HOLOMORPHIC FUNCTIONS ON INFINITE DIMENSIONAL SPACES

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COMPLEX ANALYSIS AND APPROXIMATION,
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Definitions

- X a locally convex space over $\mathbb{C}$, a function $f : X \longrightarrow \mathbb{C}$ is holomorphic if
  (a) f is continuous and
  (b) for all $a, b \in X$ the function

$$z \in \mathbb{C} \longrightarrow f(a+zb) = \sum_{n=0}^{\infty} \alpha_n z^n$$

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  is an entire holomorphic function of one complex variable (that is it can be represented by a power series that converges for all $z \in \mathbb{C}$)
  or equivalently
  $(b)'$ for all $a, b \in X$ the limit

$$\lim_{\lambda \longrightarrow 0} \frac{f(a + \lambda b) - f(a)}{\lambda}$$

  always exists.

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- X Banach, $(\varphi_n)_{n=1}^{\infty} \subset X'$ then $\sum_{n=1}^{\infty} \varphi_n^n \in \mathcal{H}(X) \Longleftrightarrow \varphi_n(z) \longrightarrow 0$ for all $z \in X$.

- Expansions: Taylor series

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- Monomial Expansion (with respect to a basis $(e_n)_{n=1}^{\infty}$):

$$f(\sum_{n=1}^{\infty} z_n e_n) = \sum_{n=0}^{\infty} \Big\{ \sum_{m, |m|=n} a_m z^m \Big\}$$

where $m = (m_1, \ldots, m_n)$, and $z^m = z_1^{m_1} \cdots z_n^{m_n}$. Note that

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- We denote by z^m the mapping

$$\sum_{n=1}^{\infty} z_n e_n \in X \longrightarrow z^m$$

- If X is a reflexive nuclear space with a basis and $f : X \longrightarrow \mathbb{C}$ is a holomorphic function, such that $|f(x)| \leq A \exp(B\|x\|)$, then for some densely defined diagonal mapping $D : X'_\beta \longrightarrow X$ and some Gaussian measure μ on X'_β

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- If X is a reflexive nuclear space with a basis $(e_n)_{n=1}^{\infty}$ then the monomials form an absolute basis for $\mathcal{H}(X)$. Moreover, $\mathcal{H}(X)$ is the closed subalgebra generated by the linear monomials and the constant functions.

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- Is the converse true? That is, if X has a basis and the monomials form an absolute basis for $\mathcal{H}(X)$, is X nuclear?

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- Is the converse true? That is, if X has a basis and the monomials form an absolute basis for $\mathcal{H}(X)$, is X nuclear?
- Let $\beta = (\beta_n)_{n=1}^{\infty}$. The compact polydiscs,

$$\mathbb{D}(\beta) := \prod_{n=1}^{\infty} \mathbb{D}(\beta_n) := \{\sum_{n=1}^{\infty} \alpha_n e_n \in X : |\alpha_n| \leq |\beta_n|\}$$

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$$\sum_{n=0}^{\infty} |a_n r^n| \leq \left\{ |\sum_{n=0}^{\infty} a_n z^n \right| : |z| \leq 1\}$$

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- (S.D, R. Timoney) For the general case one needs a generalized version of Bohr's inequality: if $r = (r_1, \ldots, r_n)$ and

$$\sum_{m \in N^n} |a_m r^m| \leq \sup_{|z_i| \leq 1} \|\sum a_m z^m\|$$

  for any finite sum $\sum_{m \in N^n} a_m z^m$ then for all $\epsilon > 0$ one can find $C(\epsilon) > 0$ such that,

$$\sum_{i=1}^{n} r_i \leq C(\epsilon) n^{\epsilon + (1/2)}.$$

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  (R. Ryan) If X has a shrinking basis then the monomials of degree n with the square order form a basis for the space of continuous polynomials which are weakly continuous on bounded sets.
  - $$\Big(\bigotimes_{n,s,\pi} X\Big)' = (\mathcal{P}(^n X), \|\cdot\|_{B_X})$$

Square Order

- $z_1;\ z_2;\ z_3;\ z_4;\ \ldots\ \mathcal{P}(^1X) = X'$

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Theorem

(V. Dimant and S.D) Let X denote a Banach space with a shrinking Schauder basis and suppose all continuous polynomials on X are weakly continuous on bounded set then, for each positive integer n, $\left\{\mathcal{P}_k({}^nX)\right\}_{k=1}^{\infty}$ is a monotone finite dimensional decomposition for $(\mathcal{P}({}^nX), \|\cdot\|_{B_X})$.

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Theorem

(A. Defant and N. Kalton) If X is an infinite dimensional Banach Space then the monomials do NOT form an unconditional basis for $(\mathcal{P}({}^nX), \|\cdot\|_{B_X})$ for any positive integer $n \geq 2$.

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$$p(f) := \sum_{n=0}^{\infty} \left\| \frac{\widehat{d}^n f(0)}{n!} \right\|_{K+\alpha_n B_X}$$

where K ranges over the compact subsets of X and $(\alpha_n)_{n=0}^{\infty}$ ranges over c_0^+.

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- If $r > 0$ then

$$p_r(f) := \sum_{n=0}^{\infty} r^n \left\| \frac{\widehat{d}^n f(0)}{n!} \right\|_{K+\alpha_n B_X} = \sum_{n=0}^{\infty} \left\| \frac{\widehat{d}^n f(0)}{n!} \right\|_{rK+r\alpha_n B_X}$$

is also τ_ω continuous.

If X has an unconditional basis τ_ω is the finest locally convex topology for which the Taylor series converges absolutely and which induces on each $\mathcal{P}(^n X)$ the topology of uniform convergence on bounded subsets of X.

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- An ordering ϕ is compatible if for all n, m_1 and m_2, $m_1 < m_2$ implies

$$\phi(n, m_1) < \phi(n, m_2)$$

Theorem

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- Using (a) that the semi-norms for τ_ω are defined by uniform convergence over polydiscs (b) the fact that the Taylor series converges absolutely (c) that we may perturb each term in the norm by r^m it can be shown that it suffices to prove the following:
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- This we prove by induction on n. $c_1 = 1$ since $c_0' = \ell_1$ isometrically.

- Let e^*_{k+1} denote evaluation at the $(k+1)^{th}$-coordinate (that is z_{k+1}). Then

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- if $f \in \mathcal{H}(c_0)$ then, for some z_0 in the distinguished boundary of the polydisc A we have $\|f\|_A = |f(z_0)|$ and hence

$$\begin{aligned}
\|e^*_{k+1}\|_A \cdot \|f\|_A &\geq \|e^*_{k+1} \cdot f\|_A \\
&\geq |e^*_{k+1}(z_0) \cdot f(z_0)| \\
&\geq |e^*_{k+1}(z_0)| \cdot |f(z_0)| \\
&= \|e^*_{k+1}\|_A \cdot \|f\|_A \\
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\end{aligned}$$

Proof

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$$\sum_{m=1}^{s} \alpha_m P_{n+1,m} = \sum_{u=1}^{k+1} \Big\{ \sum_{P_{n+1,m} \in \mathcal{P}_u({}^{n+1}X)} \alpha_m P_{n+1,m} \Big\}$$

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Proof

- Let $(\alpha_m)_{m=1}^{\infty}$ be a sequence of scalars, let s and t be positive integers $s < t$.
- $$\sum_{m=1}^{s} \alpha_m P_{n+1,m} = \sum_{u=1}^{k+1} \Big\{ \sum_{P_{n+1,m} \in \mathcal{P}_u({}^{n+1}X)} \alpha_m P_{n+1,m} \Big\} = \sum_{u=1}^{k+1} Q_{n+1,u}$$
- $$\sum_{m=1}^{t} \alpha_m P_{n+1,m} = \sum_{u=1}^{k^*} \Big\{ \sum_{P_{n+1,m} \in \mathcal{P}_u({}^{n+1}X)} \alpha_m P_{n+1,m} \Big\} = \sum_{u=1}^{k} Q_{n+1,u} + \sum_{u=k+1}^{k^*} Q^*_{n+1,u}$$

- Using the Monotone Finite Dimensional Decomposition we have

$$\|\sum_{u=1}^{k} Q_{n+1,u}\|_A \leq \|\sum_{m=1}^{t} \alpha_m P_{n+1,m}\|_A$$

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- If $m_0 = \min\{j : P_{n+1,j} \in \mathcal{P}_{k+1}({}^{n+1}X)\}$ $s^* = \max\{j : P_{n+1,j} \in \mathcal{P}_{k+1}({}^{n+1}X)\}$, then

$$\begin{aligned} Q_{n+1,k+1} &= \sum_{m=m_0}^{s} \alpha_m P_{n+1,m} \\ &= e_{k+1}^* \cdot \sum_{m=m_0}^{s} \alpha_m P_{n,m-m_0+1} \end{aligned}$$

- $$Q^*_{n+1,k+1} = Q_{n+1,k+1} + \sum_{m=s+1}^{t\wedge s^*} \alpha_m P_{n+1,m}$$

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- $$\begin{aligned}\|Q_{n+1,k+1}\|_A &= \|e^*_{k+1} \cdot \sum_{m=m_0}^{s} \alpha_m P_{n,m-m_0+1}\|_A \\ &= \|e^*_{k+1}\|_A \cdot \|\sum_{m=m_0}^{s} \alpha_m P_{n,m-m_0+1}\|_A \\ &\leq 3^n \|e^*_{k+1}\|_A \cdot \|\sum_{m=m_0}^{t\wedge s^*} \alpha_m P_{n,m-m_0+1}\|_A \\ &= 3^n \|e^*_{k+1} \cdot \sum_{m=m_0}^{t\wedge s^*} \alpha_m P_{n,m-m_0+1}\|_A\end{aligned}$$

$$\|Q_{n+1,k+1}\|_A \leq 3^n \|Q^*_{n+1,k+1}\|_A$$

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&\leq 2 \cdot 3^n \cdot \| \sum_{m=1}^{t} \alpha_m P_{n+1,m}\|_A
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$$

Hence

$$
\| \sum_{m=1}^{s} \alpha_m P_{n+1,m}\|_A \leq (1 + 2 \cdot 3^n) \| \sum_{m=1}^{t} \alpha_m P_{n+1,m}\|_A
$$