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The growth of bounded and related functions.

by W. K. Hayman

The growth of bounded and related functions in the unit disk

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Abstract: In this talk an answer supplied to the following question of Rozenblum: "How fast can a bounded analytic function in the unit disk tend to zero outside an appropriate exceptional set"? The answer is obtained first for Blaschke products and is then extended to the Nevanlinna class of functions of bounded characteristic, i.e meromorphic functions which are the ratio of two bounded functions, by the representation of such functions as the ratio of two Blaschke products multiplied by an exponential term. For Blaschke products $B(z)$ it is shown that

$$(1-|z|)\log|B(z)| \to 0 \quad (*)$$

as $|z| \to 1$ for z outside an exceptional F-set. This is defined in the introduction as the union of a suitable set of disks lying in the unit disk. It is also shown that these results are essentially sharp.

MSC 30 A 76

1 Introduction

This paper answers a question of Grigori Rozenblum [9].
"Let $f(z)$ be a bounded analytic function in the unit disk. Can we remove from the interval (0,1) some small set containing zeros or places close to zeros, so that on the remaining part of the interval f cannot tend to zero, as $x \to 1$, faster than exponentially?"
There are two basic questions.

1. What is the behaviour outside an exceptional set?

2. How small is the exceptional set?

For some earlier answers to these questions see Cargo [3], Tanaka [11] and the survey by Eiderman and Essén [5].
In this paper we give answers to these questions for Blaschke products and then deduce corresponding results for functions of bounded characteristic.

Definitions. Let $\Delta(z_0, \rho)$ denote the disk $|z - z_0| < \rho$. We write

$$r_k = 1 - 2^{-k}, k = 0, 1, 2, \ldots \tag{1.1}$$

and consider a class $\mathcal{F}$ of disks $\Delta(z_0, \rho)$ contained in $\Delta = \Delta(0,1)$, and the subclass $\mathcal{F}_k$ of those disks of $\mathcal{F}$, whose centres lie in the annulus

$$r_{k-1} \leqslant |z| < r_k, k \geqslant 1. \tag{1.2}$$

Let ρ_k be the sum of the radii of the disks in $\mathcal{F}_k$. A set of points F_0 will be called an $F-set$ if F_0 is contained in a union of disks $\mathcal{F}$, where $\mathcal{F}$ lies in $r_{k_0} \leqslant |z| < 1$ for some k_0 in $\mathbb{N}$ and is such that $\rho_k < 2^{1-k}$ for $k \geqslant k_0$ and

$$\sum_{k=k_0}^{\infty} \{\log(2^{1-k}/\rho_k)\}^{-1} < \infty. \tag{1.3}$$

Let a_μ be a sequence of points in $\Delta\backslash\{0\}$ such that

$$\sum_{\mu=1}^{\infty} (1 - |a_\mu|) < \infty, \tag{1.4}$$

and suppose that $\lambda \in \mathbb{R}, p \in \{\mathbb{N}\} \cup \{0\}$ Then

$$B(z) = e^{i\lambda} z^p \prod_{\mu=1}^{\infty} \left\{ \frac{|a_\mu|}{a_\mu} \frac{a_\mu - z}{1 - \bar{a}_\mu z} \right\} \tag{1.5}$$

is a Blaschke product.

We can now state our results.

Theorem 1 *If $B(z)$ is a Blaschke product then there exists an F-set F_0 such that*

$$(1 - |z|) \log |B(z)| \to 0 \tag{1.6}$$

as $|z| \to 1-$ with $z \notin F_0$.

Conversely we have

Theorem 2 *Let ρ_k be a sequence of positive numbers satisfying (1.3) with $k_0 = 1$ and let z_k be a sequence lying for each k in the annulus (1.2). Further suppose that $0 < \varepsilon(r) \leqslant \varepsilon_0$ for $0 \leqslant r < 1$ and that*

$$\varepsilon(r) \to 0 \text{ as } r \to 1.$$

Then there exists a Blaschke product $B(z)$ such that

$$(1 - |z|) \log |B(z)| \to -\infty \tag{1.7}$$

as $|z| \to 1-$, with z in the union of the disks $\Delta(z_k, \rho_k)$ and

$$(1 - r) \log |B(r)| < -\varepsilon(r), \; 0 \leqslant r < 1. \tag{1.8}$$

Since $\log x < x$, for $x > 1$, an immediate consequence of (1.3) is that

$$\sum \rho'_\nu / (1 - |z'_\nu|) < \infty, \tag{1.9}$$

where $\Delta(z'_\nu, \rho'_\nu)$ are the individual disks of $\mathcal{F}$; equivalently the sum of the hyperbolic radii of the disks $\Delta(z'_\nu, \rho'_\nu)$ converges. However(1.3) is significantly stronger than (1.9).

2 Consequences

We assume from now on that our functions $f(z)$ are not constant. Suppose that $f(z)$ is a meromorphic function of bounded characteristic (b.c.) i.e. the ratio of two functions bounded in Δ. Then [6, Theorem 6.13,p.179]

$$f(z) = \frac{\Pi_1(z)}{\Pi_2(z)} \exp \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta) + iD \right\} \tag{2.1}$$

for $z \in \Delta$; $\mu(\theta)$ has bounded variation, D is a real constant, and Π_1, Π_2 are Blaschke products.

Theorem 3 *If f has b.c. then there exists an F-set F_0 such that, if $\zeta = e^{i\Theta}$, then*

$$\frac{|\zeta - z|^2}{1 - |z|^2} \log |f(z)| \to \alpha(\Theta), \tag{2.2}$$

as $z \to \zeta$ nontangentially with $z \notin F_0$. Here $\alpha(\Theta) = 0$ outside a countable set $\{\Theta_\nu\}$ and

$$\sum |\alpha(\Theta_\nu)| < \infty. \tag{2.3}$$

Corollary *If*

$$M(r) = M(r, f) = \sup_{|z|=r} |f(z)| \tag{2.4}$$

then

$$(1-r)\ \log\ M(r) \to 2 \max_{\Theta} \alpha(\Theta), \tag{2.5}$$

as $r \to 1$ *through real values outside an F-set.*

Some of the results in Theorem 3 extend to more general classes of functions. For instance Rippon [8] has proved corresponding results for functions of locally bounded characteristic (l.b.c.) (For a definition see e.g. Hayman and Korenblum [7]). In this case the exceptional set F_0 may depend on the boundary point ζ. Rippon's result extends to l.b.c. subharmonic functions in a ball in R^n; it is still true that $\alpha(\zeta)$ always exists and is zero outside a countable set.

We remark that from(1.9) and in particular from (1.3) we deduce that, for every $\zeta = e^{i\theta}$ and all $\emptyset$ in $(-\pi/2, \pi/2)$ apart from a set of measure zero, (2.2) implies that, if $z = \zeta(1 - te^{i\emptyset})$ and $t \to 0$ for fixed θ and $\emptyset$,

$$\begin{aligned} (1-|z|^2)\log|f(z)| &= \frac{(1-|z|^2)^2|\zeta - z|^2}{|\zeta - z|^2(1-|z|^2)} \log|f(z)| \\ &= \frac{(2t\cos\emptyset - t^2)^2|\zeta - z|^2}{t^2(1-|z|^2)} \log|f(z)| \to 4(\cos^2\emptyset)\alpha(\theta). \end{aligned} \tag{2.6}$$

In particular, if

$$0 < |\alpha(\Theta)| < \infty, \tag{2.7}$$

then

$$u(z) = (1-|z|^2)\log|f(z)|$$

has infinitely many distinct asymptotic values at Θ. It now follows from Bagemihl's 'Ambiguous Points Theorem' (Bagemihl [1], see also Collingwood and Lohwater [4,p.83]) that the set of θ, for which $\alpha(\theta)$ exists and satisfies (2.7), is necessarily countable. Here f can be an arbitrary function from Δ to $\mathcal{C} \cup \infty$. If $f = 0$ or ∞ we write $u = 0$. This does not affect (2.6) if (2.7) holds.

3 Proof of Theorem 1. Preliminaries.

In (1.5) let $n(t)$ be the number of zeros a_μ in $0 < |z| \leqslant t$. Then by (1.4)

$$\sum_1^\infty (1 - |a_\mu|) = \int_0^1 (1-t)dn(t) = \int_0^1 n(t)dt < \infty. \tag{3.1}$$

We recall (1.1) and define

$$N_k = n(r_k).$$

Then

$$N_k 2^{-(k+1)} \leqslant \int_{r_k}^{r_{k+1}} n(t)dt.$$

So by (3.1)

$$\sum_{k=1}^{\infty} 2^{-k} N_k \leqslant 2 \int_{\frac{1}{2}}^{1} n(t)dt < \infty. \tag{3.2}$$

Suppose now that $k \geqslant 3$ and that

$$r_{k-1} \leqslant |z| < r_k. \tag{3.3}$$

Then

$$\begin{aligned} \log |z^p/B(z)| &= \sum_{\mu=1}^{\infty} \log |(1-\bar{a}_\mu z)/(z-a_\mu)| \\ &= \sum_{|a_\mu| \leqslant r_{k-2}} + \sum_{r_{k-2} < |a_\mu| \leqslant r_{k+1}} + \sum_{|a_\mu| > r_{k+1}} \\ &= \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3 . \end{aligned} \tag{3.4}$$

We note that

$$\left| \frac{1-\bar{a}z}{z-a} \right|^2 = 1 + \frac{|1-\bar{a}z|^2 - |z-a|^2}{|z-a|^2} = 1 + \frac{(1-|a|^2)(1-|z|^2)}{|z-a|^2}. \tag{3.5}$$

In $\sum_1$ we have for $a = a_\mu$

$$|a-z| \geqslant r_{k-1} - r_{k-2} \geqslant 1 - |z|,$$

and

$$|a-z| \geqslant \frac{1}{2}(1-|a|).$$

So

$$\frac{(1-|a|^2)(1-|z|^2)}{|z-a|^2} \leqslant 4 \frac{(1-|a|)(1-|z|)}{|z-a||z-a|} \leqslant 8. \tag{3.6}$$

Hence

$$\sum\nolimits_1 \leqslant \sum\nolimits_1 \frac{1}{2} \log 9 \leqslant 2N_{k-2} = o(2^k) = o(1/(1-|z|)) \tag{3.7}$$

by (3.2). Next in $\sum_3$ with $a = a_\mu$ we have

$$|z-a| \geqslant \frac{1}{2}(1-|z|), |z-a| \geqslant 1-|a|.$$

So by (3.5) and (3.6)

$$\begin{aligned} \log \left| \frac{1-\bar{a}z}{z-a} \right| &= \frac{1}{2} \log \left(1 + \frac{(1-|a|^2)(1-|z|^2)}{|z-a|^2} \right) \\ &\leqslant \frac{(1-|a|^2)(1-|z|^2)}{2|z-a|^2} \leqslant \frac{8(1-|a|)}{1-|z|} < 2^{3+k}(1-|a|). \end{aligned}$$

So by (1.4)

$$\sum\nolimits_3 < 2^{3+k} \sum\nolimits_3 (1-|a_\mu|) = o(2^k) = \frac{o(1)}{1-|z|}. \tag{3.8}$$

4 Completion of proof of Theorem 1.

To estimate $\sum_2$ in (3.4) we need two lemmas.

Lemma 1 *The Boutroux-Cartan Lemma. Let*

$$P(z) = \prod_{\nu=1}^{N} (z - a_\nu)$$

be a monic polynomial of degree N. Then if $C > 0$ we have

$$|P(z)| > C^N$$

outside a set of at most N disks, containing the zeros of P, and the sum of whose radii is at most $2eC$.

For a proof see Boas [2,p.46].

Lemma 2 *If $t_\nu, \nu = 1$ to n, are nonnegative with*

$$\sum_1^n t_\nu = T,$$

and $ø(t)$ is continuous in [0,T] with $ø(0) = 0$, and further $ø'(t)$ is continuous and nondecreasing in (0,T), then

$$\sum_{\nu=1}^{n} ø(t_\nu) \leqslant ø(T).$$

The result is trivial if $n = 1$. We now use induction on n and suppose that Lemma 2 holds for $n-1$. Since $ø'(t)$ is continuous and nondecreasing in (0,T) and $ø(t)$ is continuous in [0,T], we have, writing

$$T_{n-1} = \sum_{\nu=1}^{n-1} t_\nu,$$

$$ø(T_{n-1} + t_n) - ø(T_{n-1}) - ø(t_n) + ø(0) = \int_0^{t_n} \{ø'(T_{n-1} + x) - ø'(x)\} dx \geqslant 0.$$

Since $ø(0) = 0$, this completes the inductive step and Lemma 2 is proved.

We now recall (3.2) and, for $k \geqslant 0$, choose ε_k so that

$$0 < \varepsilon_k < 1, \quad \varepsilon_k \to 0 \quad \text{as} \quad k \to \infty, \tag{4.1}$$

and

$$\sum_{k=1}^{\infty} N_{k+1} 2^{-k} / \varepsilon_k < \infty. \tag{4.2}$$

Next we recall (3.4) and divide the annulus

$$r_{k-2} \leqslant |\zeta| < r_{k+1}$$

into sectors

$$B_{k,\nu} : \quad (\nu - 1)2^{-k} \leqslant \quad (\arg \zeta)/(2\pi) < \nu 2^{-k}, \tag{4.3}$$

so that

$$B_{k,\nu+2^k} = B_{k,\nu} \quad , \quad -\infty < \nu < +\infty.$$

If $z \in B_{k,\nu}$ and satisfies (3.3) we denote by $B'_{k,\nu}$ the set

$$\bigcup_{p=\nu-1}^{\nu+1} B_{k,p}.$$

Let $\sum_2'$ denote the sum over all zeros a_μ in $B'_{k,\nu}$ and let $\sum_2''$ denote the sum taken over the remaining a_μ in $\sum_2$. In $\sum_2''$ we have $z = \rho e^{i\Theta}$, $a_\mu = \sigma e^{i\varnothing}$, where

$$2\pi 2^{-k} \leqslant |\Theta - \varnothing| \leqslant \pi, \qquad \rho \geqslant \frac{1}{2}, \qquad \sigma \geqslant \frac{1}{2},$$

since $k \geqslant 3$ in (3.3) and so

$$|z - a_\mu| = |\sigma e^{i(\varnothing - \Theta)} - \rho| \geqslant \frac{1}{2}\sin(\pi 2^{1-k}) \geqslant 2^{1-k}.$$

It now follows from (3.5) that $a = a_\mu$ satisfies

$$\left|\frac{1 - \bar{a}z}{z - a}\right|^2 \leqslant 1 + \frac{4(1 - |a|)(1 - |z|)}{|z - a|^2} \leqslant 1 + \frac{4 \times 2^{3-2k}}{2^{2-2k}} = 9.$$

Thus

$$\sum\nolimits_2'' \log|(1 - \bar{a}_\mu z)/(z - a_\mu)| \leqslant N_{k+1} \log 3 = o(2^k) \tag{4.4}$$

by (4.2).

Next in $\sum_2'$ we have by (3.5), with $a = a_\mu$,

$$\left|\frac{1 - \bar{a}z}{z - a}\right|^2 = 1 + \frac{(1 - |a|^2)(1 - |z|^2)}{|z - a|^2},$$

and by (3.3), (4.3)

$$|z - a| \leqslant ||z| - |a|| + |\arg z - \arg a| \leqslant (4\pi + 3)2^{-k},$$

so that

$$\frac{(1 - |a|^2)(1 - |z|^2)}{|z - a|^2} \geqslant \frac{(1 - |a|)(1 - |z|)}{|z - a|^2} \geqslant 2^{-1-2k}/\{(4\pi + 3)^2 2^{-2k}\} = \frac{1}{2}(4\pi + 3)^{-2}.$$

Thus

$$\left|\frac{1 - \bar{a}z}{z - a}\right|^2 \leqslant \frac{\{2(4\pi + 3)^2 + 1\}(1 - |a|^2)(1 - |z|^2)}{|z - a|^2}$$

$$\leqslant 4(2(4\pi + 3)^2 + 1)2^{3-2k}/|z - a|^2 < 10^5 2^{-2k}/|z - a|^2.$$

Hence

$$\sum\nolimits_2' \log|(1 - \bar{a}_\mu z)/(z - a_\mu)| < \sum\nolimits_2' \log\left(\frac{10^3 2^{-k}}{|z - a_\mu|}\right). \tag{4.5}$$

Let $N = N_{k,\nu}$ be the total number of zeros in $B'_{k,\nu}$. We apply Lemma 1 with

$$P(z) = \prod_1^N (z - a_\mu),$$

where the product is taken over all the zeros in $B'_{k,\nu}$ and

$$C = C_{\nu,N} = 10^3 2^{-k} \exp(-\varepsilon_k 2^k/N),$$

so that

$$N \log(10^3 2^{-k}/C) = \varepsilon_k 2^k. \tag{4.6}$$

We deduce that

$$\sum\nolimits_2' \log(C/|z - a_\mu|) < 0,$$

outside a set of disks the sum $\rho_{k,\nu}$ of whose radii satisfies

$$\rho_{k,\nu} \leqslant 2eC. \tag{4.7}$$

By (4.5)this yields

$$\sum\nolimits_2' \log|(1 - \bar{a}_\mu z)/(z - a_\mu)| < \sum\nolimits_2' \log|10^3 2^{-k}/(z - a_\mu)|$$

$$= \sum\nolimits_2' \log|C/(z - a_\mu)| + N \log(10^3 2^{-k}/C)$$

$$\leqslant N \log(10^3 2^{-k}/C) = \varepsilon_k 2^k \tag{4.8}$$

by (4.6), outside a set of disks the sum $\rho_{k,\nu}$ of whose radii satisfies

$$\rho_{k,\nu} \leqslant 2000e2^{-k} \exp(-\varepsilon_k 2^k/N_{k,\nu}).$$

We combine (3.4),(3.7),(3.8),(4.4) and (4.8) and deduce that, for z satisfying (3.3) we have

$$\sum\nolimits_2 \log|(1 - \bar{a}_\mu z)/(z - a_\mu)| = o(2^k) = o\{1/(1 - |z|)\}, \tag{4.9}$$

outside a set of circles the sum of whose radii $\rho_{k,\nu}$ satisfies

$$\sum\nolimits_\nu \sum \rho_{k,\nu} \leqslant 2e \sum_\nu C_\nu = 2000e2^{-k} \sum_\nu \exp(-\varepsilon_k 2^k/N_{k,\nu}). \tag{4.10}$$

We now apply Lemma 2 with

$$\phi(t) = \exp(-\varepsilon_k 2^k/t) \quad, t > 0,$$

$T = 3N_{k+1}$, and $\phi(0) = 0$. Then

$$\phi'(t) = \varepsilon_k 2^k \phi(t)/t^2, \ \ \phi''(t) = \{(\varepsilon_k 2^k/t^2)^2 - 2\varepsilon_k 2^k/t^3\}\phi(t),$$

so that $\phi'(t)$ is continuous and increasing for $0 < t \leqslant \varepsilon_k 2^{k-1}$.

We note that each zero a_μ in the sum $\sum_2'$ is counted in at most 3 of the sums $N_{k,\nu}$. So

$$\sum N_{k,\nu} \leqslant 3N_{k+1} = o(\varepsilon_k 2^k),$$

by (4.2). Thus Lemma 2 is applicable for $T = 3N_{k+1}$ and $k \geqslant k_1$ say and we deduce that, for $k \geqslant k_1$,

$$\rho_k = \sum_\nu \rho_{k,\nu} \leqslant 2000\, e\, 2^{-k} \phi(\sum N_{k,\nu})$$
$$\leqslant 2000\, e\, 2^{-k} \exp(-\varepsilon_k 2^k (3N_{k+1})).$$

So

$$\log(2000\, e 2^{-k}/\rho_k) \geqslant \varepsilon_k 2^k/(3N_{k+1})$$

and

$$\sum_{k=k_1}^{\infty} (\log 2000\, e\, 2^{-k}/\rho_k)^{-1} < \infty$$

by (4.2). So

$$\rho_k 2^k \to 0 \text{ and } \log(2000\, e\, 2^{-k}/\rho_k) < 2\log(2^{1-k}/\rho_k)$$

for large k. Thus for a suitable k_2

$$\sum_{k_2}^{\infty} \{\log(2^{1-k}/\rho_k)\}^{-1} < \infty,$$

which is (1.3). Also for z in (3.3) and outside the exceptional circles we have, by (3.4), (3.7), (3.8) and (4.9)

$$\log |1/B(z)| = o(2^k) = o(1/(1 - |z|)).$$

as $|z| \to 1-$
This is (1.6) and completes the proof of Theorem 1.

5 Proof of Theorem 2.

Let ρ_k be a sequence satisfying (1.3). Thus

$$2^k \rho_k \to 0.$$

We assume from now on without loss of generality that $0 \leqslant \rho_k < 2^{1-k}$ for $k > 1$. We next choose ε_k to satisfy (4.1) and

$$\sum \{\varepsilon_k (\log 2^{1-k}/\rho_k)\}^{-1} < \infty. \tag{5.1}$$

Suppose further that z_k lies in (1.2). We then place a zero of multiplicity p_k at z_k, where p_k is the integral part of

$$1 + \varepsilon_k^{-1} 2^k \{\log(2^{-k}/\rho_k)\}^{-1}. \tag{5.2}$$

Then, since z_k lies in (1.2), we have

$$\begin{aligned} p_k(1 - |z_k|) &\leqslant p_k(1 - r_{k-1}) = p_k 2^{1-k} \\ &\leqslant 2^{1-k} + 2\varepsilon_k^{-1}\{\log(2^{-k}/\rho_k)\}^{-1}, \end{aligned} \tag{5.3}$$

so that

$$\sum p_k(1 - |z_k|) < \infty,$$

by (5.1). We can therefore form the Blaschke product

$$B_1(z) = \prod_1^{\infty} \left\{ \frac{|z_k|}{z_k} \frac{z_k - z}{(1 - \bar{z}_k z)} \right\}^{p_k}, \tag{5.4}$$

since (1.4) is satisfied.

It follows from Schwarz's Lemma that for $|z - z_k| < \rho_k < 2^{-k}$ we have

$$\begin{aligned} \log |B_1(z)| &< p_k \log\{\rho_k/(1 - r_k)\} = -p_k \log(2^{-k}/\rho_k) \\ &\leqslant -2^k/\varepsilon_k < -1/\{\varepsilon_k(1 - |z|)\}, \end{aligned}$$

by (5.2). Since $\varepsilon_k \to 0$, this proves (1.7).

We next construct another Blaschke product $B_2(z)$ to satisfy (1.8). Then $B(z) = B_1(z)B_2(z)$ will satisfy (1.7) and (1.8), and so Theorem 2. We note that, if $0 \leqslant r < a < 1$, we have

$$\begin{aligned} &\log\{(1 - ar)/(a - r)\} = \log\{1 + (1 - a)(1 + r)/(a - r)\} \\ &\geqslant \log\{1 + (1 - a)/(1 - r)\} > \frac{1}{2}(1 - a)/(1 - r), \end{aligned} \tag{5.5}$$

since $\log(1 + x) > x/(1 + x)$, if $0 < x \leqslant 1$. We now define integers k_p as follows: $k_0 = 0$, and for $p \geqslant 1$, k_p is the least integer such that $k_p \geqslant p$ and

$$\varepsilon(r) \leqslant 2^{-p-2}, \text{if} \quad r_{k_p} \leqslant r < 1. \tag{5.6}$$

Set $n_0 = 0$, let n_1 be the smallest integer such that

$$n_1 \geqslant 2^{k_1+1}\varepsilon_0 \tag{5.7}$$

and for $p > 1$ we define

$$n_p = 2^{k_p - p}. \tag{5.8}$$

We now place a zero of multiplicity n_p at $a_p = r_{k_p}$, $p = 1$ to ∞. Then

$$\sum_{p=2}^{\infty} n_p(1 - a_p) = \sum_2^{\infty} 2^{k_p - p}\, 2^{-k_p} = \sum_2^{\infty} 2^{-p} = \frac{1}{2}.$$

Thus we may form the Blaschke product

$$B_2(z) = \prod_{p=1}^{\infty} \{(a_p - z)/(1 - a_p z)\}^{n_p}.$$

Also if $r_{k_{p-1}} \leqslant r < r_{k_p}$, where $p \geqslant 1$, we have

$$\begin{aligned} &\log B_2(r) < -n_p \log\{(1 - a_p r)/(a_p - r)\} \\ &< -\frac{1}{2} n_p (1 - a_p)/(1 - r) \leqslant -\frac{1}{2} 2^{k_p - p} 2^{-k_p}/(1 - r) < -\varepsilon(r)/(1 - r) \end{aligned}$$

by (5.5) to (5.8). Hence $B(z) = B_1(z)B_2(z)$ satisfies (1.7) and (1.8) and Theorem 2 is proved.

6 Proof of Theorem 3.

We deduce from Theorem 1 that, if $\zeta = e^{i\Theta}$,

$$\frac{|1 - ze^{-i\Theta}|^2}{1 - |z|^2} \quad \log|\Pi_j(z)| \to 0 \tag{6.1}$$

as $z \to \zeta$ nontangentially outside an F-set. For if

$$1 - ze^{-i\Theta} = \rho e^{i\Psi}, \tag{6.2}$$

where

$$|\Psi| \leqslant \Psi_0 < \pi/2, \quad 0 < \rho \leqslant \cos\Psi_0, \tag{6.3}$$

then

$$\begin{aligned} z &= e^{i\Theta}(1 - \rho e^{i\Psi}), \\ |z|^2 &= 1 - 2\rho\cos\Psi + \rho^2, \\ |1 - |z|| \geqslant (1 - |z|^2)/2 &= \rho\cos\Psi - \frac{1}{2}\rho^2 \geqslant \frac{1}{2}\rho\cos\Psi_0, \\ \rho = |1 - ze^{-i\Theta}| &\leqslant 2(1 - |z|)/\cos\Psi_0. \end{aligned} \tag{6.4}$$

So (6.1) follows from Theorem 1.

In (2.1) we have

$$\mu(\Theta) = \mu_1(\Theta) - \mu_2(\Theta),$$

where μ_1, μ_2 are positive increasing functions of Θ and so have left and right limits $\mu_j^-(\Theta_0)$ and $\mu_j^+(\Theta_0)$ for all Θ_0,

$$\mu_j^+(\Theta_\circ) - \mu_j^-(\Theta_\circ) = \alpha_j \geqslant 0.$$

Also $\alpha_j = 0$ outside a countable set E_j and

$$\sum_{E_j} \alpha_j(\Theta) \leqslant \frac{1}{2\pi}\{\mu_j(2\pi) - \mu_j(0)\} < \infty. \tag{6.5}$$

We write $\alpha_2(\Theta) - \alpha_1(\Theta) = \alpha(\Theta)$. Then

$$\begin{aligned} &\frac{1}{2\pi}\int_{\Theta_0-\pi}^{\Theta_0+\pi} \frac{e^{i\Theta} + z}{e^{i\Theta} - z} d\mu(\Theta) = \alpha(\Theta_0)\frac{e^{i\Theta_0} + z}{e^{i\Theta_0} - z} \\ &+ \frac{1}{2\pi}\int_{\Theta_0-\pi}^{\Theta_0+\pi} \frac{e^{i\Theta} + z}{e^{i\Theta} - z} d\nu(\Theta) \\ &= I_1(z) + I_2(z), \end{aligned} \tag{6.6}$$

where $\nu(\Theta)$ is continuous at Θ_0. We have from (6.2), with $\Theta = \Theta_0$

$$\frac{e^{i\Theta_0} + z}{e^{i\Theta_0} - z} = \frac{2 - \rho e^{i\Psi}}{\rho e^{i\Psi}} = \frac{2}{\rho}e^{-i\Psi} - 1,$$

while

$$1 - ze^{-i\Theta_0} = \rho e^{i\Psi}, \mathcal{R}(1 - ze^{-i\Theta_0}) = \rho\cos\Psi;$$

so

$$\frac{|z-\zeta|^2}{1-|z|^2}\mathcal{R}I_1(z) = \frac{\rho^2}{2\rho\cos\Psi - \rho^2}\alpha(\Theta_0)\frac{(2\rho\cos\Psi - \rho^2)}{\rho^2} = \alpha(\Theta_0). \tag{6.7}$$

Since ν is continuous at Θ_0, we can find increasing $\nu_j, j = 1, 2$, continuous at Θ_0, such that, if $\varepsilon > 0$, there exists a positive σ satisfying

$$\nu_j(\Theta_0 + \sigma) - \nu_j(\Theta_0 - \sigma) < \varepsilon \qquad j = 1, 2$$

and

$$\nu(\Theta) = \nu_2(\Theta) - \nu_1(\Theta).$$

We now write

$$I_2 = \frac{1}{2\pi}\int_{\Theta_0-\sigma}^{\Theta_0+\sigma} + \frac{1}{2\pi}\int_{\Theta_0+\sigma}^{\Theta_0+2\pi-\sigma} = I_3 + I_4;$$

$$|I_3| \leqslant \frac{1}{\pi(1-|z|)}\int |d\nu(\Theta)| \leqslant \frac{2\varepsilon}{\pi(1-|z|)}.$$

Also

$$\mathcal{R}\left(\frac{e^{i\Theta}+e^{i\Theta_0}}{e^{i\Theta}-e^{i\Theta_0}}\right) = \mathcal{R}\left(-\frac{1+e^{i(\Theta-\Theta_0)}}{1-e^{i(\Theta-\Theta_0)}}\right) = \mathcal{R}\left(-i\cot\{\frac{1}{2}(\Theta-\Theta_0)\}\right) = 0.$$

So $\mathcal{R}I_4(z) \to 0$ as $z \to e^{i\Theta_0}$ in any manner. We deduce that

$$\lim\ \sup\ (1-|z|)|\mathcal{R}I_2(z)| \leqslant 2\varepsilon/\pi, \tag{6.8}$$

as $|z| \to 1$ in any manner, and since ε is arbitrary

$$\lim\ \sup\ (1-|z|)\mathcal{R}I_2(z) = 0 \tag{6.9}$$

as $z \to e^{i\Theta_0}$ in any manner and

$$\lim\ \sup\ \frac{|z-e^{i\Theta_0}|^2}{1-|z|^2}\mathcal{R}I_2 = 0 \tag{6.10}$$

as $z \to e^{i\Theta_0}$ nontangentially. Now (6.1),(6.5),(6.6) and (6.9) yield (2.2) and (6.4) yields (2.3). This proves Theorem 3. In fact (6.6) and (6.7) show that

$$\log|f(z)| = \alpha(\Theta)\frac{1-|z|^2}{|\zeta-z|^2} + \frac{o(1)}{1-|z|} \tag{6.11}$$

as $z \to \zeta$ in any manner outside an F-set. This is a little stronger than (2.2).

7 Proof of Theorem 3, Corollary.

We note that by (2.3)

$$\alpha(\Theta_\nu) \to 0,$$

as $\nu \to \infty$. Suppose first that $\alpha(\Theta) \leqslant 0$ for all Θ. Then it follows that $\mu(\Theta)$ is uppersemicontinuous, (u.s.c.) and so uniformly u.s.c., in (2.1). For we can write in (2.1).

$$\mu(\Theta) = \mu_1(\Theta) - \mu_2(\Theta),$$

where $\mu_1(\Theta)$ is continuous and nondecreasing and $\mu_2(\Theta)$ is nondecreasing. Hence in (2.1)

$$\log|f(z)\Pi_2(z)| = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-|z|^2}{|e^{i\Theta}-z|^2}d\mu_1(\Theta) + u(z), \tag{7.1}$$

where $u(z) \leqslant 0$. Since $\mu_1(\Theta)$ is continuous and so uniformly continuous in $[-2\pi, 2\pi]$, we can, given a positive ε, find a positive δ such that

$$\mu_1(\Theta+\delta) - \mu_1(\Theta-\delta) < \varepsilon, \quad -\pi \leqslant \Theta \leqslant \pi.$$

Then if $-\pi \leqslant \Theta_0 < \pi$ we write in (7.1)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-|z|^2}{|e^{i\Theta}-z|^2}d\mu_1(\Theta)$$

$$= \int_{\Theta_0-\delta}^{\Theta_0+\delta} + \int_{\Theta_0+\delta}^{\Theta_0+2\pi-\delta} = I_1(z) + I_2(z), \tag{7.2}$$

say. Here

$$I_1(z) \leqslant \frac{1}{2\pi}\left(\frac{1+|z|}{1-|z|}\right)\int d\mu_1(\Theta) < \frac{\varepsilon}{2\pi}\left(\frac{1+|z|}{1-|z|}\right). \tag{7.3}$$

Also if $z = re^{i\Theta_0}$ and $\delta \leqslant |\Theta - \Theta_0| \leqslant \pi$ we have in $I_2(z)$

$$|e^{i\Theta} - z| \geqslant \sin\delta$$

so that , as $r \to 1$, we have uniformly in Θ_0

$$I_2(z) \to 0. \tag{7.4}$$

On combining (7.3) and (7.4) and recalling that, by Theorem 1,

$$(1-|z|)\log|\Pi_2(z)| \to 0,$$

as $|z| \to 1$ in any manner outside an F-set F_0, we deduce that, by (7.1)

$$\limsup\,(1-|z|)\log|f(z)| \leqslant 0 \tag{7.5}$$

as $|z| \to 1$ for z outside F_0; for ε can be arbitrarily small in (7.3). Hence

$$\limsup\,(1-r)\log M(r,f) \leqslant 0, \tag{7.6}$$

as $r \to 1$ outside an F-set F_1, which is the circular projection of F_0 onto the positive axis. Also by Theorem 3 we have for almost all Θ as $r \to 1$

$$(1-r)\ \log|f(re^{i\Theta})| \to 0.$$

Thus strict inequality cannot hold in (7.6) so equality must hold with lim instead of lim sup. This proves Corollary if $\alpha(\Theta) \leqslant 0$ for all Θ. The maximum of $\alpha(\Theta)$ is zero in this case and is attained for almost all Θ.

Suppose next that $\alpha(\Theta_0) > 0$ for some $\Theta = \Theta_0$. It follows from (2.3) that $\alpha(\Theta) \geqslant \alpha(\Theta_0)$ for only finitely many values of Θ. Among these we choose Θ_1 so that $\alpha(\Theta_1)$ is maximal. Then $\alpha(\Theta) \leqslant \alpha(\Theta_1)$ for all Θ so that

$$\alpha(\Theta_1) = \max_{\Theta}\alpha(\Theta) \tag{7.7}$$

and the maximum is attained. We write $\alpha(\Theta_1) = \alpha$. It follows that, given Θ_0, $-\pi \leqslant \Theta_0 \leqslant \pi$ and $\varepsilon > 0$, we can find a positive δ such that

$$\mu_1(\Theta_0 + 2\delta) - \mu_1(\Theta_0 - 2\delta) < 2\pi(\alpha + \varepsilon). \tag{7.8}$$

The corresponding open intervals $I(\Theta_0) = (\Theta_0 - \delta, \Theta_0 + \delta)$ cover $[-\pi, \pi]$ and hence we can select a finite subcovering, $I(\Theta_\mu) = (\Theta_\mu - \delta_\mu, \ \Theta_\mu + \delta_\mu)$, $\mu = 1$ to M, whose union covers $[-\pi, \pi]$. Hence, if $\delta_0 = \min \delta_\mu$, then, for $-\pi \leqslant \Theta \leqslant \pi$, the interval $(\Theta - \delta_0, \Theta + \delta_0)$ lies in one of the intervals $(\Theta_\mu - 2\delta_\mu, \ \Theta_\mu + 2\delta_\mu)$. For we can choose μ so that Θ lies in $I(\Theta_\mu)$.

Thus $\mu_1(\Theta + \delta_0) - \mu_1(\Theta - \delta_0) < \alpha + \varepsilon, -\pi \leqslant \Theta \leqslant \pi$. We now proceed as in (7.2) with δ_0 instead of δ. Instead of (7.3) we obtain

$$I_1(z) \leqslant (\alpha + \varepsilon)\frac{(1 + |z|)}{(1 - |z|)}, \tag{7.9}$$

while

$$(1 - |z|)I_2(z) \to 0,$$

uniformly as $|z| \to 1$. Using also (2.1), Theorem 1 and (7.8) we see that

$$\lim\ \sup\ (1 - |z|)\ \log|f(z)| \leqslant 2\alpha$$

as $|z| \to 1$, while z lies outside an F set F_0. So if F_1 is the circular projection of F_0 onto the positive axis we have

$$\lim\ \sup\ (1 - r)\ \log M(r, f) \leqslant 2\alpha, \tag{7.10}$$

as $r \to 1$ outside F_1. To prove the opposite inequality, choose Θ so that $\alpha(\Theta) = \alpha$. Such a choice is possible by (7.7). Then we deduce from (2.2) that, as $r \to 1$ outside an F-set F_2

$$(1 - r)\log|f(re^{i\Theta})| \to 2\alpha.$$

Hence

$$\lim\ \inf\ (1 - r)\log M(r) \geqslant 2\alpha,$$

as $r \to 1$ outside F_2. Combining this with (7.10) we obtain (2.5) and the Corollary is proved.

By applying the result to $1/f$, we deduce that

$$\lim_{r\to 1}(1 - r)\ \inf_{|z|=r}\ \log|f(z)| = \min \alpha(\Theta).$$

In particular

$$(1 - |z|)\ \log|f(z)|$$

is bounded above and below as $|z| \to 1$, while z lies outside an $F - set$. As a special case we obtain the following result of Šaginyan [10].

Theorem A. Let $p(t)$ be a continuous function for $0 \leqslant t < 1$, such that $p(t) \to \infty$ as $t \to 1$. Let Γ be a continuous curve in Δ ending at a point in C. If f is bounded in Δ and

$$(1 - |z|)\ \log|f(z)| < -p(|z|), \quad z \in \Gamma$$

then $f \equiv 0$.

In fact by Theorem 1 and (2.1) it is only necessary to assume that Γ is a subset of Δ which is not an $F - set$, and that f is a meromorphic function of bounded characteristic in Δ.

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