On The η-Invariant of Dirac Operators on Manifolds with Free Circle Action

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1. Introduction

Given a Riemannian $Spin^c$-manifold M with free isometric action of the circle we can rescale the metric of M in the direction of the orbits. The main aim of the present thesis is to compute the limit of the η-invariant of equivariant twisted Dirac operators when the orbits are shrinked. For that purpose the Atiyah-Patodi-Singer index theorem for manifolds with boundary will be applied to the disc bundle associated to M. The index of the Dirac operator will be shown to vanish if the scalar curvature M and the connections on the canonical complex line bundle of the $Spin^c$-structure and on the twisting bundle satisfy Hitchin-Lichnerowicz's estimate. O'Neill's formulae for the curvature of Riemannian submersions yield formulae for the limit of the characteristic integral in the Atiyah-Patodi-Singer index formula.
In some cases the η-invariant of the Dirac operator has been computed directly out of the Dirac spectrum, e.g. by Hitchin (see[**Hit**]) for the Berger spheres, by Seade-Steer (see [**SS**]) for quotients of $PSL_2(\mathbb{R})$ by Fuchsian groups. There are also general formulae by Bismut-Cheeger ([**BC**]) and Dai ([**Dai**]) for the adiabatic limit of the η-invariant in fibrations. This has recently been made explicit for S^1-bundles by W. Zhang in [**Zh**] thus also deriving formulae for the adiabatic limit of η-invariants.
For some invariants on zero bordant manifolds which are defined by choosing a zero bordism one can find expressions involving η-invariants. These are more intrinsic in the sense that one can compute them within the manifold. Instead of choosing a zero bordism one has to choose a Riemannian metric.
Examples:

- The Rohlin invariant of an $(8k+3)$-dimensional *Spin*-boundary is the reduction modulo 16 of the signature of a *Spin*-zero bordism. This is well-defined because by a theorem of S. Ochanine (see [**Oc**]) the signature of a closed $(8k+4)$-dimensional manifold is divisible by 16 and by a result of Novikov the signature is additive under the operation of glueing two manifolds along a common boundary. In [**ML**] the Rohlin invariant is expressed as a linear combination of η-invariants of Dirac operators twisted with certain tensor powers of the complexified tangent bundle.
- the Eells-Kuiper invariant classifying 7- and 11-dimensional spheres up to diffeomorphism (see [**EK**]). In [**Don**] the Eells-Kuiper invariant of a stably parallelizable *Spin*-boundary is shown to be a linear combination of the η-invariants of the Dirac operator and the signature operator for a metric which is induced by an immersion in Euclidean space such that the induced connection on the normal bundle is trivial. Such metrics exist for stably parallelizable manifolds.
- the relative index on cylinders of Gromov-Lawson ([**GL2**]) which is related to the diffeomorphism invariants of Kreck and Stolz as we will describe in more detail now.

There is the following diffeomorphism classification for a certain type of 7-dimensional simply-connected manifolds in [**KS1**] and [**KS2**], Theorem 3.1.

THEOREM 1. *Let M and N be 7-dimensional simply-connected simultaneously Spin- or non-Spin-manifolds with $H^2(\,\cdot\,;\mathbb{Z}) \cong \mathbb{Z}$, generated by u, say, and $H^4(\,\cdot\,;\mathbb{Z}) \cong \mathbb{Z}/n$ generated by u^2. Then M is diffeomorphic to N if and only if $s_x(M) = s_x(N)$ for $x = 0, u, 2u$.*

For a generalization of this theorem to the case of arbitrary finitely generated free $H^2(M;\mathbb{Z})$, see [**Ber**]. The invariant s_0 is a generalization of the Eells-Kuiper invariant. The diffeomorphism invariants s_x are reductions modulo $\mathbb{Z}$ of real valued invariants of twisted $Spin^c$-Dirac structures defined as follows: Consider $Spin^c$-manifolds M of odd dimension $n = 2k-1$ with $Spin^c$-structure α_M and a unitary vector bundle ζ of rank $r < k$ over M. Define rational numbers a_k by

$$a_k = \begin{cases} 1/(2^{k+1}(2^{k-1}-1)) & \text{if } k \equiv 0 \bmod 2, \\ 0 & \text{if } k \equiv 1 \bmod 2. \end{cases}$$

For even k the rational number a_k is minus the quotient of the coefficients of $p_{k/2}$ in $\hat{A}$ and L, so that $\mathrm{ch}(\zeta)e^{c_1(\alpha)/2}\hat{A}(p(M)) + ra_kL(p(M))$ does not involve $p_{k/2}$. Assume that (M,α,ζ) has the following properties:

(i) (M,α,ζ) is zero bordant in $\Omega^{Spin^c}(BU(r))$.

(ii) Expand $\mathrm{ch}(\zeta)e^{c_1(\alpha)/2}\hat{A}(p(M)) + ra_kL(p(M)) = P(p(M),c(\zeta),c_1(\alpha))$ as a polynomial P in the Pontrjagin classes $p(M) = \sum p_i(M)$ and the Chern classes $c(\zeta) = \sum_{i=0}^r c_i(\zeta)$ and $c_1(\alpha)$. If m is a monomial of degree $2k$ in this polynomial P we require that $m = abc$ with monomials a, b, c of positive degree such that $a(p(M),c(\zeta),c_1(\alpha))$ and $b(p(M),c(\zeta),c_1(\alpha))$ vanish rationally.

Given a Riemannian metric g on M and connections ω^α and ω^ζ on the canonical complex line bundle of α and on ζ, define, following [**KS3**], a rational number:

$$(2) \qquad \begin{aligned} s(M,\alpha,\zeta,g,\omega^\alpha,\omega^\zeta) \;:=\; & \operatorname{index} D_W^+ + ra_k \operatorname{sign}(W,M) \\ & - \langle e^{c_1(\alpha_W)/2}\,\mathrm{ch}(\zeta_W)\hat{A}(p) + ra_kL(p) \mid [W,M]\rangle \end{aligned}$$

where (W,α_W,ζ_W) is a $Spin^c$-$BU(r)$-manifold with boundary (M,α,ζ). The operator D_W^+ is the twisted Dirac operator on W constructed with extensions of $(g,\omega^\alpha,\omega^\zeta)$ to W whose restrictions to a collar neighbourhood $M \times I$ of $M = \partial W$ in W are induced from $(g,\omega^\alpha,\omega^\zeta)$ on M. If k is even then $\operatorname{sign}(W,M)$ is the signature of the quadratic form on $H^k(W,M;\mathbb{R})$ given by the relative cup product. For odd k define $\operatorname{sign}(W,M) := 0$.

For the evaluation of $\langle e^{c_1(\alpha_W)/2}\,\mathrm{ch}(\zeta_W)\hat{A}(p) + ra_kL(p) \mid [W,M]\rangle = P(p(M),c(\zeta),c_1(\alpha))$ in (2) we have, for every monomial $m = abc$ of degree $2k$ of P, to replace $m(p(M),c(\zeta),c_1(\alpha))$ by $\bar{a} \cup a(p(M),c(\zeta),c_1(\alpha)) \cup c(p(M),c(\zeta),c_1(\alpha))$ to get a relative cohomology class. Here $\bar{a}$ is any inverse image of $a(p(M),c(\zeta),c_1(\alpha))$ under the restriction map $H^{4i}(W,M;\mathbb{R}) \to H^{4i}(W;\mathbb{R})$.

Since the index is always an integer the reduced invariant

$$s(M,\alpha,\zeta) := s(M,\alpha,\zeta,g,\omega^\alpha,\omega^\zeta) \bmod \mathbb{Z}$$

does not depend on the choice of the metric g nor on the choice of the connections ω^α and ω^ζ on M. The invariants of the classification theorem are $s_x = s(M,\alpha,\zeta)$ where

ζ is the complex line bundle with first Chern class x and α is a *Spin*^{c}-structure with canonical complex line bundle ζ. Since $H^3(M;\mathbb{Z})=0$ the manifolds M admit a *Spin*^{c}-structure which is determined by its canonical complex line bundle because the manifolds are simply-connected (see section 2, [**LM**]).
Applying the Atiyah-Patodi-Singer index formula for manifolds with boundary to (W,M) (see 2.4) the s-invariant may be expressed in terms of the η-invariants of twisted Dirac operators $D^M_{\omega^\alpha,\omega^\zeta}$ and the signature operator S on M: We denote by $c_1(\omega^{\alpha_W})$, $\mathrm{ch}(\omega^{\zeta_W})$ and $p(g_W)$ the first Chern form, the Chern character form and the Pontrjagin form of the connections ω^{α_W}, ω^{ζ_W} and the Levi-Civita connection of the metric g_W on W. Then

$$\begin{aligned}
(3)\qquad s(M,\alpha,\zeta,g,\omega^\alpha,\omega^\zeta) &= \int_W \left(e^{c_1(\omega^{\alpha_W})/2}\,\mathrm{ch}(\omega^{\zeta_W})\hat{A}(p(g_W))+ra_kL(p(g_W))\right)\\
&\quad -\frac{\eta(D^M_{\omega^\alpha,\omega^\zeta})+\dim\ker D^M_{\omega^\alpha,\omega^\zeta}}{2}-ra_k\eta(S)\\
&\quad -\langle e^{c_1(\alpha_W)/2}\,\mathrm{ch}(\zeta_W)\hat{A}(\bar{p})+ra_kL(\bar{p})\mid[W,M]\rangle\\
(4)\qquad &= -\frac{\eta(D^M_{\omega^\alpha,\omega^\zeta})+\dim\ker D^M_{\omega^\alpha,\omega^\zeta}}{2}-ra_k\eta(S)\\
&\quad +\int_M d^{-1}\left(e^{c_1(\omega^\alpha)/2}\,\mathrm{ch}(\omega^\zeta)\hat{A}(p(g))+ra_kL(p(g))\right).
\end{aligned}$$

Here $d^{-1}\left(e^{c_1(\omega^\alpha)/2}\,\mathrm{ch}(\omega^\zeta)\hat{A}(p(g))+ra_kL(p(g))\right)$ is defined as follows: Every monomial m of degree $2k$ of P as in condition (ii) factors as $m=abc$ where we can choose $\bar{a},\bar{b}\in\Omega^*(M)$ such that $a(p(g),c((\omega^\zeta),c_1(\omega^\alpha))=d\,\bar{a}$ and $b(p(g),c((\omega^\zeta),c_1(\omega^\alpha))=d\,\bar{b}$. Define

$$d^{-1}(m(p(g),c((\omega^\zeta),c_1(\omega^\alpha))):=\bar{a}\wedge b(p(g),c((\omega^\zeta),c_1(\omega^\alpha))\wedge c(p(g),c((\omega^\zeta),c_1(\omega^\alpha)).$$

Then by Stoke's Theorem $\int_M d^{-1}(m(p(g),c((\omega^\zeta),c_1(\omega^\alpha)))$ does not depend on the choice of $\bar{a}$. This also shows that s is well-defined by (2). Moreover (4) extends the definition (2) to non zero bordant twisted *Spin*^{c}-manifolds (M,α,ζ).
For a compact manifold X of odd dimension with *Spin*^{c}-structure α and carrying a vector bundle ζ let $\{g_\tau,\omega^\alpha_\tau,\omega^\zeta_\tau)\}$ be a smooth family of metrics and connections defined for $\tau\in I=[0,1]$ and constant near $\tau=0$ and $\tau=1$. The family $\{g_\tau,\omega^\alpha_\tau,\omega^\zeta_\tau)\}_{\tau\in[0,1]}$ determines a metric on the cylinder $Z=X\times I$ and a connection for the *Spin*^{c}-structure induced from X and a connection on the pull-back of ζ to Z. The index of the twisted *Spin*^{c}-Dirac operator D_Z on Z only depends on the values of $(g_\tau,\omega^\alpha_\tau,\omega^\zeta_\tau)$ for $\tau=0$ and $\tau=1$. Following Gromov-Lawson ([**GL1**]) it therefore makes sense to define

$$i((g_0,\omega^\alpha_0,\omega^\zeta_0),(g_1,\omega^\alpha_1,\omega^\zeta_1)):=\mathrm{index}\,D^+_Z.$$

If (M,α,ζ) satisfies (ii) we can use (2) to get

$$\begin{aligned}
i((g_0,\omega^\alpha_0,\omega^\zeta_0),(g_1,\omega^\alpha_1,\omega^\zeta_1)) &= s((M,\alpha,\zeta,g_0,\omega^\alpha_0,\omega^\zeta_0)\dot{\cup}-(M,\alpha,\zeta,g_1,\omega^\alpha_1,\omega^\zeta_1))\\
&= s(M,\alpha,\zeta,g_0,\omega^\alpha_0,\omega^\zeta_0))+s(-(M,\alpha,\zeta,g_1,\omega^\alpha_1,\omega^\zeta_1))\\
&= s(M,\alpha,\zeta,g_0,\omega^\alpha_0,\omega^\zeta_0))-s(M,\alpha,\zeta,g_1,\omega^\alpha_1,\omega^\zeta_1)\\
&\quad -\dim\ker D^M_{\omega^\alpha_1,\omega^\zeta_1}
\end{aligned}$$

because s is additive under disjoint union and the η-invariant and the integral in (4) alter their signs if the orientation is reversed whereas $\dim\ker D^M_{\omega_1^\alpha,\omega_1^\zeta}$ remains unchanged.
For a twisted *Spin*^{c}-manifold (X,α,ζ) let $\mathcal{S}^+(X,\alpha,\zeta)$ be the space of all triples $(g,\omega^\alpha,\omega^\zeta)$ for which Hitchin-Lichnerowicz's estimate of Theorem 2.2.6 holds. If the family $\{g_\tau,\omega_\tau^\alpha,\omega_\tau^\zeta)\}_{\tau\in[0,1]}$ stays in $\mathcal{S}^+(X,\alpha,\zeta)$ then the twisted *Spin*^{c}-Dirac structure on the cylinder also satisfies Hitchin-Lichnerowicz's estimate and we have $\operatorname{index} D_Z^+ = 0$ and also $\dim\ker D^X_{\omega_i^\alpha,\omega_i^\zeta} = 0$ for $i = 0,1$. Thus (see [**KS3**]):

THEOREM 5. *For a manifold (M,α,ζ) satisfying (ii) the real valued function*

$$\begin{aligned} s(M,\alpha,\zeta,g,\omega^\alpha,\omega^\zeta) &= -\frac{1}{2}\,\eta(D^M_{\omega^\alpha,\omega^\zeta}) - ra_k\eta(S) \\ &\quad + \int_M d^{-1}\left(e^{c_1(\omega^\alpha)/2}\operatorname{ch}(\omega^\zeta)\hat{A}(p(g)) + ra_kL(p(g))\right) \end{aligned}$$

is constant on the path components of the space $\mathcal{S}^+(M,\alpha,\zeta)$ of all triples $(g,\omega^\alpha,\omega^\zeta)$ for which the Hitchin-Lichnerowicz estimate of Theorem 2.2.6 holds.

In the special case of untwisted *Spin*-manifolds we get that $s(g) := s(g,\omega^\alpha,\omega^\zeta) \in \mathbb{R}$, where ω^α and ω^ζ are trivial connections, is constant on the path components of $\mathcal{S}^+$, the space of metrics with positive scalar curvature. In [**KS3**] this fact is used to prove that on some Wallach spaces (see section 8) the space of metrics of positive sectional curvature is not connected.
For a Riemannian manifold M with a free isometric and geodesic action of the circle S^1 let $(g,\omega^\alpha,\omega^\zeta) \in \mathcal{S}^+(M,\alpha,\zeta)$ be a strictly equivariant twisted *Spin*^{c}-Dirac structure. By Theorem 4.2.1 the index of the Dirac operator of an extension of $(g,\omega^\alpha,\omega^\zeta)$ over the associated disc bundle vanishes. By (2) the s-invariant is therefore determined by the characteristic classes of a zero bordism for (M,α,ζ). With regard to the formulae (3) and (4) for the s-invariant Theorem 4.1.1 calculates the defect of s from being asymptotically under canonical variation a spectral invariant. For example the limit of $s_0(g)$ for $g \in \mathcal{S}^+(M)$ on an equivariantly parallelizable S^1-manifold M is determined by the limits of η-invariants because the quotient manifold is then also parallelizable and it is immediate from Theorem 4.1.1 that the integral in 4 vanishes.

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Leitfaden

The first part of the present thesis is designed firstly to recall the basic concepts of *Spin*^{c}-manifolds, the Chern-Weil homomorphism, the Atiyah-Patodi-Singer index formula for Dirac operators and Riemannian submersions and secondly to provide some elementary facts concerning *Spin*^{c}-Dirac structures on disc bundles. In the second part we state and prove formulae for the η-invariant of Dirac-operators on manifolds M carrying a free S^1-action and a strictly S^1-equivariant *Spin*^{c}-Dirac structure. To that end we show that the index of twisted Dirac operators on the associated disc bundle vanishes if the *Spin*^{c}-Dirac structure on M fulfills the Hitchin-Lichnerowicz estimate. The computation of the adiabatic limit of the integral in the index formula applied to DE then yields the desired formulae for the η-invariant. The third part presents a recipe to compute the η-invariant of a compact normal homogeneous Riemannian manifold admitting a non-trivial homogeneous action of the circle. As an example the η-invariant on the Wallach spaces is given for the normal metric induced from the Cartan-Killing form on $SU(3)$.

1

Basic Concepts

2. Preliminaries & Notation

Throughout this thesis we deal with smooth oriented compact manifolds which we usually assume connected and smooth maps between them. A smooth map $f : X \to Y$ has the differential $df : TX \to TY$. The set of smooth real-valued functions on X will be written as $C^\infty X$. The set of sections of a fibre bundle $\pi : E \to X$ over a manifold X is denoted by $\Gamma\pi = \Gamma E$. If $\pi : E \to X$ is an oriented metric vector bundle we will also write $\pi : P_{SO}(E) = P_{SO}(\pi) \to X$ for its oriented orthonormal frame bundle. The k-forms on X with values in a vector bundle E over X are $\Omega^k(X;E) = \Gamma\,\mathrm{Hom}(\Lambda^k TX, E)$. By d we also denote the exterior derivative $d : \Omega^k(X;E) \to \Omega^{k+1}(X;E)$ for a trivialized vector bundle E.

2.1. Principal Fibre Bundles and Connections.

Let $\pi : P \to B$ be a principal G-fibre bundle, where G is a Lie group acting from the right on P. The Lie group G acts on its Lie algebra $\mathfrak{g}$ via the adjoint representation. The tangent bundle along the fibres of P is isomorphic to $\ker d\pi = P \times \mathfrak{g}$ so we can identify $\Omega^*(P;\mathfrak{g}) := \Omega^*(P; P\times\mathfrak{g}) \cong \Omega^*(P; \ker d\pi)$. A 1-form $\omega \in \Omega^1(P;\mathfrak{g})$ is a (principal) connection on π if it is vertical (i.e. $\omega(v) = v$ if $d\pi(v) = 0$ for $v \in TP$) and G-equivariant (i.e. $\omega(vg) = Ad_g\omega(v)$ for all $g \in G$ and $v \in TP$). A connection ω can also be viewed as a vertical projection $\mathcal{V} : TP \to \ker d\pi$. The horizontal projection complementary to $\mathcal{V}$ is $\mathcal{H} = 1 - \mathcal{V}$ and we define $\Omega = \mathcal{H}^* d\omega$. We use the definition of [**KN1**] for d. Especially we have $d\omega(x,y) = 1/2(x\omega(y) - y\omega(x) - \omega([x,y]))$. This 2-form $\Omega \in \Omega^2(P;\mathfrak{g}) = \Omega^2(P; P\times\mathfrak{g})$ is horizontal and equivariant, hence it is pulled back via π from a form $\Omega \in \Omega^2(B; P\times_G \mathfrak{g})$ called the (principal) curvature of ω.
For the linear Lie groups over $F = \mathbb{R}$ or $\mathbb{C}$ there is an equivalent notion of covariant derivative on an F-vector bundle $\zeta : E \to X$ over a manifold X: This is an F-linear map

$$\begin{aligned} \nabla : \Gamma E \otimes \Gamma TX &\longrightarrow \Gamma E \\ s \otimes x &\longmapsto \nabla_x s \end{aligned}$$

satisfying

$$\nabla_{fx} gs = f(g\nabla_x s + x(g)s)$$

for all vector fields x on X, sections s of E and smooth funcions f, g on X.
The curvature tensor $R \in \Omega^2(X; \mathrm{End}(E))$ of ∇ is defined as

$$R_{x,y} = \nabla_{[x,y]} - [\nabla_x, \nabla_y].$$

A straightforward calculation shows that $R_{x,y}s$ is $C^\infty X$-linear in x, y and s, so we really get a tensor field.

Let W be a representation of G. If $\zeta : E = P \times_G W \to X$ is associated to the principal G-fibre bundle π then ω induces a covariant derivative ∇ on ζ by $\nabla_x s = \bar{x} s$ where $\bar{x}$ is the horizontal lift of $x \in TX$ to TP with respect to ω and the section s of ζ is considered as a G-equivariant function $P \to W$. If W is faithful then ω is determined by ∇. The representation induces a map $P \times_G \mathfrak{g} \to \mathrm{End}(E)$. The curvature tensor R of ∇ and the image Ω of the principal curvature form under this map are related by

$$R = -2\Omega.$$

The metric g on a Riemannian manifold X will sometimes be written as $g(x, y) = \langle x \mid y \rangle$. The Levi-Civita connection (covariant derivative) ∇ on X is the unique metric torsion free connection on the tangent bundle of a Riemannian manifold and is given by the formula

$$\begin{aligned} 2\langle \nabla_x y \mid z \rangle &= x\langle y \mid z \rangle + y\langle x \mid z \rangle - z\langle x \mid y \rangle \\ &\quad + \langle [x, y] \mid z \rangle - \langle [x, z] \mid y \rangle - \langle [y, z] \mid x \rangle \end{aligned} \tag{2.1.1}$$

for arbitrary vector fields x, y and z on X.

Recall the notions of equivariant bundles and connections over a manifold X carrying an action of a group H: An equivariant bundle over X is a fibre bundle $\pi : P \to X$ together with an action of H on its total space P covering the action on X. A principal G-fibre bundle $\pi : P \to X$ is called equivariant if this action of H on P commutes with the action of G.

An equivariant connection $\omega \in \Omega^1(P; \mathfrak{g})$ on an H-equivariant principal G-fibre bundle will be called strictly equivariant if the orbits of H are horizontal with respect to ω. As an example consider a Riemannian H-manifold X. If the action of H is geodesic i.e. the orbits of H are totally geodesic in X then the Levi-Civita connection on X is strictly equivariant.

Fixing a homomorphism $\rho : K \to G$ of Lie groups a K-structure for π is a principal K-fibre bundle $\chi : Q \to X$ together with an isomorphism $Q \times_K G \xrightarrow{\bar{\alpha}} P$ of principal G-bundles or equivalently a K-equivariant map $Q \xrightarrow{\alpha} P$ over X. An H-equivariant K-structure is a K-structure together with an H-action on Q commuting with the action of K and such that α is H-equivariant. A (H-equivariant, strictly H-equivariant) K-Dirac structure for a principal fibre bundle π with connection ω^π is a K-structure χ for π together with a (H-equivariant, strictly H-equivariant) connection ω^χ on χ for which $d\rho \circ \omega^\chi = \omega^\pi \circ d\alpha$.

We will confine our discussion to the case of free H-actions on X. Then equivariant (principal) bundles and strictly equivariant connections over X are induced from (principal) bundles and connections over the quotient X/H. This correspondence is biunique.

2.2. Dirac Operators.

References for this section are [**LM**], [**ABS**], [**AS3**].

2.2.1. *$Spin^c$-Dirac structures.*

Let X be a Riemannian manifold and ζ a real oriented metric vector bundle over X of rank n with oriented orthonormal frame bundle $\zeta : P_{SO}(\zeta) \to X$. Let $Spin(n) \overset{\rho_{Spin}}{\to} SO(n)$ be the non-trivial double covering and define $Spin^c(n) = Spin(n) \times_{\mathbb{Z}/2} U(1)$. In the notation of the previous section consider the representation

$$K = Spin^c(n) \overset{\rho_{Spin^c}}{\longrightarrow} SO(n) = G$$

induced by ρ_{Spin}. For $n > 2$, ρ is the non-trivial principal $U(1)$-bundle over $SO(n)$. Thus a $Spin^c$-structure on ζ is a principal $U(1)$-bundle

$$P_{Spin^c}(\zeta) \overset{\alpha}{\longrightarrow} P_{SO}(\zeta),$$

whose restriction to any fibre of $P_{SO}(\zeta)$ is the canonical principal $U(1)$-bundle ρ_{Spin^c}. For $\zeta : P_{SO}(X) \to X$ we call $\alpha : P_{Spin^c}(X) \to P_{SO}(X)$ a $Spin^c$-structure on X.
The canonical $U(1)$-bundle of α is

$$\xi(\alpha) : P_{U(1)}(\alpha) := P_{Spin^c}(\zeta)/Spin(n) = P_{Spin^c}(\zeta) \times_{Spin^c(n)} U(1) \longrightarrow X.$$

Thus we have a commutativ diagram

$$\begin{array}{ccccc} P_{SO}(X) & \overset{\tilde{\xi}}{\longleftarrow} & P_{SO}(X) \times_X P_{U(1)}(\alpha) & \overset{\alpha'}{\longleftarrow} & P_{Spin^c}(X) \\ \downarrow & & \downarrow & & \\ X & \overset{\xi(\alpha)}{\longleftarrow} & P_{U(1)}(\alpha) & & \end{array}$$

where α' is a twofold covering and $\alpha = \tilde{\xi} \circ \alpha'$ and the square is a pull back diagram.
The reduction modulo 2 of the first Chern class $c_1(\alpha) := c_1(\xi(\alpha)) \in H^2(X, \mathbb{Z})$ is the second Stiefel-Whitney class $w_2(\zeta)$ and $Spin^c$-structures on ζ exist if $w_2(\zeta)$ is the reduction modulo 2 of an integral class $c \in H^2(X, \mathbb{Z})$. The group $H^2(X, \mathbb{Z}) = Vect_1^{\mathbb{C}}(X) = Prin_{U(1)}(X)$ of isomorphism classes of principal $U(1)$-bundles over X acts transitively and effectively on the set $Spin^c(\zeta)$ of isomorphism classes of $Spin^c$-structures on ζ. A principal $U(1)$-bundle μ over X maps α to $\zeta^*\mu \otimes \alpha$. The canonical $U(1)$-bundle $\xi(\alpha)$ is mapped to $\xi(\zeta^*\mu \otimes \alpha) = \mu^2 \otimes \xi(\alpha)$.
A $Spin^c$-Dirac structure (α, ω^α) on X is a $Spin^c$-structure α together with a connection on $P_{Spin^c}(X) \to X$ which is compatible with the Levi-Civita connection. Such connections correspond biuniquely to arbitrary connections ω^α on $\xi(\alpha)$. This correspondence is given as follows:
Denoting the principal Levi-Civita connection on $P_{SO}X$ by ω_X we get a connection $\mu = j(\alpha^*\omega_X + q^*\omega^\alpha)$ on $P_{Spin^c}(X)$ where q is the quotient map $P_{Spin^c}(\zeta) \to P_{Spin^c}(\zeta)/Spin(n)$ and j is the isomorphism of Lie algebras $j : \mathfrak{so}(n) \oplus \mathfrak{u}(1) \cong \mathfrak{spin}^c(n)$.
If α is an H-equivariant $Spin^c$-structure on the Riemannian H-manifold X then $\xi(\alpha)$ is an H-equivariant bundle. Furthermore if H is connected the action of H on $P_{Spin^c}(X)$ is determined by the H-action on $P_{U(1)}(\alpha)$. Equivariant $Spin^c$-Dirac structures X correspond to equivariant connections on $\xi(\alpha)$. If the action of H on X is geodesic then strictly

equivariant *Spin*^{c}-Dirac structures on X match with strictly equivariant connections on $\xi(\alpha)$.
On $Y = \partial X$ the boundary *Spin*^{c}-structure $\partial\alpha$ is the restriction of α to $P_{SO}Y \hookrightarrow P_{SO}X$.

2.2.2. *Dirac operators.*

Let Σ be a complex module for the Clifford algebra $Cl(n)$ of the vector space $\mathbb{R}^n$ with the negative definite quadratic form $Q(x) = -\sum x_i^2$. A module for the Clifford algebra is always assumed to have a Hermitian scalar product such that multiplication by unit vectors of $\mathbb{R}^n \subset Cl(n)$ is unitary. Consider a Riemannian *Spin*^{c}-Dirac manifold X of dimension n with *Spin*^{c}-structure α and a connection ω^α on $\xi(\alpha)$. Then we have an associated spinor bundle $S = P_{Spin^c}(X) \times_{Spin^c(n)} \Sigma$ with a covariant derivative ∇^S determined by ω^α and the Levi-Civita connection on X. Given a Hermitian vector bundle ζ over X with a unitary covariant derivative ∇^ζ the tensor product connection on $S \otimes \zeta$ is $\nabla = 1 \otimes \nabla^\zeta + \nabla^S \otimes 1$. We then define the Dirac operator twisted with ζ as the composition

$$D_\zeta : \Gamma(S \otimes \zeta) \xrightarrow{\nabla} \Gamma(S \otimes \zeta \otimes T^*X) \xrightarrow{cl} \Gamma(S \otimes \zeta),$$

where cl is the Clifford multiplication of $T^*X \subset Cl(X)$ on $S \otimes \zeta$ induced from the action of the Clifford-bundle $Cl(T^*X)$ on S.
Every complex module for the Clifford algebra $Cl(n)$ of finite dimension over $\mathbb{C}$ decomposes in irreducible modules. So it suffices to consider spinor bundles $P_{Spin^c}(X) \times_{Spin^c(n)} \Sigma$ where Σ is an irreducible module for $Cl(n)$.
In even dimensions $n = 2k$ there is a unique irreducible $Cl(2k)$-module Σ_{2k}. When viewed as a *Spin*$^c(2k)$-representation Σ splits in two different irreducible representations $\Sigma_{2k} = \Sigma_{2k}^+ \oplus \Sigma_{2k}^-$ which give rise to a splitting $S = S^+ \oplus S^-$ of the corresponding spinor bundle. With respect to this splitting D has the form $\begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$ where

$$D^\pm : \Gamma S^\pm \longrightarrow \Gamma S^\mp.$$

These operators twisted with a coefficient bundle (ζ, ∇^ζ) are denoted by D_ζ, $D_\zeta^\pm$.
In odd dimensions $n = 2k-1$ the modules Σ_{2k}^+ and Σ_{2k}^- obtained from the irreducible module Σ_{2k} for $Cl(2k)$ are the irreducible modules for $Cl(2k-1)$ and yield equivalent irreducible representations of *Spin*$^c(2k-1)$: Let v be a unit vector in $\mathbb{R}^{2k}$ orthogonal to $\mathbb{R}^{2k-1}$. The Clifford action of $u \in \mathbb{R}^{2k-1} \subset Cl(2k-1)$ on $s \in \Sigma_{2k}^+$ is given by $s \mapsto -vus$. The representations of $Spin(2k-1)$ are isomorphic because v commutes with $Spin(2k-1)$.
Let X be a $2k$-dimensional *Spin*^{c}-manifold with boundary Y. If $S_X = S^+ \oplus S^-$ is the spinor bundle on X then the spinor bundle S_Y on the boundary Y is isomorphic to $S^+|_Y$. The Clifford multiplication with $u \in TY \subset TX$ under this isomorphism is given by $u \cdot s = -vus$ for $s \in S^+|_Y$ where v is the inward normal vector field in $TX|_Y$. The tangential Dirac operator D^Y (see 2.4) is the Dirac operator on $S^+|_Y$ with this Clifford multiplication.

2.2.3. *Lichnerowicz's vanishing theorem.*

Let $\mu \in \Omega^2(X; \mathfrak{u}(S \otimes \zeta))$ be a 2-form on X with values in the skew-Hermitian endomorphisms of $S \otimes \zeta$. Choose an orthonormal basis $\{e_1, \dots, e_n\}$ of T_xX and define a Hermitian endomorphism $\mathcal{E}(\mu)$ of $S \otimes \zeta$ as

$$\mathcal{E}(\mu)(\sigma \otimes \varepsilon) = \frac{i}{2} \sum_{j<k} \mu(e_j, e_k)(e_j e_k \sigma \otimes \varepsilon)$$

for $\sigma \in S_x$, $\varepsilon \in \zeta_x$. Also define $||\mu|| \in C^\infty(X)$ by

$$||\mu||(x) = -\min\{\langle \mathcal{E}(\mu)s \mid s\rangle \mid s \in (S \otimes \zeta)_x, ||s|| = 1\}.$$

This is minus the smallest eigenvalue of $\mathcal{E}(\mu)$. For 2-forms $\mu, \nu \in \Omega^2(X; \mathfrak{u}(S \otimes \zeta))$ we have $||\mu|| + ||\nu|| \geq ||\mu + \nu||$.

Let Ω^ζ and Ω^α be the principal curvature forms of ∇^ζ and ω^α respectively. Note that $\Omega^\alpha \in \Omega^2(X; i\mathbb{R})$ induces the skew-Hermitian endomorphism $s \mapsto \frac{1}{2}\Omega^\alpha(x,y)s$ on S, so that

$$\mathcal{E}(\Omega^\alpha \otimes 1 + 1 \otimes \Omega^\zeta)(\sigma \otimes \varepsilon) = \frac{i}{2} \sum_{j<k} \frac{1}{2}\Omega^\alpha(e_j, e_k) e_j e_k \sigma \otimes \varepsilon + e_j e_k \sigma \otimes \Omega^\zeta(e_j, e_k)\varepsilon$$

As an example look at the untwisted $Spin^c$-case (take the trivial complex line bundle for $\zeta = X$ with its trivial connection). Hitchin ([**Hit**]) has computed the function $||\Omega^\alpha|| := ||\Omega^\alpha \otimes 1||$ as follows: Choose an orthonormal basis $\{e_1, \ldots, e_n\}$ of $T^*_x X$ such that the curvature form Ω^α at the point $x \in X$ has the form

$$\Omega^\alpha = \sum_{1 \leq i \leq [n/2]} \lambda_i e_{2i-1} \wedge e_{2i}.$$

Then

$$||\Omega^\alpha||(x) = \frac{1}{2} \sum_{1 \leq i \leq [n/2]} |\lambda_i|. \tag{2.2.4}$$

For the Dirac Laplacian D^2_ζ we have Bochner's formula

$$D^2_\zeta = \nabla^*\nabla + \frac{1}{4}s + \mathcal{E}(\Omega^\alpha \otimes 1 + 1 \otimes \Omega^\zeta).$$

On a closed manifold X the operator $\nabla^*\nabla$ is nonnegative since for sections s of $S \otimes \zeta$ we have that

$$\int_X \langle \nabla^*\nabla s \mid s\rangle = \int_X \langle \nabla s \mid \nabla s\rangle \geq 0. \tag{2.2.5}$$

The endomorphism $\frac{1}{4}s + \mathcal{E}(\Omega^\alpha \otimes 1 + 1 \otimes \Omega^\zeta)$ is positive if $s - 4\,||\Omega^\alpha \otimes 1 + 1 \otimes \Omega^\zeta||$ is positive. In this case the kernel of $D_\zeta = D^+_\zeta \oplus D^-_\zeta$ must be trivial and $\operatorname{index} D^+_\zeta = \dim\ker D^+_\zeta - \dim\ker D^-_\zeta = 0$.

In [**APS2**], before Theorem (3.9), the same argument is shown to work on the compact manifold X with boundary Y if we impose the Atiyah-Patodi-Singer boundary conditions (2.4.1) on the sections s: If the kernel of the tangential operator D^Y (see 2.4) is trivial then the index of D^+_ζ equals the index of the Dirac-operator $\mathcal{D}^+_\zeta$ on L^2-sections of the extension of $S \otimes \zeta$ to the elongation $\hat{X} = X \cup_Y Y \times [0, \infty[$ of X. By [**LM**], Chapter II, Proposition 8.1, the difference between the two integrands in (2.2.5) is the divergence of a vector field w: $\langle \nabla s \mid \nabla s\rangle - \langle \nabla^*\nabla s \mid s\rangle = \operatorname{div} w$. At a point $x \in X$ this vector field is given by $w = \sum_i e_i \langle \nabla_{e_i} s \mid s\rangle$ in terms of a local orthonormal framing $\{e_i\}$ of $T\hat{X}$ around x with $(\nabla e_i)(x) = 0$. If $s \in \ker \mathcal{D}_\zeta$ then the boundary conditions (2.4.1) imply that s

and its covariant derivative decrease exponentially on the cylinder $Y \times [0, \infty[$. Therefore the vector field w is exponentially small at infinity. Now Green's formula shows that $\int_{\hat{X}} \langle \nabla^* \nabla s \mid s \rangle = \int_{\hat{X}} \langle \nabla s \mid \nabla s \rangle \geq 0$.
Thus the following vanishing theorem of Hitchin ([**Hit**]), Lichnerowicz ([**Li**]) holds for Dirac operators on closed manifolds X as well as for Dirac operators on compact manifolds with boundary and acting on sections fulfilling the Atiyah-Patodi-Singer boundary conditions (2.4.1):

THEOREM 2.2.6. [**Hit**], [**Li**] *Let $(X, g, \alpha, \omega^\alpha)$ be a Spin^c-Dirac manifold with metric g of scalar curvature s, Spin^c-structure α and a connection ω^α on the canonical $U(1)$-bundle $\xi(\alpha)$. Let D_ζ be the Dirac operator on $(X, g, \alpha, \omega^\alpha)$ twisted with a Hermitian bundle (ζ, ∇^ζ). If*

$$\frac{1}{4}s - ||\Omega^\alpha \otimes 1 + 1 \otimes \Omega^\zeta||$$

is positive somewhere and nonnegative everywhere on X then

$$\operatorname{index} D^+_\zeta = 0.$$

On closed manifolds we actually have $\ker D_\zeta = 0$.

2.3. The Chern-Weil Homomorphism.

A Lie group G acts on the $\mathbb{R}$- or $\mathbb{C}$-dual of the k-fold tensor product of its Lie algebra $\mathfrak{g}$ via the adjoint representation. We also have an action of the symmetric group on k elements by permuting the factors. An invariant polynomial p of degree k on $\mathfrak{g}$ is a k-form $p \in (\underbrace{\mathfrak{g} \otimes \ldots \otimes \mathfrak{g}}_{k})^*$ on $\mathfrak{g}$ which is invariant under these actions.
For a principal G-bundle $\mu : P \to X$ with connection form ω we can substitute the curvature form Ω into an invariant polynomial p of degree k to get a form $p(\Omega) \in \Omega^{2k}(X; \mathbb{R})$ or in $\Omega^{2k}(X; \mathbb{C})$.
Examples:

(1) For an $SO(n)$-principal bundle P the Pontrjagin forms $p_k(\omega) \in \Omega^{4k}(X; \mathbb{R})$ are defined by
$$\sum_{k=1}^{n} p_k(\omega) x^{n-2k} = \det\left(x - \frac{\Omega}{2\pi}\right).$$
The Pontrjagin forms of the Levi-Civita connection on the tangent bundle of a Riemannian manifold X with metric g will be denoted by $p(g)$.

(2) For a $U(n)$-principal bundle P the Chern forms $c_k(\omega) \in \Omega^{2k}(X; \mathbb{R})$ are given by
$$\sum_{k=1}^{n} c_k(\omega) x^{n-k} = \det\left(x - \frac{\Omega}{2\pi i}\right).$$
Especially we have
$$c_1(\omega) = \operatorname{trace} \frac{i\Omega}{2\pi}.$$

The Chern character form $\mathrm{ch}(\omega) \in \Omega^{2*}(X;\mathbb{R})$ is

$$\mathrm{ch}(\omega) = \mathrm{trace}\exp\left(\frac{i\Omega}{2\pi}\right).$$

2.4. The Atiyah-Patodi-Singer Index Theorem for Manifolds with Boundary.

Let W be a compact Riemannian manifold with boundary M and E be a vector bundle over W. Assume that some collar neighbourhood of M is isometric to the product $M \times I_\varepsilon$ with metric $g_M \oplus dt^2$ where $I_\varepsilon = [0,\varepsilon]$ for some $\varepsilon > 0$, $\partial W = M \times \{0\}$ and $\frac{\partial}{\partial t}$ is the inward normal vector field on $M \times I_\varepsilon$. We consider elliptic first order differential operators D on E which in this collar neighbourhood have the form $D = \sigma(\frac{\partial}{\partial t} + D^M)$ for a bundle isomorphism σ of E and a self-adjoint differential operator D^M on $E|_M$ called the tangential operator to D. We consider the action of D on sections s satisfying the boundary condition

$$P(s|_M) = 0, \tag{2.4.1}$$

where P is the projection onto the span of all eigenvectors of D^M with non-negative eigenvalues.
Define $h(D^M) = \dim\ker D^M$ and let $\eta(D^M,s)$ be the meromorphic extension of the function

$$\sum_{\lambda\in Spec D^M} |\lambda|^{-s} sgn(\lambda)$$

to the complex plane. The η-invariant of D^M is $\eta(D^M) = \eta(D^M,0)$.
By $\hat{A}$ and L we denote the multiplicative sequences corresponding to the characteristic power series $x/(2\sinh(x/2))$ and $x/\tanh(x)$ respectively.

THEOREM 2.4.2. [**APS1**] *Let (W,g,α,ω) be an even-dimensional Riemannian $Spin^c$-manifold W with boundary M, metric g and a $Spin^c$-Dirac structure (α,ω). Let (ζ,∇^ζ) be a twisting bundle. Assume that g is isometric to the product $g_M \oplus dt^2$ near M and that the restrictions of ω and ∇^ζ to some collar neighbourhood $M \times I_\varepsilon$ are induced from the connections ω_M and ∇^ζ_M over M.*
Consider the $Spin^c$-Dirac operator D_ζ on W, twisted with ζ and acting on sections s of $S \otimes \zeta$ satisfying the boundary condition (2.4.1). *Let D^M_ζ be the $Spin^c$-Dirac operator on $(M,g_M,\partial\alpha,\omega_M)$ twisted with $(\zeta,\nabla^\zeta)|_M$.*
Then

$$\mathrm{index}\, D^+_\zeta = \int_W e^{c_1(\omega)/2}\,\mathrm{ch}(\nabla^\zeta)\hat{A}(p(g)) - \frac{\eta(D^M_\zeta) + h(D^M_\zeta)}{2}.$$

Note: This formula is not stated littarally in [**APS1**] but it follows from the discussion before formula (6.13) in [**ABP**]: The point is that the integrand is computed locally. But locally a *Spin*^{c}-Dirac structure (α,ω^α) is the same as a *Spin*-structure twisted with a complex line bundle (χ,ω^χ) such that $\chi\otimes\chi = \xi(\alpha)$ and $\omega^\chi\otimes\omega^\chi = \omega^\alpha$. So $\mathrm{ch}(\omega^\chi) = e^{c_1(\omega^\alpha)/2}$ as forms.

The signature operator on a $(4k-1)$-dimensional Riemannian manifold (M, g_M) is the operator $S(g_M) = \oplus_p S_p : \Omega^{2*}(M) \to \Omega^{2*}(M)$ constructed out of the Hodge $*$-operator and the exterior derivative d (see [**AS3**]):

$$\begin{array}{rcl} S_p : \Omega^{2p}(M) & \longrightarrow & \Omega^{4k-2p-2}(M) \oplus \Omega^{4k-2p}(M) \\ \chi & \longmapsto & (-1)^{k+p+1}(*d - d*)\chi. \end{array}$$

THEOREM 2.4.3. [**APS1**] *Let (W, g) be a Riemannian manifold of dimension $4k$ with boundary (M, g_M). The signature of (W, M) is*

$$sign(W, M) = \int_W L(p(g)) - \eta(S(g_M)).$$

2.5. Riemannian Submersions.

The ensuing formulae can be found in [**Be**], [**O'N**]. Let $\pi : (M, g) \to (B, g_B)$ be a Riemannian submersion with totally geodesic fibres (F, g_F). Denote by $V = \ker d\pi \subset TM$ and $H = (\ker d\pi)^\perp \subset TM$ the vertical and horizontal distributions of π and by $\mathcal{V}$, $\mathcal{H}$ the corresponding orthogonal projections. Then $d\pi|_H$ is isometric and $d\pi \nabla_u v = 0$ for all vertical vector fields u and v.
Let ∇, R, ∇^F, R^F and ∇^B, R^B be the Levi-Civita connections and curvature tensors on (M, g), (F, g_F) and (B, g_B). Note that by a theorem of Hermann [**Her**] all fibres of a Riemannian submersion with totally geodesic fibres are isometric.
We will always denote horizontal tangent vectors of M by $a, b, c, \dots, h$ and vertical vectors by r, s, t, u, v, w.

2.5.1. *O'Neill's formulae.*

For vector fields $x, y \in \Gamma TM$ define the A-tensor of O'Neill by

$$A_x y = \mathcal{V}\nabla_{\mathcal{H}x}\mathcal{H}y + \mathcal{H}\nabla_{\mathcal{H}x}\mathcal{V}y.$$

If a, b are horizontal vector fields, then formula (2.1.1) shows that

$$A_a b = \frac{1}{2}\mathcal{V}[a, b].$$

For a Riemannian submersion the Riemannian curvature tensor of the total space M is given by O'Neill's formulae which for submersions with totally geodesic fibres simplify to

$$\begin{array}{rcl} \langle R_{a,b}c|h\rangle & = & \langle R^B_{a,b}c|h\rangle - 2\langle A_a b|A_c h\rangle + \langle A_a h|A_b c\rangle - \langle A_a c|A_b h\rangle, \\ \langle R_{a,b}c|u\rangle & = & \langle(\nabla_c A)_a b|u\rangle, \\ \langle R_{a,b}u|v\rangle & = & \langle(\nabla_u A)_a b|v\rangle - \langle(\nabla_v A)_a b|u\rangle + \langle A_a u|A_b v\rangle - \langle A_a v|A_b u\rangle, \\ \langle R_{a,u}b|v\rangle & = & \langle(\nabla_u A)_a b|v\rangle + \langle A_a u|A_b v\rangle, \\ \langle R_{u,v}w|a\rangle & = & 0, \\ \langle R_{u,v}w|t\rangle & = & \langle R^F_{u,v}w|t\rangle. \end{array}$$

2.5.2. *The Canonical Variation.*

The canonical variation of a metric g on M is the family of metrics on M given by

$$g^\tau = \mathcal{H}^* g + \tau \mathcal{V}^* g$$

for $\tau \in \mathbb{R}^+$. Let $\langle \cdot \mid \cdot \rangle^\tau = g^\tau$ and $\nabla^\tau, R^\tau, A^\tau$ be the covariant derivative, Riemannian curvature tensor and A-tensor for the metric g^τ. Then

$$\begin{aligned}
A^\tau_a b &= A_a b,\\
A^\tau_a u &= \tau A_a u,\\
\langle (\nabla^\tau_c A^\tau)_a b | u\rangle^\tau &= \tau \langle (\nabla_c A)_a b | u\rangle,\\
\langle (\nabla^\tau_u A^\tau)_a b | v\rangle^\tau &= \tau \langle (\nabla_u A)_a b | v\rangle + (\tau - \tau^2)(\langle A_a u | A_b v\rangle - \langle A_a v | A_b u\rangle).
\end{aligned}$$

The dependence of R^τ on τ is given by (see [**Be**])

$$\begin{aligned}
\langle R^\tau_{a,b} c | h\rangle^\tau &= \langle R^B_{a,b} c | h\rangle + \tau(\langle A_a h | A_b c\rangle - \langle A_a c | A_b h\rangle - 2\langle A_a b | A_c h\rangle),\\
\langle R^\tau_{a,b} c | u\rangle^\tau &= \tau \langle (\nabla_c A)_a b | u\rangle,\\
\langle R^\tau_{a,b} u | v\rangle^\tau &= \tau(\langle (\nabla_u A)_a b | v\rangle - \langle (\nabla_v A)_a b | u\rangle),\\
&\quad +(2\tau - \tau^2)(\langle A_a u | A_b v\rangle - \langle A_a v | A_b u\rangle),\\
\langle R^\tau_{a,u} b | v\rangle^\tau &= \langle (\nabla_u A)_a b | v\rangle \tau + (\tau - \tau^2)(\langle A_a u | A_b v\rangle - \langle A_a v | A_b u\rangle),\\
&\quad +\tau^2 \langle A_a u | A_b v\rangle,\\
\langle R^\tau_{u,v} w | a\rangle^\tau &= 0,\\
\langle R^\tau_{u,v} w | t\rangle^\tau &= \tau \langle R^F_{u,v} w | t\rangle^F.
\end{aligned}$$

For a function $f : \mathbb{R} \to \mathbb{R}$ we shall write $f(\tau) = O(\tau^k)$ for $\tau \to 0$ if $\lim_{\tau \to 0} f(\tau)/\tau^k$ exists and is finite. The asymptotic expansion of R^τ for $\tau \to 0$ is then

$$\begin{aligned}
\langle R^\tau_{a,b} c | h\rangle^\tau &= \langle R^B_{a,b} c | h\rangle + O(\tau),\\
\langle R^\tau_{a,b} c | u\rangle^\tau &= O(\tau),\\
\langle R^\tau_{a,b} u | v\rangle^\tau &= O(\tau),\\
\langle R^\tau_{a,u} b | v\rangle^\tau &= O(\tau),\\
\langle R^\tau_{u,v} w | a\rangle^\tau &= 0,\\
\langle R^\tau_{u,v} w | t\rangle^\tau &= \tau \langle R^F_{u,v} w | s\rangle^F = O(\tau).
\end{aligned}$$

For the scalar curvature we have

$$s_{g^\tau} = \frac{1}{\tau} s_F + s_B - \tau ||A||^2,$$

where the function $||A||$ on M is defined at the point $p \in M$ by

$$||A||^2(p) = \sum_{i,j} \langle A_{h_i} h_j \mid A_{h_i} h_j\rangle$$

for an orthonormal basis $\{h_1, \dots h_n\}$ of H_p.
On the bundle $T_F M = V$ along the fibres of π there is a connection

$$\nabla^{\mathcal{V}} := \mathcal{V} \nabla^\tau. \tag{2.5.3}$$

We have suppressed the rescaling factor τ in the notation here because it follows immediately from (2.1.1) that $\nabla^{\mathcal{V}}$ does not depend on τ.
Its curvature tensor $R^{\mathcal{V}}$ is given by

$$\begin{aligned}\langle R^{\mathcal{V}}_{x,y}u \mid v\rangle &= \langle \nabla_{[x,y]}u \mid v\rangle - \langle \nabla_x \mathcal{V}\nabla_y u \mid v\rangle + \langle \nabla_y \mathcal{V}\nabla_x u \mid v\rangle \\ &= \langle R_{x,y}u \mid v\rangle + \langle \nabla_x \mathcal{H}\nabla_y u \mid v\rangle - \langle \nabla_y \mathcal{H}\nabla_x u \mid v\rangle.\end{aligned}$$

By definition $A_y u = \mathcal{H}\nabla_y u$. Using the formulae for the canonical variation of A we therefore can compute

$$\begin{aligned}\langle \nabla^\tau_x \mathcal{H}\nabla^\tau_y u \mid v\rangle^\tau &= x\langle \mathcal{H}\nabla^\tau_y u \mid v\rangle^\tau - \langle \mathcal{H}\nabla^\tau_y u \mid \nabla_x v\rangle^\tau \\ &= \langle A^\tau_y u \mid A^\tau_x v\rangle^\tau \\ &= O(\tau^2)\end{aligned} \tag{2.5.4}$$

to obtain

$$R^{\mathcal{V}} = \mathcal{V}R^\tau + O(\tau).$$

At a point $p \in M$ choose a basis $\mathcal{B} = \{h_1, \dots, h_n, u_1, \dots, u_m\}$ of T_pM consisting of horizontal vectors $\{h_1, \dots, h_n\}$ and vertical vectors $\{u_1, \dots, u_m\}$. If $\mathcal{B}$ is an orthonormal basis with respect to $g = g^1$ the asymptotic behaviour for $\tau \to 0$ of a matrix representation of the curvature tensor is:

$$\begin{aligned}R^\tau &= \begin{pmatrix} \langle R^\tau u_i \mid u_j\rangle^\tau/\tau & \langle R^\tau h_k \mid u_j\rangle^\tau/\tau \\ \langle R^\tau u_i \mid h_l\rangle^\tau & \langle R^\tau h_k \mid h_l\rangle^\tau \end{pmatrix} \\ &\overset{\tau\to 0}{\sim} \begin{pmatrix} R^{\mathcal{V}} + O(\tau) & O(1) \\ O(\tau) & R^B + O(\tau)\end{pmatrix}\end{aligned}$$

and the limit of R^τ is

$$\lim_{\tau\to 0} R^\tau = \begin{pmatrix} R^{\mathcal{V}} & * \\ 0 & \pi^* R^B \end{pmatrix}. \tag{2.5.5}$$

3. Disc Bundles

In this section $(M, g_M, \alpha_M, \omega_M)$ will always be a Riemannian manifold of dimension $n+1$ with metric g_M, carrying a free isometric and geodesic action of the circle S^1 and an equivariant *Spin*^{c}-Dirac structure (α_M, ω_M). Then the orbit space $B = M/S^1$ is a manifold and there is a metric g_B on B such that the quotient map $\pi : M \to B$ becomes a principal S^1-bundle and a Riemannian submersion with totally geodesic fibres. By the theorem of Hermann there is a positive real number ρ such that all fibres of π are isometric to $S^1_\rho \hookrightarrow \mathbb{C}$, a circle of radius ρ in $\mathbb{C} \cong \mathbb{R}^2$ with its standard metric. We construct extensions of the metric and the *Spin*^{c}-Dirac structure on M to the disc bundle $\pi_D : DE = M \times_{S^1} \mathbb{D} \to B$ of the associated complex line bundle $\pi_{\mathbb{C}} : E = M \times_{S^1} \mathbb{C} \to B$ to π. The disc $\mathbb{D} \subset \mathbb{C}$ of radius $\delta > \rho$ will be endowed with a metric such that $\partial\mathbb{D}$ is isometric to S^1_ρ.

3.1. Associated Bundles.

A tool for constructing Riemannian submersions with totally geodesic fibres is the following

THEOREM 3.1.1. [**Vi**], *see also* [**Be**] *Let $\pi : P \to B$ be a principal G-bundle, G a Lie group, with connection ω over a Riemannian manifold (B, g_B). Also let (F, g_F) be a Riemannian G-manifold (i.e. G acts isometrically for g_F). On the total space of the associated fibre bundle $\pi_{\otimes F} : E := P \times_G F \to B$ there is then a unique metric g_E such that:*

1. *the fibres of π are totally geodesic submanifolds of (E, g_E) and isometric to (F, g_F),*
2. *$\pi_{\otimes F} : (E, g_E) \to (B, g_B)$ is a Riemannian submersion and*
3. *the horizontal distribution of TE is associated to ω.*

3.2. Some Metrics on Disc Bundles.

Let ω^π be the connection on π whose horizontal distribution is the horizontal distribution of the Riemannian submersion π, i.e. $(\ker d\pi)^\perp = \ker \omega^\pi$. On $\mathbb{C} \setminus \{0\}$ consider the vectorfields $\tilde{u}$ and v given by polar coordinates:

$$\tilde{u}\phi(re^{it}) = \frac{\partial}{\partial t}\phi(re^{it})$$

$$v\phi(re^{it}) = \frac{\partial}{\partial r}\phi(re^{it})$$

for functions ϕ on $\mathbb{C} \setminus \{0\}$, $r \in \mathbb{R}^+$ and $t \in \mathbb{R}$. Let the metric $g_{\mathbb{C}}$ on $\mathbb{C} \setminus \{0\}$ expressed in the basis $\{\tilde{u}, v\}$ be given by the matrix

$$\begin{pmatrix} f(r)^2 & 0 \\ 0 & 1 \end{pmatrix}$$

for some function

$$f : \mathbb{R}_0^+ \longrightarrow [0, \rho]$$

which extends to an odd smooth function f on $\mathbb{R}$ with $f'(0) = 1$ to ensure that the metric $g_{\mathbb{C}}$ extends from $\mathbb{C} \setminus \{0\}$ to the whole of $\mathbb{C}$. We also fix some real number γ with $0 < \gamma < \delta$ and require that $f(r) = \rho$ if $r \geq \gamma$. Then near its boundary S^1_ρ the disc $\mathbb{D} \subset \mathbb{C}$ with radius δ equipped with this metric becomes isometric to a cylinder $S^1 \times [\gamma, \delta]$.
Take ω^π and this metric on $\mathbb{C}$ to construct the metric g_E on E by the theorem of [**Vi**]. The horizontal distribution associated to ω^π is then the horizontal distribution of the Riemannian submersion $\pi_{\mathbb{C}}$ and we have isometric embeddings into E of B as the zero section and of the canonical variation of M by

$$(M, g^{f(\tau)^2/\rho^2}) = M_\tau = \{x \in E \mid d(x, B) = \tau\} \hookrightarrow E,$$

where d is the distance function on (E, g_E). Especially (M, g) is embedded as the boundary of the associated disc bundle $DE = \{x \in E \mid d(x, B) \leq \delta\}$.

Associated to the vector fields $\tilde{u}$, $u = \tilde{u}/||\tilde{u}||$ and v are vertical vectorfields on $DE \setminus B$ which we denote by the same letters. Using that v commutes with all basic vectorfields of π we get

$$\begin{aligned}
\nabla_v v &= \nabla_v u = 0, \\
\nabla_u v &= \frac{f'}{f} u, \\
\nabla_u u &= -\frac{f'}{f} v, \\
\nabla_a v &= 0 \text{ for every horizontal vector field } a.
\end{aligned}$$

At a point with distance τ from the zero section in E the scalar curvature of the fibre $F = \mathbb{C}$ is

$$s_F = -2\frac{f''(\tau)}{f(\tau)}.$$

In order to compute the function $||A||$ on E we fix a point $e \in E$ with $\pi(e) = b \in B$, $d(e, B) = \tau$ and choose an orthonormal basis $\{h_1, \dots, h_n\}$ for T_bB for some $b \in B$.
For arbitrary vectors $x, y \in T_bB$, their horizontal lifts $\bar{x}, \bar{y} \in T_eE$, and extensions to horizontal vector fields also denoted by $\bar{x}$, $\bar{y}$, the A-tensor is given by $A_{\bar{x}}\bar{y} = \frac{1}{2}\mathcal{V}[\bar{x}, \bar{y}]$. We have that $\langle [\bar{x}, \bar{y}] \mid v \rangle = 0$, since the vectorfields $\bar{x}$, $\bar{y}$ are vectorfields on the submanifold M_τ which is perpendicular to the vectorfield v. Hence

$$\begin{aligned}
A_{\bar{x}}\bar{y} &= \frac{1}{2}\langle [\bar{x}, \bar{y}] \mid u \rangle u = -\frac{1}{i}\langle \Omega^\pi(x, y)\tilde{u} \mid u \rangle u \\
&= i\Omega^\pi(x, y) f(\tau) u,
\end{aligned}$$

where Ω^π is the curvature form of ω^π. By definition,

$$||A||^2 = \sum_{i,j} \langle A_{\bar{h}_i}\bar{h}_j \mid A_{\bar{h}_i}\bar{h}_j \rangle = \sum_{i,j} -\Omega^\pi(h_i, h_j)^2 f(\tau)^2.$$

Since $f(\tau) \leq \rho$ if $e \in DE$ we have

$$s_{M_\tau} = s_B - \frac{f(\tau)^2}{\rho^2}||A||^2 \geq s_B - ||A||^2 = s_{M_\delta} = s_M$$

and the scalar curvature of DE is estimated by

$$s_{DE} \geq s_F + s_M = -2\frac{f''}{f} + s_M. \tag{3.2.1}$$

3.3. *Spin^c*-Dirac Structures.

We will consider two *Spin^c*-Dirac sructures on a fixed equivariant *Spin^c*-structure α_M on M. Such *Spin^c*-structures M are obtained from *Spin^c*-structures on B and vice versa. A *Spin^c*-structure $\alpha_B : P_{Spin^c}(B) \to P_{SO}(B)$ on B induces the *Spin^c*-structure

$$\begin{array}{c}
P_{Spin^c}(M) := \pi^*(P_{Spin^c}(B) \times_{Spin^c(n)} Spin^c(n+1)) \\
\big\downarrow \alpha_M \\
P_{SO}(M) = \pi^*(P_{SO}(B) \times_{SO(n)} SO(n+1))
\end{array}$$

on M. The canonical bundle of α_M is $\xi(\alpha_M) = \pi^*\xi(\alpha_B)$. If ω_M is a strictly equivariant connection on $\xi(\alpha_M)$ then $\omega_M = \pi^*\omega_B$ for some connection ω_B on $\xi(\alpha_B)$.
The equivariant $Spin^c$-structure α_M extends to a $Spin^c$-structure α_{DE} on the disc bundle DE which is induced from the $Spin^c$-structure α_B and the canonical $Spin^c$-structure $\alpha^\pi : P_{Spin^c}(\pi) = M \times_{U(1)} Spin^c(2) \to M = P_{SO}(\pi)$ of the principal S^1-bundle π:

$$\begin{array}{c} P_{Spin^c}(DE) = \pi^*(P_{Spin^c}(B) \times_B P_{Spin^c}(\pi)) \times_{Spin^c(n)\times Spin^c(2)} Spin^c(n+2) \\ \big\downarrow \alpha_{DE} \\ P_{SO}(DE) = \pi^*(P_{SO}(B) \times_B P_{SO}(\pi)) \times_{SO(n)\times SO(2)} SO(n+2) \end{array}$$

Its canonical bundle is $\xi(\alpha_{DE}) = \pi_D^*(\xi(\alpha_B) \otimes \pi)$. The canonical isomorphism to $\pi_D^*\xi(\alpha_B)$ outside the zero section is not equivariant. Putting $\omega_{DE} = \pi_D^*(\omega_B \otimes \omega^\pi)$ we get an equivariant $Spin^c$-Dirac structure $(\alpha_{DE}, \omega_{DE})$ on DE. Restrictions to M of such $Spin^c$-Dirac structures will be referred to as boundary $Spin^c$-structures. They are equivariant but not strictly equivariant.
In order to extend the strictly equivariant $Spin^c$-Dirac structure $(\alpha_M, \omega_M = \pi^*\omega_B)$ from M over DE we pick a smooth function

$$\psi : \mathbb{R}_0^+ \longrightarrow [0,1]$$

such that for the same real number γ as in section 3.2 and some $\alpha \in]0,\gamma[$ we have

(1) $\psi(\tau) = 1$ if $\tau \in [0,\alpha]$,
(2) $\psi(\tau) = 0$ if $\tau \geq \gamma$.

Then the function $\psi(d(\cdot, B))$ which we will also denote by ψ is smooth on DE.
By ω^0 we denote the trivial connection on $\pi_D^*\pi|_{DE\setminus B}$ induced from its canonical trivialisation. The $Spin^c$-Dirac structures $(\alpha_M, \pi^*\omega_B)$ and $(\alpha_M, \pi^*\omega_B \otimes \omega^0)$ on M are equivariantly isomorphic and we can extend $(\alpha_M, \pi^*\omega_B \otimes \omega^0)$ to all of DE by

$$\omega_{DE} = \pi_D^*\omega_B \otimes (\psi\pi_D^*\omega^\pi + (1-\psi)\omega^0).$$

For the curvature $\Omega_{DE} = d\omega_{DE}$ of ω_{DE} we therefore get

$$\Omega_{DE} = \pi_D^*\Omega_B \otimes 1 + 1 \otimes (d\psi(\pi_D^*\omega^\pi - \omega^0) + \psi\pi_D^*\Omega^\pi). \tag{3.3.1}$$

Let $\{d\tilde{u}, dv\}$ and $\{du, dv\}$ be the framings of the vertical distribution dual to $\{\tilde{u}, v\}$ and $\{u, v\}$ respectively. In terms of the metric these 1-forms are given by $du = \langle u \mid \cdot\rangle$, $dv = \langle \tilde{v} \mid \cdot \rangle$ and $d\tilde{u} = \langle \tilde{u} \mid \cdot\rangle / \langle \tilde{u} \mid \tilde{u}\rangle = i\langle u \mid \cdot\rangle / f$. Then

$$\pi_D^*\omega^\pi - \omega^0 = id\tilde{u} = \frac{i}{f}\, du \in \Omega^1(DE \setminus B; i\mathbb{R})$$

and the form $d\psi(\pi_D^*\omega^\alpha - \omega^0)$ is given by

$$d\psi(\pi_D^*\omega^\xi - \omega^0) = -\frac{\psi'}{f} i du \wedge dv \tag{3.3.2}$$

3.4. *Spin*-Structures.

Recall that a *Spin*-structure $\alpha_X : P_{Spin}(X) \to P_{SO}(X)$ on a manifold X of dimension n canonically induces a $Spin^c$-Dirac structure $(\alpha_X^{\mathbb{C}}, \omega)$ by taking the associated bundle

$$\alpha_X^{\mathbb{C}} : P_{Spin^c}(X) = P_{Spin}(X) \times_{Spin(n)} Spin^c(n) = P_{Spin}(X) \times_{\mathbb{Z}/2} U(1) \to P_{SO}(X)$$

for $\alpha_X^{\mathbb{C}}$ and the trivial connection ω^0 on

$$\xi(\alpha_X^{\mathbb{C}}) = P_{Spin}(X) \times_{Spin(n)} Spin^c(n)/Spin(n) = X \times U(1)$$

for ω. Conversely given a $Spin^c$-Dirac structure (α_X, ω) with trivialized canonical bundle $X \times S^1$ and trivial conection ω we can construct a *Spin*-structure α_X^0 as restriction of the principal *Spin*-bundle $P_{Spin^c}(X) \to P_{U(1)}(\alpha_X^{\mathbb{C}}) = X \times S^1$ to $X = X \times \{1\}$.
If M is a *Spin*-manifold then $w_2(M) = 0$ and from the exactness of the Gysin sequence of π we see that $w_2(B) = 0$ or $w_2(B) = w_2(\pi) = c_1(\pi) \bmod 2$.
The *Spin*-structure α_M on M is equivariant if and only if B is a *Spin*-manifold with *Spin*-structure α_B and $\alpha_M = \pi^*\alpha_B$. Such a *Spin*-structure does not extend to a *Spin*-structure on the associated disc bundle DE. The induced $Spin^c$-Dirac structure $(\alpha_M^{\mathbb{C}}, \omega^0)$ is strictly equivariant.
If α_M is not equivariant then there is a *Spin*-structure α_{DE} on DE with $\alpha_M = \partial\alpha_{DE}$: α_M induces a *Spin*-structure on $\pi^* P_{SO}(B)$ which gives an equivariant $Spin^c$-structure $\alpha_M^{\mathbb{C}}$ if we endow $\xi(\alpha_M^{\mathbb{C}}) : M \times S^1 \to M$ with the diagonal action of S^1 on $M \times S^1$. The canonical $U(1)$-bundle of the quotient $Spin^c$-structure α_B on B of $\alpha_M^{\mathbb{C}}$ is $\xi(\alpha_B) = \xi(\alpha_M^{\mathbb{C}})/S^1 = -\pi$ i.e. π with the $U(1)$-action reversed. Therfore α_B induces a $Spin^c$-structure α_{DE} on DE with $\xi(\alpha_{DE}) = \pi \otimes (-\pi)$ trivial. The desired *Spin*-structure is α_{DE}^0. Given a connection ω^π on π we also get an induced $Spin^c$-Dirac structure (α_B, ω_B) on B with $\xi(\alpha_B) = -\pi$ and connection $\omega_B = -\omega^\pi$. In this case DE is a *Spin*-manifold and we must have $w_2(B) = c_1(\pi) \bmod 2$.
Thus the set $Spin(M)$ of isomorphism classes of *Spin*-structures on M is given by

$w_2(B)$	$w_2(\pi)$	$Spin(M)$
0	0	$\pi^* Spin(B) \,\dot\cup\, \partial Spin(DE)$
0	$\neq 0$	$\pi^* Spin(B)$
$w_2(B) =$	$w_2(\pi) \neq 0$	$\partial Spin(DE)$

This can also be seen from the Gysin sequence of π because $H^1(X;\mathbb{Z}/2)$ acts transitively and effectively on the set $Spin(X)$ of isomorphism classes of *Spin*-structures on a *Spin*-manifold X. So $|H^1(X;\mathbb{Z}/2)| = |Spin(X)|$. The table above follows by counting $H^1(M;\mathbb{Z}/2)$ and comparing with $|H^1(B;\mathbb{Z}/2)| = |H^1(DE;\mathbb{Z}/2)|$ in the three cases. The sets $\pi^* Spin(B)$ and $\partial Spin(DE)$ are disjoint because the restrictions to a neighbourhood of a fibre which has the form $U \times S^1$ for some contractible $U \subset B$ give the two different *Spin*-structures of S^1.

2

The Adiabatic Limit of η-Invariants

4. Statement of Results

For the statement of the formula for the η-invariant of S^1-manifolds we will adopt the following conventions:

(1) (M, g_M) is a Riemannian manifold of odd dimension $2k+1$ with free S^1-action and a metric g_M which is invariant under this action. Furthermore the orbits of the S^1-action are geodesic in (M, g_M). Then on the orbit manifold $B = M/S^1$ there is a (unique) metric g_B such that the quotient map π becomes a Riemannian submersion with totally geodesic fibres. The connection ω^π on π is as in section 3.2 i.e. ω^π is the vertical projection with respect to g_M.

(2) $\pi_D : DE = M \times_{S^1} D^2 \to B$ is the associated disc bundle.

(3) M is identified with the boundary of DE. g_{DE} is a metric on DE such that $M \times I_\epsilon$ with the product metric $g_M \oplus dt^2$ is isometric to a collar neighbourhood of (DE, g_{DE}) for some $\epsilon > 0$. By ω^0 we denote the trivial connection on $\pi^*\pi$ over this collar-neighbourhood.

(4) g^τ_{DE} and g^τ_M are the canonical variations of the metrics g_{DE} and g_M (see section 2.5.2).

(5) $e = c_1(\pi) \in H^2(B; \mathbb{Z})$ is the Euler class (first Chern class) of the principal S^1-bundle $\pi : M \to B$.

(6) Let S_e be the bilinear form on $H^{k-1}(B; \mathbb{Z})$ given by

$$(x, y) \longmapsto \langle x \cup y \cup e \mid [B] \rangle$$

If k is odd then S_e is symmetric and we define $\operatorname{sign}(S_e)$ to be its signature. For even k we set $\operatorname{sign}(S_e) = 0$. Note that $\operatorname{sign}(S_e)$ is the signature of (DE, M).

(7) $S(g^\tau_M)$ is the signature operator on (M, g^τ_M) as in section 2.4

(8) K is a multiplicative sequence with characteristic power series k (see [**Hi**]). The examples we need here are
   (a) $K = L$, $k(x) = x/\tanh(x)$ and
   (b) $K = \hat{A}$, $k(x) = x/(2\sinh(x/2))$.

(9) $p(g^\tau_{DE})$ is the (total) Pontrjagin form given by the Levi-Civita connection of the metric g^τ_{DE} on DE.

4.1. The Integral in The Atiyah-Patodi-Singer Index Formula.

THEOREM 4.1.1. *The limit under canonical variation of integrals of the type as in the Atiyah-Patodi-Singer index formula is given by*

$$\lim_{\tau\to 0}\int_{DE} K(p(g^{\tau}_{DE}))\, f(c_1(\omega))\, \pi^*_D\beta = \langle K(p(TB))\beta\frac{(k(c_1(\pi))f(c_1(\pi))-1)}{c_1(\pi)} \mid [B]\rangle.$$

where

(1) f *is an arbitrary power-series in one variable starting with* 1.
(2) ω *is a connection on* $\pi^*_D\pi$ *which extends* ω^0. *Its first Chern form is* $c_1(\omega)\in\Omega^2(DE;\mathbb{R})$.
(3) $\beta\in\Omega^*(B;\mathbb{R})$ *is arbitrary.*

COROLLARY 4.1.2. *Under the canonical variation the* η*-invariant of the signature operator tends to*

$$\lim_{\tau\to 0}\eta(S(g^{\tau}_M))) = \langle L(p(TB))\left(\frac{1}{\tanh(c_1(\pi))}-\frac{1}{c_1(\pi)}\right) \mid [B]\rangle - sign(S_e).$$

4.2. Dirac Operators.

From now on we additionally assume:

(1) s_M, s_B are the scalar curvatures of g_M, g_B respectively.
(2) (α_M,ω_M) is a strictly equivariant or a boundary *Spin*^{c}-Dirac structure on M. Ω_M is the curvature form of ω_M.
(3) (B,α_B,ω_B) is the induced *Spin*^{c}-Dirac structure on B as in section 3.3. Ω is the curvature of ω_B if ω_M is strictly equivariant and the curvature of $\omega_B\otimes\omega^{\pi}$ if (α_M,ω_M) is an equivariant boundary *Spin*^{c}-Dirac structure. Note that $\Omega_M=\pi^*\Omega$.
(4) ζ is a Hermitian vector bundle over B equipped with a Hermitian covariant derivative ∇^{ζ} with curvature form Ω^{ζ}. ∇^M is the induced connection over M.
(5) ∇^{DE} is a connection on the bundle $\pi^*\zeta$ over DE.
(6) $(\alpha_{DE},\omega_{DE})$ is a *Spin*^{c}-Dirac structure on DE extending (α_M,ω_M) to DE. Over the collar neighbourhood $M\times I_{\epsilon}$ the connection ∇^{DE} and the *Spin*^{c}-Dirac structure $(\alpha_{DE},\omega_{DE})$ are induced from ∇^M and (α_M,ω_M) by the projection $M\times I_{\epsilon}\to M$.
(7) $D_{\pi^*\zeta}$ is the *Spin*^{c}-Dirac operator on $(DE,\alpha_{DE},\omega_{DE})$ twisted with the coefficient bundle $(\pi^*\zeta,\nabla^{DE})$ acting on spinors over DE satisfying the Atiyah-Patodi-Singer boundary conditions (2.4.1). The twisted Dirac operator on M is denoted by $D^M_{\pi^*\zeta}$. It coincides with the tangential operator to $D_{\pi^*\zeta}$ by section 2.2.2.

THEOREM 4.2.1. *If*

$$s_M(x) > 4||\Omega_M\otimes 1 + 1\otimes\pi^*\Omega^{\zeta}||(x) \textit{ for all } x\in M$$

then the index of $D^+_{\pi^*\zeta}$ *vanishes.*

Since $\lim_{\tau\to 0} s_{(M,g^{\tau}_M)} = s_B$ and $s_B\geq s_{(M,g^{\tau}_M)}$ for all τ we therefore have:

COROLLARY 4.2.2. *If*

$$s_B(b) > 4||\Omega\otimes 1 + 1\otimes\Omega^{\zeta}||(b) \textit{ for all } b\in B$$

then

$$\lim_{\tau\to 0} \operatorname{index} D^+_{\pi^*\zeta}(g^\tau_{DE}) = 0.$$

By Theorems 4.2.1 and 4.1.1 the Atiyah-Patodi-Singer index theorem applied to the manifold (DE, M) yields:

THEOREM 4.2.3. *Define $c \in H^2(B;\mathbb{Z})$ by*

$$c = \begin{cases} c_1(\pi) & \text{if } (\alpha_M, \omega_M) \text{ is a boundary } Spin^c\text{-Dirac structure} \\ 0 & \text{if } (\alpha_M, \omega_M) \text{ is a strictly equivariant } Spin^c\text{-Dirac structure} \end{cases}$$

Then

$$\lim_{\tau\to 0} \frac{1}{2}(\eta(D^M_{\pi^*\zeta}, g^\tau_M) + \dim\ker(D^M_{\pi^*\zeta}, g^\tau_M))$$
$$= \langle \hat{A}(B)\, e^{c_1(\alpha_B)/2}\, \mathrm{ch}(\zeta) \left(\frac{e^{c_1(\pi)/2}}{2\sinh(c_1(\pi)/2)} - \frac{e^{c/2}}{c_1(\pi)} \right) \mid [B]\rangle \bmod \mathbb{Z}.$$

If in addition

$$s_B(b) > 4\,||\Omega\otimes 1 + 1\otimes\Omega^\zeta||(b)$$

for all $b \in B$, then

$$\lim_{\tau\to 0} \frac{1}{2}\,\eta(D^M_{\pi^*\zeta}(g^\tau_M)) = \langle \hat{A}(B)\, e^{c_1(\alpha_B)/2}\, \mathrm{ch}(\zeta) \left(\frac{e^{c_1(\pi)/2}}{2\sinh(c_1(\pi)/2)} - \frac{e^{c/2}}{c_1(\pi)} \right) \mid [B]\rangle.$$

The η-invariant of the *Spin*-Dirac operator of a *Spin*-structure α_M on M is the η-invariant of the $Spin^c$-Dirac operator of the associated $Spin^c$-Dirac structure $(\alpha^{\mathbb{C}}_M, \omega_M)$. For the quotient $Spin^c$-structure α_B on B we have that $c_1(\alpha_B) = 0 = c$ if α_M is equivariant and $c_1(\alpha_B) = -c_1(\pi)$ if not. (see section 3.4)

5. Computation of The Integral

The aim of this section is to prove Theorem 4.1.1. The value of the integral in Theorem 4.1.1 does not depend on the extensions g_{DE} and ω of the metric g_M and the connection ω^0 to the interior of DE. Hence we may take a metric as in section 3 for g_{DE}. For this metric let the vector fields u, v be as in 3.2. The tangent bundle $T_F E = \ker d\pi_{\mathbb{C}}$ along the fibres of E is the associated complex line bundle to $\pi^*_{\mathbb{C}}\pi$. The collar neighbourhood $M \times I_\epsilon$ was required to carry the product metric. Therefore $\nabla^{\mathcal{V}}$ is the covariant derivative of the principal connection ω^0 and provides an extension of ω^0 to all of DE which we take for ω. We also have that $\mathcal{H}\nabla^\tau_y v = 0$ for all τ and $y \in TDE$, thus the difference term (2.5.4) vanishes and the curvature of $\nabla^{\mathcal{V}}$ is $\mathcal{V}R^\tau$.
By (2.5.5) we have

$$\lim_{\tau\to 0} R^\tau_{x,y} = \begin{pmatrix} R^{\mathcal{V}} & * \\ 0 & \pi^*_D R^B \end{pmatrix}.$$

The invariant polynomial P defining the Pontrjagin forms has the property that $P\left(\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix}\right) = P(A)P(C)$. Hence the total Pontrjagin form of g^τ_{DE} converges to

$$\lim_{\tau\to 0} p(g^\tau_{DE}) = p(\pi^*_D R^B) \wedge p(R^{\mathcal{V}}) = \pi^*_D p(g_B) \wedge (1 + e(R^{\mathcal{V}})^2).$$

Let $e \in \Omega^2(B;\mathbb{R})$ be an Euler form of π. Then both $\acute{e} := e(R^{\mathcal{V}})$ and $\pi_D^* e$ are Euler forms of connections on the tangent bundle along the fibres $T_F DE$, hence they must be cohomologous. The restriction to the collar $M \times I_\epsilon$ of $\acute{e}$ vanishes because $v|_{M\times I_\epsilon}$ is the derivative in I-direction and $\nabla_x v = 0$ for any vector field x. The integral along the fibres is $\pi_{D!}\acute{e}_\tau = 1$: We have to compute $\int_{D^2} \acute{e}(u,v)$ for an oriented orthonormal basis u, v of $T_F E$. Since the metric on E was required to be product near the boundary, the integral is half the corresponding integral over S^2 which is the Euler characteristic of S^2. Now consider an integral

$$\int_{DE} K(p(g^\tau_{DE}))f(c_1(\omega))\pi_D^*\beta$$

where we may assume that $c_1(\omega) = \acute{e}$. As $\tau \to 0$ the integral converges to

$$\int_{DE} \pi_D^*(K(p(TB))\beta)K(1+\acute{e}^2)f(\acute{e}) = \int_{DE} \pi_D^*(K(p(TB))\omega)(K(1+\acute{e}^2)f(\acute{e})-1)$$

In view of

$$K(1+x^2)f(x)-1 = k(x)f(x)-1 = x + O(x^2)$$

we may split off $\acute{e}$ to get

$$\int_{DE} \pi_D^*(K(p(TB))\beta)\,\frac{(k(\acute{e})f(\acute{e})-1)}{\acute{e}}\,\acute{e}.$$

The form

$$\frac{(k(\acute{e})f(\acute{e})-1)}{\acute{e}}\,\acute{e}$$

is cohomologous to

$$\frac{(k(\pi_D^* e)f(\pi_D^* e)-1)}{\pi_D^* e}\,\acute{e}$$

which still vanishes near the boundary. Because of $\pi_{D!}\acute{e} = 1$ integration over the fibre yields

$$\langle K(p(TB))\beta\frac{(k(e)f(e)-1)}{e} \mid [B]\rangle$$

to establish the formula of the theorem.

6. Vanishing of The Index on Disc Bundles

In this section we are going to prove Theorem 4.2.1. Since the index does not depend on the metric nor on the curvature form in the interior of DE it suffices to construct a specific extension of the *Spin*^{c}-Dirac structure on M to DE of the type as in section 3.2 for which the Hitchin-Lichnerowicz estimate holds.

In the case of boundary *Spin*^{c}-Dirac structures (α_M, ω_M) this is easy, since we can then take connections on $\pi_D^*\zeta$ and $\xi(\alpha_{DE}) = \pi_D^*(\xi(\alpha_B) \otimes \pi$ which are induced from those on the corresponding bundles over B. Hence their curvature forms are $\Omega^{\pi_D^*\zeta} = \pi_D^*\Omega^\zeta$ and $\Omega_{DE} = \pi_D^*\Omega$. Thus the function

$$||\Omega_{DE} \otimes 1 + 1 \otimes \Omega^{\pi_D^*\zeta}||\,(x) = ||\Omega \otimes 1 + 1 \otimes \Omega^\zeta||\,(\pi_D(x))$$

is constant on the fibres of the disc bundle. Since by assumption it is dominated by the scalar curvature of M all we must arrange for is that the fibres of DE have nonnegative

curvature. This can be achieved by using any function f in the construction of section 3.2 with $f'' \leq 0$ and $f = \rho$ outside some nonempty interval $[0, \gamma]$.
A strictly equivariant $Spin^c$-Dirac structure (α_M, ω_M) has an extension $(\alpha_{DE}, \omega_{DE})$ over the disc bundle as in section 3.3. But here we need to extend the connection ω^0 to $\pi_D^*\pi$ induced from the canonical trivialization of $\pi^*\pi$. By (3.3.1) the curvature form of ω_{DE} on $\xi(\alpha_{DE}) = \pi_D^*(\xi(\alpha_B) \otimes \pi)$ is

$$\Omega_{DE} = \pi_D^*\Omega \otimes 1 + 1 \otimes (d\psi(\pi_D^*\omega^\pi - \omega^0) + \psi\pi^*\Omega^\pi).$$

We search functions f and ψ such that

$$4\,||\Omega_{DE} \otimes 1 + 1 \otimes \Omega^{\pi_D^*\zeta}|| \leq -\frac{f''}{f} + s_M,$$

because by (3.2.1) the scalar curvature of DE is estimated by $s_{DE} \geq -f''/f + s_M$. By the triangular inequality for $||\cdot||$ we estimate

$$\begin{aligned} ||\Omega_{DE} \otimes 1 + 1 \otimes \Omega^{\pi^*\zeta}|| \;\; &\leq \;\; ||\pi_D^*\Omega \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \Omega^\zeta|| \\ &\quad + ||1 \otimes (d\psi(\pi_D^*\omega^\pi - \omega^{MC}) + \psi\pi_D^*\Omega^\pi) \otimes 1||. \end{aligned}$$

For the strictly equivariant $Spin^c$-Dirac structure we have $\Omega_M = \pi^*\Omega$. So the assumption of the theorem is that the first term is dominated by the scalar curvature of M:

$$s := \min(s_M - 4\,||\pi_D^*\Omega \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \Omega^\zeta||) > 0.$$

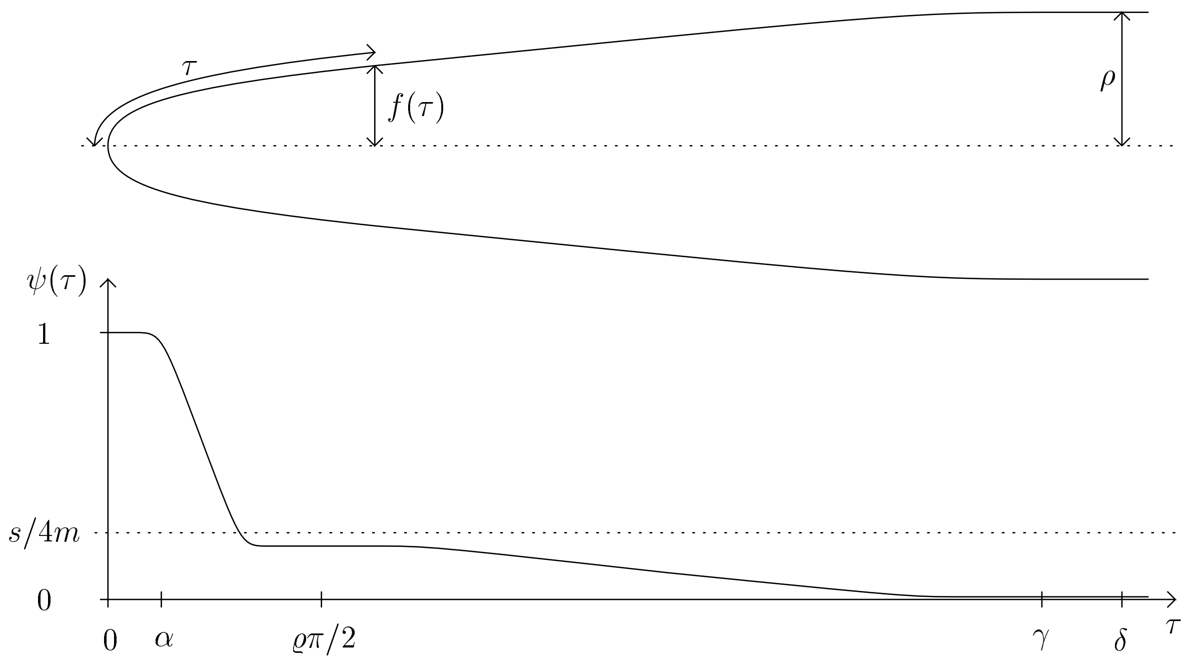
By (3.3.2) and (2.2.4) the second term equals

$$-\psi'/2f + \psi\,||\pi_D^*\Omega^\pi||.$$

Let m be a real number with $m > s/4$ and $m > ||\pi_D^*\Omega^\pi||(b)$ for all $b \in B$. The theorem is proved if we can solve the differential inequality

$$-\frac{f''}{f} + \frac{s}{2} \geq 2\left(-\frac{\psi'}{2f} + \psi m\right) = -\frac{\psi'}{f} + 2\psi m \tag{6.0.1}$$

for functions f and ψ satisfying the conditions of section 3.2 and 3.3.
The metric on the disc we are going to construct is a modification of the "torpedo" metric of [**GL1**] and [**Ga**] looking like the rotation of the following picture around the horizontal axis. The type of functions ψ that will do is plotted below:



Let ρ be the radius of the orbits of S^1 on M. For every ϱ, β with $0 < \varrho < \rho$ and $0 < \beta < \varrho\pi/2$ there is a real number δ and a function $f : \mathbb{R}_0^+ \longrightarrow [0, \rho]$ such that

$$\begin{aligned} f(r) &= \varrho \sin(r/\varrho), \text{ if } r \in [0, \beta], \\ f''(r) &\leq 0 \text{ for all } r, \\ f(r) &\geq \varrho, \text{ if } r \geq \varrho\pi/2, \\ f(r) &\equiv \rho \text{ near } \delta, \text{ i.e for some } \gamma < \delta \text{ we have } f \equiv \rho \text{ on } [\gamma, \delta]. \end{aligned}$$

Such a function f satisfies the conditions of section 3.2 for δ sufficiently large and thus provides a smooth metric on the disc with radius δ. We will show that one can find $\varrho \in]0, \rho]$, $\beta \in]0, \varrho\pi/2[$ and $\alpha \in]0, \beta[$ and a function $\psi : \mathbb{R}_0^+ \longrightarrow [0, 1]$ with $\psi \equiv 1$ on $[0, \alpha]$ and $\psi \equiv 0$ near δ such that (f, ψ) solve (6.0.1). (f, ψ) solve (6.0.1) on $[0, \alpha]$ if $-f''/f = 1/\varrho^2 > 2m$ so we need

$$\text{condition 1: } 2m\varrho^2 < 1.$$

There is the following obvious fact about smooth functions:

Lemma 6.0.2. *Let F be a smooth real function such that $F' \geq 0$ and let $b > a$, $\Psi_b > \Psi_a > 0$ be real numbers with $F(b) - F(a) > \Psi_b - \Psi_a > 0$. Then there is a smooth real function Ψ which is constant near a and near b with $\Psi(b) = \Psi_b$, $\Psi(a) = \Psi_a$ and $0 \leq \Psi' \leq F'$*

Clearly

$$0 \leq -\psi' \leq -f'' - 2mf \tag{6.0.3}$$

implies (6.0.1) on $[0,\beta]$. In view if the lemma we can extend ψ to $[0,\beta]$ such that ψ is constant near β and $\psi(\beta) < s/4m$ and 6.0.3 hold if

$$\text{condition 2: } 1 - s/4m < \int_\alpha^\beta -f'' - 2mf = (1-2m\varrho^2)(\cos(\alpha/\rho) - \cos(\beta/\rho))$$

is fulfilled. If we set $\psi \equiv \psi(\beta)$ on $[\beta, \varrho\pi/2]$ then (f,ψ) solve (6.0.1) on $[0,\varrho\pi/2]$. In order to get a solution on $[0,\delta]$ with $\psi \equiv 0$ near δ for some δ we solve $s/2 \geq -\psi'/\varrho + 2\psi(\beta)m$ on $[\varrho\pi/2, \infty[$ for some extension of ψ which is constant near $\varrho\pi/2$ and δ. Again applying the lemma we need to find δ such that

$$\text{condition 3: } \int_{\varrho\pi/2}^{\delta} s/2 - 2\psi(\varrho\pi/2)m = (s/2 - 2\psi(\varrho\pi/2)m)(\delta - \varrho\pi/2) > \psi(\varrho\pi/2)$$

holds.

Now choose ϱ sufficiently small to achieve that $1 - s/4m < 1 - 2m\varrho^2$. Then condition 1 holds and we can accomplish condition 2 by choosing α sufficiently close to 0 and β close to $\varrho\pi/2$. The values of $\psi(\varrho\pi/2) < s/4m$ and ϱ now being fixed we can take δ sufficiently large to ensure that condition 3 holds.

3

The Homogeneous Case

7. η-Invariants of Some Homogenous Spaces

In this part we will compute the η-invariant of the Atiyah-Patodi-Singer operator $D^+(g_M^\tau)$ on compact homogeneous Riemannian *Spin*-manifolds $M = G/V$ carrying a nontrivial homogeneous S^1-action. The family of Riemannian metrics g_M^τ on M we will deal with is the canonical variation in the direction of this S^1-action of a normal homogeneous metric $g = g^1$ which is not flat. Under these assumptions we get a principal S^1-bundle $\pi : M \to B = G/K$ for some closed subgroup $K \subset G$ with $K \rhd V$ and $K/V = S^1$. We apply the Atiyah-Patodi-Singer index formula to the Dirac operator on the associated disc bundle DE with boundary M.

7.1. Preliminaries.

(see [**Be**], [**CE**], [**KN1**], [**KN2**]) Let G be a connected Lie group and $V \subset G$ a closed subgroup. Also assume that the circle S^1 acts homogeneously on $M = G/V$ i.e. the S^1-action commutes with the action of G. Since G acts transitively on M the isotropy group $I_p \subset S^1$ of a point $p \in M$ does not depend on the point p so we may assume the S^1-action free. The preimage of the orbit of S^1 through the image o of $1 \in G$ under the quotient map $G \to M$ is a closed subgroup $K \subset G$ such that $K \rhd V$ and $S^1 \cong T = K/V$ acts by $(kV, gV) \mapsto gkV$ for $k \in K$, $g \in G$. The Lie algebras of these groups will be denoted by $\mathfrak{g}$, $\mathfrak{v}$, $\mathfrak{k}$, $\mathfrak{t}$.
Normal metrics on M and B are induced from bi-invariant metrics on G. These metrics on G correspond to scalar products on $\mathfrak{g}$ which are invariant under the adjoint action Ad_G of G on $\mathfrak{g}$. Denoting the orthogonal complements of $\mathfrak{v}$ and $\mathfrak{k}$ in $\mathfrak{g}$ by $\mathfrak{m} = \mathfrak{v}^\perp$ and $\mathfrak{b} = \mathfrak{k}^\perp$ and identifying $\mathfrak{t}$ with the orthogonal complement of $\mathfrak{v}$ in $\mathfrak{k}$ we get orthogonal splittings of $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{m} = \mathfrak{v} \oplus \mathfrak{t} \oplus \mathfrak{b} = \mathfrak{k} \oplus \mathfrak{b} \tag{7.1.1}$$

which are invariant under the adjoint actions Ad_V and Ad_K respectively. The tangent bundle of M is associated to the principal V-bundle $G \to M$:

$$TM = G \times_{(V, Ad_V)} \mathfrak{m}. \tag{7.1.2}$$

The curvature of normal metrics can be computed by O'Neill's formulae. By left invariance it suffices to do so at the point $o \in M$. Let $\bar{x}, \bar{y}, \bar{z}, \bar{w} \in \mathfrak{m}$ be the left invariant vector fields corresponding to $x, y, z, w \in T_o M \cong \mathfrak{m}$. The vertical projection of the Riemannian submersion $G \to M$ corresponds to the projection of $\mathfrak{g}$ onto $\mathfrak{v}$. The Riemannian curvature tensor of a normal metric on M is then

$$\langle R_{x,y} z \mid w \rangle = -\frac{1}{4} \langle [\bar{y}, \bar{w}] \mid [\bar{x}, \bar{z}] \rangle + \frac{1}{4} \langle [\bar{x}, \bar{z}] \mid [\bar{y}, \bar{w}] \rangle + S_{\mathfrak{v}}(x, y, z, w)$$

with

$$S_{\mathfrak{v}}(x,y,z,w) = -\frac{1}{4}\langle[\bar{x},\bar{w}]^{\mathfrak{v}} \mid [\bar{y},\bar{z}]^{\mathfrak{v}}\rangle + \frac{1}{4}\langle[\bar{x},\bar{z}]^{\mathfrak{v}} \mid [\bar{y},\bar{w}]^{\mathfrak{v}}\rangle + \frac{1}{2}\langle[\bar{x},\bar{y}]^{\mathfrak{v}} \mid [\bar{z},\bar{w}]^{\mathfrak{v}}\rangle.$$

Especially the sectional curvature of M is

$$K_M(x,y) = \langle R_{x,y}x \mid y\rangle = \frac{1}{4}||[x,y]||^2 + \frac{3}{4}||[\bar{x},\bar{y}]^{\mathfrak{v}}||^2 \geq 0$$

for $x, y \in \mathfrak{m}$. Hence the scalar curvature of M is positive iff there are $x, y \in \mathfrak{m}$ with $[x,y] \neq 0$.

We can always replace G by its universal covering and extend V and K appropriately without changing M or B to achieve that G is simply connected. Then *Spin*-structures on M are given by lifts of the adjoint representation $V \to SO(\mathfrak{m})$ over $Spin(\mathfrak{m}) \to SO(\mathfrak{m})$ (see also [**Bär**]): The differential of the action of G on M gives an action on the orthonormal frame bundle of M. Since G is simply connected this action lifts to an action on the principal *Spin*-bundle of the *Spin*-structure of M. By restricting we get the isotropy representation of $V \to SO(\mathfrak{m})$ and a lift $V \to Spin(\mathfrak{m})$. Conversely given such a lift $V \to Spin(\mathfrak{m})$ we use the V-structure (7.1.2) of the orthonormal frame bundle of M to get an associated *Spin*-structure on M.

The *Spin*-structure on M is S^1-equivariant if and only if the lift $V \to Spin(\mathfrak{m})$ of isotropy representation $V \to SO(\mathfrak{m})$ extends to a lift $K \to Spin(\mathfrak{b})$ of the isotropy representation $K \to SO(\mathfrak{b})$.

7.2. Computation of The η-Invariant.

If the metric on M is normal then the orbits of one-parameter subgroups of G are geodesic, thus $M \to B$ has totally geodesic fibres. Let E be the associated complex line bundle of the principal S^1-bundle $\pi : M = G/V \to B = G/K$ and consider $M = G/V$ as the boundary of the disc bundle DE. By assumption M has positive scalar curvature and Theorem 4.2.1 shows that the index of the *Spin*-Dirac operator on DE vanishes. So it remains to compute the integral in the Atiyah-Patodi-Singer index formula.

We will do so for the metric g_{DE} and the $Spin^c$-Dirac structure on the disc bundle DE of the type considered in section 3. The vector field u of section 3 corresponds to the generator of $\mathfrak{t}$ in (7.1.1). As before $v \in TDE$ is the radial derivative. Horizontal vectors $a \in TDE$ correspond to left invariant vector fields $a \in \mathfrak{b}$. We will explicitly make use of the formulae in section 3 for the covariant derivatives on the disc: $\nabla_v v = \nabla_v u = 0$, $\nabla_u v = \frac{f'}{f}\, u$, $\nabla_u u = -\frac{f'}{f}\, v$ and $\nabla_a v = 0$.

The A-tensor of the submersion $M \to B$ is

$$\langle A_a b \mid u\rangle = \frac{1}{2}\,\langle[\bar{a},\bar{b}] \mid \bar{u}\rangle$$

for horizontal vectors $a, b \in TM$. For the Riemannian curvature tensor on DE at a point with distance r from the zero section $B \subset DE$ we get

$$\begin{aligned}
\langle R_{u,v}u \mid v\rangle &= -\frac{f''(r)}{f(r)},\\
\langle R_{a,u}u \mid v\rangle &= 0,\\
\langle R_{a,v}u \mid v\rangle &= 0,\\
\langle R_{a,v}b \mid v\rangle &= 0,\\
\langle R_{a,b}v \mid u\rangle &= \langle \nabla_{[a,b]}v \mid u\rangle = 2f'(r)\langle A_a b \mid u\rangle^M,\\
\langle R_{a,b}c \mid v\rangle &= 0,\\
\langle R_{a,u}a \mid u\rangle &= \langle A_a u \mid A_a u\rangle = f(r)^2\langle A_a u \mid A_a u\rangle^M,\\
\langle R_{a,v}b \mid u\rangle &= \langle \nabla_v \nabla_a b \mid u\rangle = v\langle \nabla_a b \mid u\rangle = f'(r)\langle A_a b \mid u\rangle^M,\\
\langle R_{a,b}c \mid u\rangle &= f(r)\langle R_{a,b}c \mid u\rangle^M,\\
\langle R_{a,b}c \mid h\rangle &= \langle R^B_{a,b}c \mid h\rangle - 2\langle A_a b \mid A_c h\rangle + \langle A_a h \mid A_b c\rangle - \langle A_a c \mid A_b h\rangle,\\
&= \langle R^B_{a,b}c \mid h\rangle + f(r)^2(-2\langle A_a b \mid A_c h\rangle^M + \langle A_a h \mid A_b c\rangle^M - \langle A_a c \mid A_b h\rangle^M).
\end{aligned}$$

The exponent in $\langle \cdot \mid \cdot\rangle^M$ is to indicate that this term is computed in M with its normal metric $g = g^1$. We need to extend the *Spin*-structure α_M on M to a *Spin*^{c}-Dirac structure $(\alpha_{DE}, \omega_{DE})$ on the disc bundle DE and compute the integral over DE of the characteristic form $e^{c_1(\omega_{DE})/2}\hat{A}(p(g_{DE}))$. If α_M is not equivariant then we can extend it to a *Spin*-structure to get $c_1(\omega_{DE}) = 0$ (see section 3.4). If α_M is equivariant then we let ω_{DE} be the connection on $\pi_D^*\pi$ with covariant derivative $\nabla^{\mathcal{V}}$ of (2.5.3). As in section 4.1.1 the curvature of $\nabla^{\mathcal{V}}$ is $R^{\mathcal{V}} = \mathcal{V}R$ and is therefore given by the above formulae for the Riemannian curvature tensor R on DE. Hence for the first Chern form we obtain:

$$c_1(\omega_{DE}) = \frac{i\Omega_{DE}}{2\pi} = -\frac{\langle Ru \mid v\rangle}{4\pi} = \frac{1}{2\pi}\left(\frac{f''}{f}du \wedge dv + f'\langle A \mid u\rangle^M\right). \tag{7.2.1}$$

The characteristic form $e^{c_1(\omega_{DE})/2}\hat{A}(p(g_{DE}))$ at a point $x \in DE$ with distance r from the zero section can then be written as

$$e^{c_1(\omega_{DE})/2}\hat{A}(p(g_{DE}))(x) = P(f(r), f'(r), \frac{f''(r)}{f(r)})\ \mathrm{vol}(DE, g_{DE}),$$

where P is a polynomial whose coefficients are polynomials in the entries of A and R^M, R^B, the latter being given by the formulae in 7.1. The volume form $\mathrm{vol}(DE, g_{DE})$ is $\mathrm{vol}(DE, g_{DE}) = \pi^*\,\mathrm{vol}(B) \wedge du \wedge dv$ and we finally get

$$\begin{aligned}
\frac{1}{2}\eta(M) &= \int_{DE} e^{c_1(\omega_{DE})/2}\hat{A}(p(g_{DE})) = \int_{DE} P(f(r), f'(r), \frac{f''(r)}{f(r)})\,\mathrm{vol}(DE, g_{DE})\\
&= \mathrm{vol}(B)\int_0^\delta P(f(r), f'(r), \frac{f''(r)}{f(r)})\,2\pi f(r)\,dr
\end{aligned}$$

Now take any function f satisfying the conditions of section 3.2 and compute this integral.

8. An Example: The Wallach Spaces

As an example look at the Wallach spaces $SU(3)/\Delta_{k,l}U(1)$ for coprime integers k and l, where the embedding $\Delta_{k,l} : U(1) \to SU(3)$ is given by

$$z \longmapsto \begin{pmatrix} z^k & 0 & 0 \\ 0 & z^l & 0 \\ 0 & 0 & z^{-k-l} \end{pmatrix}.$$

These spaces are S^1-bundles over the flag manifold $B = SU(3)/K$ where $K = T^2$ is a maximal torus in $SU(3)$ and the S^1-action is the action of $S^1 \cong K/\Delta_{k,l}U(1)$ as in section 7. Since M is simply-connected there is a unique *Spin*-structure on M and this must be equivariant for the S^1-action because B is *Spin*. Therefore $c = c_1(\omega_{DE})$ is given by (7.2.1). The normal metric $g = g^1$ on M is induced from the Cartan-Killing form on $\mathfrak{su}(3)$ which for 3×3 matrices $A, B \in \mathfrak{su}(3)$ is defined as

$$\langle A \mid B \rangle = -1/2 \operatorname{trace} AB.$$

The radius ρ_τ of the orbits of S^1 on (M, g^τ_M) is $\rho_\tau = \tau\sqrt{3}/2\sqrt{k^2+kl+l^2}$. An explicit computation on a computer gave:

$$\begin{aligned}
\int_{DE} p_1(g_{DE})^2 &= 3\,k\,l\,(k+l)\,(16 - 412\,{\rho_\tau}^4 + 340\,{\rho_\tau}^6 - 63\,{\rho_\tau}^8)/16, \\
\int_{DE} p_1(g_{DE})c_1(\omega_{DE})^2 &= 3\,k\,l\,(k+l), \\
\int_{DE} c_1(\omega_{DE})^4 &= 3\,k\,l\,(k+l), \\
\int_{DE} p_2(g_{DE}) &= 27\,k\,l\,(k+l)\,{\rho_\tau}^4\,(-20 + 21\,{\rho_\tau}^2 - 4\,{\rho_\tau}^4)/32.
\end{aligned}$$

The η-invariant of $(M_{k,l}, g^\tau_M)$ for $\tau \le 1$ is therefore

$$\begin{aligned}
\eta(D^+(g^\tau_M), k, l) &= 2\int_{DE} e^{c_1(\omega_{DE})/2}\hat{A}(p_2(g_{DE}), p_1(g_{DE})) \\
&= k\,l\,(k+l)\,(-128 - 2524\,{\rho_\tau}^4 + 2002\,{\rho_\tau}^6 - 369\,{\rho_\tau}^8)/15360.
\end{aligned}$$

Bibliography

[ABS] M. F. Atiyah, R. Bott and A. Shapiro, Clifford modules, Topology 3 (Suppl. 1), 1964, 3–38

[ABP] M. F. Atiyah, R. Bott and V. K. Patodi, On the heat equation and the index theorem, Invent. Math. 19, 1973, 297–330

[AH] M. F. Atiyah and F. Hirzebruch, Spin manifolds and group actions, Essays in topology and related topics, Proceedings, Springer, 1970, 18–28

[APS1] M. F. Atiyah, V. K. Patodi and I. Singer, Spectral asymmetry and Riemannian geometry I, Math. Proc. Camb. Philos. Soc. 77, 1975, 43–69

[APS2] M. F. Atiyah, V. K. Patodi and I. Singer, Spectral asymmetry and Riemannian geometry II, Math. Proc. Camb. Philos. Soc. 78, 1975, 405–432

[AS3] M. F. Atiyah and I. Singer, The index of elliptic operators III, Ann. of Math. 87, 1968, 546–604

[AW] S. Aloff and N. Wallach, An infinite family of distinct 7-manifolds admitting positively curved Riemannian structures, Bull. Amer. Math. Soc. 81, 1975, 93–97

[Bär] C. Bär, Das Spektrum von Dirac-Operatoren, Doktorarbeit, Bonn, 1990

[Ber] F. Bermbach, On simply-connected 7-manifolds, Doktorarbeit, Joh. Gutenberg-Universität Mainz 1991

[Be] A. L. Besse, Einstein manifolds, Springer Verlag, Berlin and New York, 1986

[BC] J.-M. Bismut and J. Cheeger, η-invariants and their adiabatic limits, J. Amer. Math. Soc. 2, 1989, 33–70

[CE] J. Cheeger, D. Ebin, Comparison Theorems in Riemannian Geometry, North Holland, Amsterdam, 1975

[Dai] X. Dai, Adiabatic limits, nonmultiplicativity of signature, and Leray spectral sequence, J. Amer. Math. Soc., ,

[Don] H. Donnelly, Spectral Geometry and Invariants from differential Topology, Bull. London Math. Soc. 7, 1975, 147–150

[EK] J. Eells, N.H.Kuiper, An invariant for certain smooth manifolds, Ann. Mat. Pura Appl. 60, 1962, 93-110

[Ga] P. Gaier, Riemannian metrics of positive scalar curvature on compact manifolds with boundary, Ann. Global Anal. Geom. 5 (1987), 179-191

[GL1] M. Gromov and H. B. Lawson, Jr., The classification of simply-connected manifolds of positive scalar curvature, Ann. of Math. 111, (1980), 423-434

[GL2] M. Gromov and H. B. Lawson, Jr., Positive scalar curvature and the Dirac operator on complete manifolds, Publ. Math. I.H.E.S. no. 58 (1983), 83-196

[Her] R. Hermann, A sufficient condition that a mapping of Riemannaian manifolds be a fibre bundle, Proc. A. M. S. 11 (1960), 236–242

[Hi] F. Hirzebruch, Neue topologische Methoden in der algebraischen Geometrie, Grundlehren der mathematischen Wissenschaften Bd. 131, Springer, 1966

[Hit] N. Hitchin, Harmonic spinors, Adv. Math. 14, 1974, 1–55

[KN1] S. Kobayashi, K.Nomizu, Foundations of Differential Geometry Vol. 1., Interscience Publishers, New York, 1963

[KN2] S. Kobayashi, K.Nomizu, Foundations of Differential Geometry Vol. 2., Interscience Publishers, New York, 1969

[KS1] M. Kreck and S. Stolz, A diffeomorphism classification of 7-dimensional homogeneous Einstein manifolds with $SU(3) \times SU(2) \times U(1)$-symmetry, Ann. of Math. (2) 127, 1988, 373–388

[KS2] M. Kreck and S. Stolz, Some nondiffeomorphic homeomorphic homogeneous 7-manifolds with positive sectional curvature, J. Diff. Geom. 33, 1991, 465–486

[KS3] M. Kreck and S. Stolz, Nonconncected Moduli Spaces of Positive Sectional Curvature Metrics, preprint of the Max-Planck-Institut für Mathematik, Bonn, no. 92-15

[LM] H. B. Lawson and M.-L. Michelson, Spin Geometry, Princeton University Press, Princeton, 1989

[Li] A. Lichnerowicz, Spineurs harmoniques, C. R. Acad. Sci. Paris, Sér. A-B 257, 1963, 7–9

[ML] E. Y. Miller and R. Lee, Some Invariants of Spin Manifolds, Toplogy and its applications 25, 1987, 301–311

[MS] J. W. Milnor and J. D. Stasheff Characteristic classes, Princeton University Press, Annals of Mathematics Studies No. 76, 1974

[Oc] S. Ochanine, Signature modulo 16, invariants de Kervaire généralisé et nombres caractéristiques dans la K-théorie reélle, Bull. SMF 5, 1981

[O'N] B. O'Neill, The fundamental equations of a submersion, Mich. Math. J. 13, 459–469, 1966

[SS] J. Seade and B. Steer, a note on the Eta function for quotients of $PSL_2(\mathbb{R})$ by co-compact Fuchsian groups, Topology 26, 1987, 79–91

[Vi] J.Vilms, Totally geodesic maps, J. of Diff. Geom. 4 (1970), 73-79

[Wi] E. Witten, Search for a realistic Kaluza-Klein theory, Nucl. Physics B 186, 1981, 412–428

[Zh] W. Zhang, η-invariants and Rokhlin congruences, C. R. Acad. Sci. Paris A 315, 1992, 305–308