ ,$$

leads to $\alpha = -2$,

$$M(z) = \frac{-2z-2}{z-9} = M_A(z) \quad \text{with} \quad A = \begin{pmatrix} -2 & -2 \\ 1 & -9 \end{pmatrix} \in \mathrm{GL}^+(2, \mathbb{R})$$

Under the correspondence $A \overset{\mu}{\longmapsto} M_A$, matrix multiplication becomes composition of Möbius transforms, the map $\mu\colon \mathrm{GL}\,(2,\mathbb{C}) \to \mathcal{M}$ is a **group homomorphism**. To see this, let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, and compute

$$M_A(M_B(z)) = \frac{a_{11}\frac{b_{11}z+b_{12}}{b_{21}z+b_{22}} + a_{12}}{a_{21}\frac{b_{11}z+b_{12}}{b_{21}z+b_{22}} + a_{22}} = \frac{(a_{11}b_{11}+a_{12}b_{21})z + a_{11}b_{12} + a_{12}b_{22}}{(a_{21}b_{11}+a_{22}b_{21})z + a_{21}b_{12} + a_{22}b_{22}} = M_{AB}(z) \ .$$

Theorem 4.22 *The Möbius transformations mapping the upper half plane to itself correspond to real matrices of positive determinant. Thus for $M \in \mathcal{M}$,*

$$M(\mathcal{H}) = \mathcal{H} \quad \Longleftrightarrow \quad \exists A \in \mathrm{GL}^+(2,\mathbb{R}),\ M = M_A$$

Proof: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}\,(2,\mathbb{C})$. If $c = 0$, then $d \neq 0$ and

$$M_A = M_{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}} = M_{\begin{pmatrix} a/d & b/d \\ 0 & 1 \end{pmatrix}}$$

so we may assume $d = 1$. But

$$M_{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}}(z) = az + b \in \mathcal{H} \quad \text{for all} \quad z \in \mathcal{H}$$

only if $a \in \mathbb{R}^+$ and $b \in \mathbb{R}$.

If $c \neq 0$, we can assume that $c = 1$ because

$$M_A = M_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = M_{\begin{pmatrix} a/c & b/c \\ 1 & d/c \end{pmatrix}} .$$

Then

$$M_A(z) = \frac{az+b}{z+d} = \frac{a(z+d) - (ad+b)}{z+d} = a + \det(A)\iota(z+d) .$$

If $\Im(d) < 0$ then $M_A(z)$ would not be defined for $z = -d \in H$. If $\Im(d) > 0$ then we would have

$$|z+d| > |\Im(d)| \quad \text{hence} \quad |\iota(z+d)| < \frac{1}{|\Im(d)|}$$

for all $z \in \mathcal{H}$. But this implies that

$$|M_A|(z) \leq |a| + \frac{1}{|\Im(d)|} .$$

Thus $M_A(\mathcal{H})$ is bounded and therefore can not be equal to $\mathcal{H}$. Thus we must have $\Im(d) = 0$, $d \in \mathbb{R}$. Furthermore, for all $x \in \mathcal{H}$ we must have

$$a + \det(A)x \in \mathcal{H}$$

But his forces $a \in \mathbb{R}$, $\det(A) \in \mathbb{R}^+$ and hence $b = \det(A) - ad \in \mathbb{R}$. •

Theorem 4.23 *The set $\mathcal{M}$ of Möbius transformations M_A with $A \in \mathrm{GL}^+(2,\mathbb{R})$ is the same as the set of all compositions of of translations, scalings and the inversion. More precisely,*

$$\mathcal{M} = \left\{\tau_u \circ \sigma_\alpha \,\middle|\, u \in \mathbb{R}, \alpha \in \mathbb{R}^+\right\} \cup \left\{\tau_u \circ \sigma_\alpha \circ \iota \circ \tau_w \,\middle|\, u, w \in \mathbb{R}, \alpha \in \mathbb{R}^+\right\} .$$

Proof: Clearly, the inversion, the translations and scalings as in example 4.19 are such Möbius transformations:

$$\tau_b = \mu\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad , \quad \sigma_\alpha = \mu\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad \iota = \mu\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

To show the converse, let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}^+(2,\mathbb{R}) = \{A \in \mathrm{GL}\,(2,\mathbb{R}) \,|\, \det(A) > 0\}$. If $c = 0$, then we must have $d \neq 0$, $a/d > 0$, and

$$M_A(z) = \frac{az+b}{cz+d} = \frac{a}{d}z + \frac{b}{d} = \tau_{b/d}(\sigma_{a/d}(z))$$

If $c \neq 0$, then since $\det(A) = ad - bc > 0$,

$$\frac{az+b}{cz+d} = \frac{\frac{a}{c}\left(z+\frac{d}{c}\right) - \frac{ad-bc}{c^2}}{z+\frac{d}{c}} = \frac{a}{c} + \frac{ad-bc}{c^2}\frac{-1}{z+\frac{d}{c}} = \tau_{a/c}(\sigma_{\det(A)/c^2}(\iota(\tau_{d/c}(z))))$$

In all cases, M_A is a composition of translations, scalings and inversions. •

Via this Theorem, mapping properties of Möbius transformations can be proved by showing them for translations, scalings and the inversion.

A **hyperbolic line** is a subset $C(a,b) \subset \mathcal{H}$, $a,b \in \mathbb{R} \cup \{\infty\}$, $a < b$ of the form

$$C(a,b) = \left\{ z \in \mathcal{H} \,\middle|\, \left| z - \frac{a+b}{2} \right| = \frac{b-a}{2} \right\} = \left\{ \frac{a+b}{2} + \frac{b-a}{2} e^{it} \,\middle|\, 0 < t < \pi \right\} \quad \text{if} \quad b < \infty$$

and

$$C(a,\infty) = \left\{ a + it \,\middle|\, t \in \mathbb{R}^+ \right\} \ .$$

The **ideal endpoints** a, b of $C(a,b)$, respectively a, ∞ of $C(a,\infty)$ are do not points of the hyperbolic plane.

Corollary 4.24 *A Möbius transform maps hyperbolic lines to hyperbolic lines.*

Proof: This is immediately clear for the translations and the scalings. We check it for the inversion. We will show that

$$\iota(C(a,b)) = C(1/a, 1/b)$$

where the cases $b = \infty$ or $a = 0$ have to be read benevolently. For $p, r \in \mathbb{R}$, $r > 0$, the equation of the circle $|z - p| = r$. i.e. the circle with center p and radius r, is

$$z\overline{z} - p(z + \overline{z}) + p^2 - r^2 = 0 \tag{4.25}$$

hence

$$\frac{p^2 - r^2}{z\overline{z}} - p\left(\frac{1}{z} + \frac{1}{\overline{z}}\right) + 1 = 0$$

which is equivalent to

$$z\overline{z} - \underbrace{\frac{p}{p^2 - r^2}}_{q} \left(\frac{1}{z} + \frac{1}{\overline{z}}\right) + \underbrace{\frac{1}{p^2 - r^2}}_{q^2 - s^2} = 0 \ .$$

But this is of the form (4.25) with

$$\text{center} \quad q = \frac{p}{p^2 - r^2} \quad \text{and radius} \quad s^2 = \frac{p^2}{(p^2 - r^2)^2} - \frac{1}{p^2 - r^2} = \frac{r^2}{(p^2 - r^2)^2}$$

i.e.

$$s = \frac{r}{|p^2 - r^2|} \ .$$

The ideal endpoints of this circle are

$$q \pm s = \frac{p}{p^2 - r^2} \pm \frac{r}{|p^2 - r^2|} = \frac{p \pm r}{p^2 - r^2} = \frac{a \text{ or } b}{ab} = \frac{1}{a} \text{ or } \frac{1}{b} \ .$$

(The $\pm$ signs here need not correspoind.) •

Since translations, scalings and the inversion are isometries of the hyperbolic upper half plane, we immedately get

Corollary 4.26 *Möbius transformations in $\mathcal{M}$ are isometries of $(\mathcal{H}, g^{hyp})$.*

The hyperbolic plane is homogeneous, i.e. for every two points there is an isometry mapping one to the other. We have an even stronger transitivity:

Theorem 4.27 *In the hyperbolic upper half plane let (P_i, ℓ_i, H_i), $i = 1, 2$, be triples consisting of a point, a line through this point and a half plane of this line. Such triples are called* **flags**. *Then there is a Möbius transformation M_A in $\mathcal{M}$ (in particular an isometry) of $(\mathcal{H}, g^{hyp})$ so that*

$$M_A(P_1) = P_2 \quad , \quad M_A(\ell_1) = \ell_2 \quad , \quad M_A(H_1) = H_2 \ .$$

Proof: We will show that we can map any flag (Q, k, H) to the flag $(i, i\mathbb{R}^+, \{z \,|\, \Im(z) > 0, \Re(z) > 0\})$ consisting of i, the positive real axis and the right quadrant. We first map k to $i\mathbb{R}^+ = C(0, \infty)$. If $k = C(a, b) = \frac{a+b}{2} + \frac{b-a}{2} e^{i(0,2\pi)}$, $a, b \in \mathbb{R}$, $a < b$, we map the ideal endpoints $b \mapsto 0$ and $a \mapsto \infty$,

$$M_{\begin{pmatrix} \alpha & -a\alpha \\ 1 & -b \end{pmatrix}}(z) = \alpha \frac{z-b}{z-a}$$

with any $\alpha \in \mathbb{R}^+$. If $k = C(b, \infty) - b + i\mathbb{R}^+$ we can take

$$M_{\begin{pmatrix} \alpha & -b\alpha \\ 0 & 1 \end{pmatrix}}(z) = \alpha(z - b) \ .$$

In both cases we can choose α so that the point P is mapped to i.

Finally, if the Möbius transform constructed above maps the given half plane to the left upper quadrant, we adjust by composing with the inversion map ι, which maps

$$\{z \in \mathbb{C} \,|\, \Re(z) < 0 < \Im(z)\} \overset{\iota}{\longleftrightarrow} \{z \in \mathbb{C} \,|\, 0 < \Im(z),\ 0 < \Re(z)\}$$

•

Homogeneity greatly facilitates calculations in $(\mathcal{H}, g^{\text{hyp}})$. Thus, for instance, computing distances via the definiton (4.12) by taking the infimum can be very difficult. We therefore first compute distances only along $C(0, \infty) = i\mathbb{R}^+$ and then use a Möbius transform. Thus let $p = i$, $q = \lambda i$, and $c\colon [a, b] \to \mathcal{H}$, $c(a) = i$, $c(b) = \lambda i$ be a C^1-curve with coordinates $c(t) = (x(t), y(t))$. Then

$$\textbf{length}\,(c) = \int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)}\, dt \geq \int_a^b \frac{\sqrt{y'(t)^2}}{y(t)}\, dt \geq \left| \int_a^b \frac{y'(t)}{y(t)}\, dt \right| = |\ln(y(b)) - \ln(y(a))| = |\ln(\lambda)| \ .$$

But this is the length of the curve $c\colon [0, \lambda] \to \mathcal{H}$, $c(t) = it$. We thus have shown

Theorem 4.28 *The curve $c_\lambda\colon [1, \lambda] \to \mathcal{H}$, $t \mapsto it$, is a minimizing geodesic in $(\mathcal{H}, \mathrm{d}^{hyp})$. Its lenght is*

$$\textbf{length}\,(c_\lambda) = \mathrm{d}^{hyp}(1, \lambda i) = |\ln(\lambda)| \ .$$

In order to compute the distance $\mathrm{d}^{\mathrm{hyp}}(P,Q)$ of arbitrary points $P, Q \in \mathcal{H}$, we find a Möbius transformation M mapping $P \mapsto i$ and $Q \mapsto i\lambda \in i\mathbb{R}^+$ with some $\lambda \in \mathbb{R}^+$. Then $\mathrm{d}^{\mathrm{hyp}}(P,Q) = |\ln(\lambda)|$.

Example 4.29 *We will compute the hyperbolic distance between*

$$P = 2 + i \quad \text{and} \quad Q = 2 + \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \ .$$

First observe that $(PQ) = C(1,3) = \{2 + e^{it} \mid t \in \mathbb{R}\}$. Then the curve

$$\gamma\colon [\pi/4, \pi/2] \to \mathcal{H} \quad , \quad \gamma(t) = 2 + e^{it}$$

is a minimizing geodesic from $Q = \gamma(i/4)$ to $P = \gamma(\pi/2)$, hence

$$\begin{aligned}
\mathrm{d}^{hyp}(P,Q) = \mathbf{length}\,(\gamma) &= \int_{\pi/4}^{\pi/2} g^{hyp}_{\gamma(t)}(\gamma'(t), \gamma'(t))\ dt \\
&= \int_{\pi/4}^{\pi/2} \sqrt{\frac{\gamma'(t) \bullet \gamma'(t)}{\Im(\gamma(t))^2}}\ dt \\
&= \int_{\pi/4}^{\pi/2} \frac{1}{\sin(t)}\ dt \\
&= \left[\frac{1}{2}\ln\left(\frac{1-\cos(u)}{1+\cos(u)}\right)\right]_{u=\pi/4}^{u=\pi/2} = -\frac{1}{2}\ln\left(\frac{1-\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}}\right) = \frac{1}{2}\ln(3+2\sqrt{2}) \ .
\end{aligned}$$

Alternatively, we can use Möbius transformations to move the line (PQ) to $C(0,\infty)$: To find such a Möbius transformation we map the ideal endpoints, for instance $1 \mapsto 0$, $3 \mapsto \infty$, via

$$M(z) = \alpha\frac{z-1}{z-3}$$

where only the sign of α needs to be chosen correctly, i.e. so that M preserves the upper half plane, equivalently so that the corresponding matrix has positive determinant. In the case at hand, we need to choose $\alpha < 0$, say -1. Then

$$M(z) = \frac{-z+1}{z-3} \quad \text{maps} \quad C(1,3) \to C(0,\infty)$$

and since M is an isometry of g^{hyp},

$$\mathrm{d}^{hyp}(P,Q) = \mathrm{d}^{hyp}(M(P), M(Q)) = \left|\ln\left(\frac{M(P)}{M(Q)}\right)\right| = \ln(\sqrt{2}+1)$$

because

$$M(P) = \frac{-(2+i)+1}{2+i-3} = \frac{-i-1}{i-1} = i \ ,$$

$$M(Q) = \frac{-\left(2+\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)+1}{2+\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}-3} = \frac{-1-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}}{-1+\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}} = \frac{\sqrt{2}+1+i}{\sqrt{2}-1-i} = \frac{i2\sqrt{2}}{4-2\sqrt{2}} = \frac{i}{\sqrt{2}-1} = i(\sqrt{2}+1) \ .$$

Check that this coincides with the result computed above.

We determine the **isotropy group** $\mathcal{M}_i$ of i, the group of all Möbius transformations $M \in \mathcal{M}$ fixing i, i.e. with $Mi = i$.

Proposition 4.30

$$\begin{aligned}\mathcal{M}_i &= \{M_A \,|\, A \in \mathbb{C} \setminus \{0\}\} \\ &= \left\{\mu\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \,\middle|\, a, b \in \mathbb{R}, a^2 + b^2 > 0\right\} \\ &= \left\{\mu\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \,\middle|\, \theta \in \mathbb{R}\right\}\end{aligned}$$

Proof: This is because

$$\frac{ai+b}{ci+d} = i \quad \Longleftrightarrow \quad ai + b = di - c \quad \Longleftrightarrow \quad a = d \text{ and } b = -c\ .$$

The last identity holds because for any $\alpha \in \mathbb{R}^+$ the matrices $A, \alpha A \in \mathrm{GL}^+(2, \mathbb{R})$ yield the same Möbius transform, i.e. $M_A(z) = M_{\alpha A}(z)$ for all z. •

Problem 4.30 Let $r > 0$. Sketch the hyperbolic circle of radius $\ln(2)$ around i, i.e.

$$C_{\ln(2)}(i) = \left\{p \in \mathcal{H} \,\middle|\, \mathrm{d}^{\mathrm{hyp}}(i, p) = \ln(2)\right\}\ .$$

This is a euclidean circle. Compute its euclidean center and radius.

5 The Gauss-Bonnet Theorem

5.1 Area and Angle Sum

The **Gauss curvature of a triangle** ΔABC **in a surface** is

$$K(\Delta ABC) = \frac{\alpha + \beta + \gamma - \pi}{\mathbf{area}\,\Delta ABC}$$

where α, β, γ are the inner angles of ΔABC, and the Gauss curvature at a point p on this surface is the limit

$$K(p) = \lim_{A \to p, B \to p, C \to p} K(\Delta ABC)\ .$$

A triangulation of a surface is a decomposition of the surface in triangles so that two triangles meet in one vertex or in an edge and its two vertices or not at all and so that every edge lies in two triangles. If T is a triangulation T of a surface S, then the integral of the Gauss curvature over S is

$$\begin{aligned}\int_S K &= \sum_{\Delta ABC \in T} K(\Delta ABC) \times \textbf{area}\,\Delta ABC \\ &= \sum_{\Delta ABC \in T} (\alpha + \beta + \gamma - \pi) \\ &= \sum \text{all interior angles} - \pi \times \#\text{triangles} \\ &= 2\pi \times \#\text{vertices} - \pi \times \#\text{triangles} \\ &= 2\pi \times \left(\#\text{vertices} - \frac{1}{2} \#\text{triangles} \right) .\end{aligned}$$

With the notation

$$v = \#\text{vertices} \quad , \quad e = \#\text{edges} \quad , \quad t = \#\text{triangles}$$

this becomes

$$\int_S K = 2\pi \left(v - \frac{1}{2}\, t \right) . \tag{5.1}$$

By the conditions set out above, in a triangulation we always have

$$3t = 2e$$

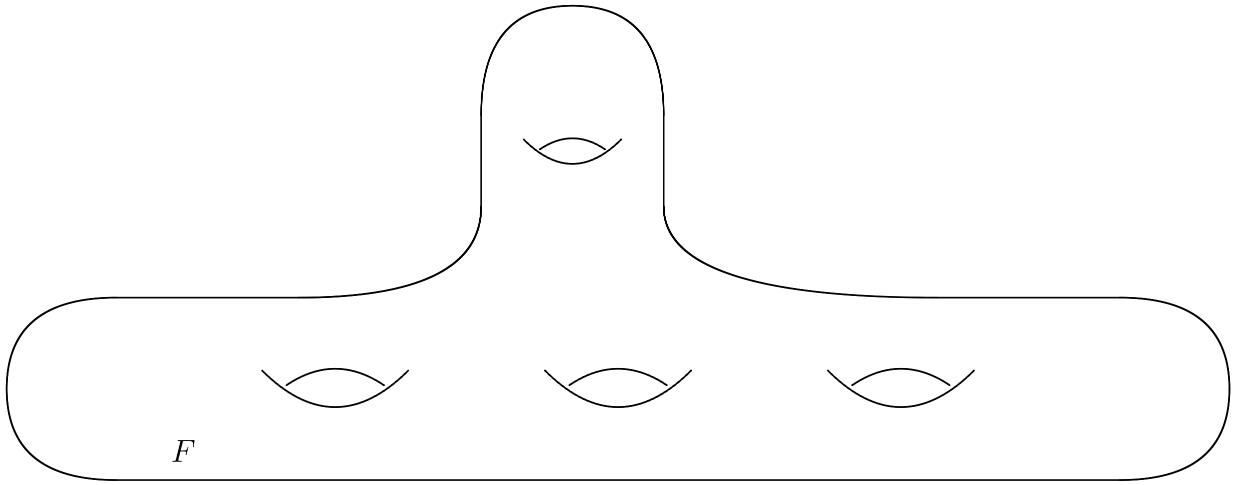
Since every triangle has three edges, and each edge is in two triangles. We can thus rewrite (5.1) as

$$\int_S K = 2\pi\ (\underbrace{v - e + t}_{\chi(S)}) . \tag{5.2}$$

This is the **Gauss Bonnet Theorem** for closed surfaces. The number $\chi(S)$ is called the **Euler characteristic** of the surface. Since the left hand side of (5.2) does not depend on the triangulation, the Euler characteristic does is the same for all triangulations of a given surface. Since the right hand side of (5.2) is independent on th choice of a metric on a given surface, the curvature integral is the same for all metrics on a given surface.

One advantage, and our reason for rewriting (5.1) in the form (5.2) is that (5.2) still holds if we work with decompositions of a surface in polygons.

Example 5.3 *Let $F \subset \mathbb{R}^3$ be the surface with boundary shown in the picture below. Compute the integral of the Gauss curvature over F.*



Summing over the triangles of a triangulation of F, we get the Gauss Bonnet Theorem,

$$\int_F K = \sum_{\substack{\Delta ABC \text{ a triangle} \\ \text{in the triangulation}}} K(\Delta ABC) \times \mathbf{area}\, \Delta ABC = \sum (\textit{interior angles} - \pi)$$

$$= 2\pi \# \textit{vertices} - \pi \times \# \textit{triangles}$$

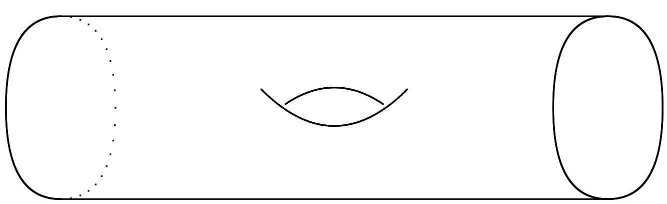
$$= 2\pi \# \textit{vertices} - 2\pi \times \# \textit{edges} + 2\pi \times \# \textit{triangles}$$

$$2\pi \times (\# \textit{vertices} - \# \textit{edges} + \# \textit{triangles}) = 2\pi \chi(F) \quad \textit{Euler Characteristic}$$

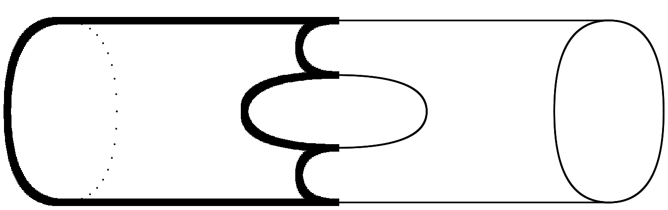
since each edge is in two triangles, we also have

$$2 \times \# \textit{edges} = 3 \times \# \textit{triangles} \ .$$

We now need to decompose the surface F into triangles. The basic block is

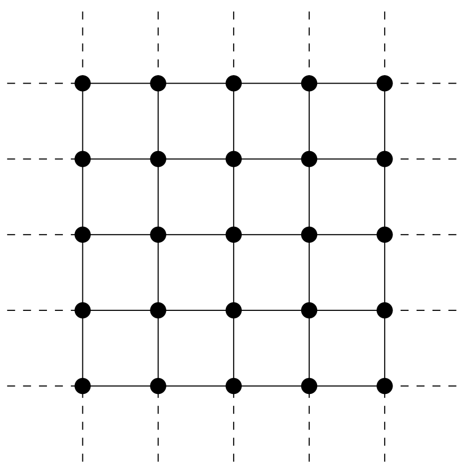


This can be split into four hexagons,



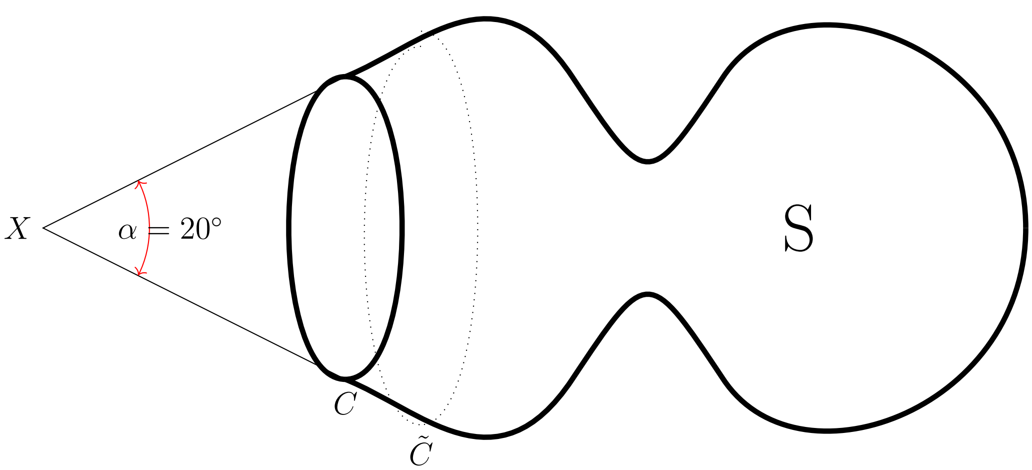
Problem 5.3 Show that a surface must have Euler characteristic 0 if it can be decomposed into squares so that

1. Each edge lies in two squares,
2. Two squares meet in a edge (and two vertices, in one vertex, or not at all.
3. Each vertex is common to four squares.



locally the

Problem 5.4 Compute the integral of the Gauss curvature over the surface S show in the picture.



The boundary curve C of S is a circle. Near this circle the surface the metric of the surface is that of a cone, shown in the picture between the vertex X and the circle $\tilde{C}$. The angle of the cone is $20°$.