Recent results on finely monogenic functions

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Introduction

- **Borel monogenic functions** (É. Borel 1917)
  $\equiv$ 'holomorphic' $f : U \to \mathbb{C}$ where $U \subset \mathbb{C}$ is **not** necessarily **open**, unique continuation property.
- **Finely holomorphic functions** (B. Fuglede, A. Debiard and B. Gaveau, T. J. Lyons and A. G. O'Farrell; 1970's)
  $\equiv$ 'holomorphic' $f : U \to \mathbb{C}$ where $U \subset \mathbb{C}$ is open in the **fine topology**
- B. Fuglede (1986), J. Wiegerinck: an extension to functions of several complex variables
- **Finely monogenic functions** (R. Lávička 2006)
  a generalisation of finely holomorphic functions to higher dimensions in hypercomplex analysis

Fine topology

Definition

The **fine topology** $\mathcal{F}$ in $\mathbb{R}^m$, $m \geq 2$, is the weakest topology making all superharmonic functions in $\mathbb{R}^m$ continuous.

Example: If $G \subset \mathbb{R}^m$ is open and K is countable and dense in G, then $G \setminus K \in \mathcal{F}$.

Let $U \subset \mathbb{R}^m$ be **finely open** (i.e., $U \in \mathcal{F}$) and $f : U \to \mathbb{R}^n$.

We call f **finely continuous** if

$$f : (U,\ \text{fine top.}) \to (\mathbb{R}^n,\ \text{Euclid. top.})$$

is continuous. Similarly, we define **fine limit**.

Let $U \subset \mathbb{R}^m$ be **finely open** (i.e., $U \in \mathcal{F}$) and $f : U \to \mathbb{R}^n$.

Fine differential $d_\mathrm{f} f(\tilde{x})$ is a linear map L on $\mathbb{R}^m$ such that

$$\operatorname{fine-lim}_{x \to \tilde{x}} \frac{f(x) - f(\tilde{x}) - L(x - \tilde{x})}{|x - \tilde{x}|} = 0.$$

Fine partial derivatives: For $j = 0, \ldots, m-1$ and $\tilde{x} \in U$, denote

$$\frac{\partial_\mathrm{f} f}{\partial x_j}(\tilde{x}) := d_\mathrm{f} f(\tilde{x})(e_j).$$

Here $(e_0, \ldots, e_{m-1})$ is the standard basis of $\mathbb{R}^m$, $x = (x_0, \ldots, x_{m-1}) \in \mathbb{R}^m$.

Function spaces

Let $U \subset \mathbb{R}^m$ be finely open and $f : U \to \mathbb{R}^n$.

Notation:

- $f \in \text{fine-}\mathcal{C}^1(U)$ if $d_\text{f}\, f$ is finely continuous on U.
- $\text{fine-}\mathcal{C}^k(U)$ are defined inductively for $k \in \mathbb{N}_0 \cup \{\infty\}$
- $f \in \mathcal{C}^k_{\text{f-loc}}(U)$ if $\forall x \in U\ \exists V \in \mathcal{F}_x\ \exists F \in \mathcal{C}^k(\mathbb{R}^m) : F = f$ on V

  where $\mathcal{F}_x$ is the set of (compact) fine neighbourhoods at x.

Obviously, we have that $\mathcal{C}^k_{\text{f-loc}}(U) \subset \text{fine-}\mathcal{C}^k(U)$ for $k \in \mathbb{N}_0 \cup \{\infty\}$.

The Brelot property: $\mathcal{C}^0_{\text{f-loc}}(U) = \text{fine-}\mathcal{C}^0(U)$

Questions: What happens for higher k?

Fine differentiability

Theorem

Let $U \subset \mathbb{R}^m$ be a finely open set and $k \in \mathbb{N} \cup \{\infty\}$. Then

$$\mathcal{C}^k_{\text{f-loc}}(U) = \text{fine-}\mathcal{C}^k(U).$$

In addition, a function f is constant on a fine domain U if $d_{\mathrm{f}} f = 0$ on U.

Proof:

(i) For the case $m = 2$ and $k = 1$, see (R. Lávička, 2007).
If $m = 2$, then any $V \in \mathcal{F}_x$ contains small circles around x.
Not true when $m > 2$! (complement of the Lebesgue spine)

(ii) For a partial solution of the case $m > 2$ and $k = 1$, see (R. Lávička, 2008).

(iii) For a general solution of the case $m > 2$, see (S. J. Gardiner, 2010).

Finely harmonic functions

Let $U \subset \mathbb{R}^m$ be finely open and $f : U \to \mathbb{R}$.

Definition (B. Fuglede 1972)

f is **finely harmonic** if $\forall x \in U \; \exists V \in \mathcal{F}_x$:
$f|_V$ is uniform limit of functions f_j harmonic on open sets $G_j \supset V$.

Properties:

- For **open** U, f is harmonic iff f is finely harmonic and (in case $m > 2$) locally bounded on U.
- In general, $f \notin \text{fine-}\mathcal{C}^1(U)$ and f need not satisfy the **unique continuation property**! (See Fuglede (1982), Lyons (1984).)

Finely holomorphic functions

(B. Fuglede, A. Debiard, B. Gaveau, T. J. Lyons, A. G. O'Farrell; 1970's)

Let $U \subset \mathbb{C} \simeq \mathbb{R}^2$ be **finely open** and $f : U \to \mathbb{C}$.

A function f is **finely holomorphic** if one of the following equivalent conditions holds:

(FH1) f has finely continuous f' on U;
in other words, $f \in \mathrm{fine}\text{-}\mathcal{C}^1(U)$ and $\bar{\partial}_{\mathrm{f}} f = 0$ on U. Here $x = x_0 + ix_1$,

$$f'(\tilde{x}) = \mathrm{fine}\text{-}\lim_{x \to \tilde{x}} \frac{f(x) - f(\tilde{x})}{x - \tilde{x}} \quad \text{and} \quad \bar{\partial}_{\mathrm{f}} f = \frac{1}{2}\left(\frac{\partial_{\mathrm{f}} f}{\partial x_0} + i \frac{\partial_{\mathrm{f}} f}{\partial x_1}\right).$$

(FH2) $f \in \mathcal{C}^1_{\mathrm{f\text{-}loc}}(U)$ and $\bar{\partial}_{\mathrm{f}} f = 0$ on U.

(FH3) $\forall x \in U \ \exists V \in \mathcal{F}_x : f|_V$ is uniform limit of functions f_j holomorphic on open sets $G_j \supset V$.

(FH4) f is finely harmonic and $\bar{\partial}_{\mathrm{f}} f = 0$ a.e. on U.

(FH5) f and $xf(x)$ are finely harmonic on U.

Properties:

- For **open** U, f is holomorphic iff f is finely holomorphic.
- If f is finely holomorphic, so is f'. Thus $f \in \mathrm{fine}\text{-}\mathcal{C}^{\infty}(U)$.
- The **unique continuation property**:
  If U is a fine domain and $\tilde{x} \in U$, then a finely holomorphic function f on U is uniquely determined by $f^{(n)}(\tilde{x})$, $n \geq 0$.

Let $\mathbb{H}$ be the **real quaternions** (W. R. Hamilton 1843) and $x \in \mathbb{H} \simeq \mathbb{R}^4$. Then

$$x = x_0 + x_1 i + x_2 j + x_3 k, \quad x_t \in \mathbb{R}.$$

Non-commutative product: $i^2 = j^2 = k^2 = ijk = -1$

Let $G \subset \mathbb{H}$ be open, $f : G \longrightarrow \mathbb{H}$ and $f \in \mathcal{C}^1(G)$.

Definition (R. Fueter, G. C. Moisil, N. Théodoresco, 1930s)

We call f **monogenic** if $\bar{\partial} f = 0$ on G. Here

$$\bar{\partial} f = \frac{\partial f}{\partial x_0} + i \frac{\partial f}{\partial x_1} + j \frac{\partial f}{\partial x_2} + k \frac{\partial f}{\partial x_3}.$$

Hypercomplex analysis

Let $\mathcal{C}\ell_{m-1}$ be the **Clifford algebra** generated by $e_1, \ldots, e_{m-1}$ satisfying

$$e_j^2 = -1, \;\; e_j e_k = -e_k e_j \;\; \text{for } j \neq k.$$

Then $x \in \mathbb{R}^m \subset \mathcal{C}\ell_{m-1}$: $x = x_0 + x_1 e_1 + x_2 e_2 + \cdots + x_{m-1} e_{m-1}$.

Let $G \subset \mathbb{R}^m$ be open, $\mathcal{H} = \mathcal{C}\ell_{m-1}$ or a $\mathcal{C}\ell_{m-1}$-module.
Let $f : G \to \mathcal{H}$ and $f \in \mathcal{C}^1(G)$.

Definition (R. Delanghe 1970s)

We call f **monogenic** if $\bar{\partial} f = 0$ on G. Here

$$\bar{\partial} f = \frac{\partial f}{\partial x_0} + e_1 \frac{\partial f}{\partial x_1} + e_2 \frac{\partial f}{\partial x_2} + \cdots + e_{m-1} \frac{\partial f}{\partial x_{m-1}}.$$

- Complex analysis ($m = 2$)
- Quaternionic analysis ($m = 4$)

Properties:

- Monogenic functions do **not** form an **algebra**!
- Cauchy's theorem, Cauchy's integral formula, Taylor and Laurent series, residue theory etc.
- $\Delta = \bar{\partial}\partial = \partial\bar{\partial}$ where

$$\Delta = \sum_{j=0}^{m-1} \frac{\partial^2}{\partial x_j^2}, \ \bar{\partial} = \frac{\partial}{\partial x_0} + \sum_{j=0}^{m-1} e_j \frac{\partial}{\partial x_j}, \ \partial = \frac{\partial}{\partial x_0} - \sum_{j=0}^{m-1} e_j \frac{\partial}{\partial x_j}.$$

- f is monogenic iff f and $xf(x)$ are both harmonic.

Possible definitions of finely monogenic functions

Let $U \subset \mathbb{R}^m$ be finely open, $\mathcal{H}$ be $\mathcal{C}\ell_{m-1}$-module and $f : U \to \mathcal{H}$.

(FM1) $f \in \text{fine-}\mathcal{C}^1(U)$ and $\bar{\partial}_\mathrm{f} f = 0$ on U. Here

$$\bar{\partial}_\mathrm{f} f = \frac{\partial_\mathrm{f} f}{\partial x_0} + e_1 \frac{\partial_\mathrm{f} f}{\partial x_1} + \cdots + e_{m-1} \frac{\partial_\mathrm{f} f}{\partial x_{m-1}}.$$

(FM2) $f \in \mathcal{C}^1_{\text{f-loc}}(U)$ and $\bar{\partial}_\mathrm{f} f = 0$ on U.

(FM3) $\forall x \in U \ \exists V \in \mathcal{F}_x : f|_V$ is uniform limit of functions f_j monogenic on open sets $G_j \supset V$.

(FM4) f is finely harmonic and $\bar{\partial}_\mathrm{f} f = 0$ a.e. on U.

(FM5) f and $xf(x)$ are finely harmonic on U.

Theorem (R. Lávička 2008 and 2011)

$(FM1) \iff (FM2) \implies (FM3) \implies (FM4) \iff (FM5)$;
for $f \in \text{fine-}\mathcal{C}^1(U)$, all conditions are equivalent to each other.

Finely monogenic functions

Let $U \subset \mathbb{R}^m$ be finely open, $\mathcal{H}$ be $\mathcal{C}\ell_{m-1}$-module and $f : U \to \mathcal{H}$.

Definition (R. Lávička 2006)

We call f **finely monogenic** if f and $xf(x)$ are both finely harmonic.

Properties:

- For $m = 2$ we have B. Fuglede's finely holomorphic functions.
- For **open** U, f is monogenic iff f is finely monogenic and (when $m > 2$) locally bounded on U.

Open questions:

- Does any finely monogenic function f belong to $\text{fine-}\mathcal{C}^1(U)$?
- Does any finely monogenic function f satisfy the unique continuation property?

Remark: Finely monogenic functions do not belong to $\mathrm{fine}\text{-}\mathcal{C}^\infty(U)$, in general.

Theorem (S. J. Gardiner 2010)

Let $h \in \mathrm{fine}\text{-}\mathcal{C}^2(U)$. Then we have

(i) *h is finely harmonic if and only if $\Delta_{\mathrm{f}}\, h = 0$.*

(ii) *If $m = 2$ and h is finely harmonic, then $h \in \mathrm{fine}\text{-}\mathcal{C}^\infty(U)$.*

Proof of (ii): By (i), $f = \partial_{\mathrm{f}}\, h$ is finely monogenic if h is finely harmonic!

Example (S. J. Gardiner 2012) Let $m > 2$.

There is $h \in \mathrm{fine}\text{-}\mathcal{C}^2(U) \setminus \mathrm{fine}\text{-}\mathcal{C}^3(U)$ such that $\Delta_{\mathrm{f}}\, h = 0$.

Then $f = \partial_{\mathrm{f}}\, h$ is finely monogenic and $f \in \mathrm{fine}\text{-}\mathcal{C}^1(U) \setminus \mathrm{fine}\text{-}\mathcal{C}^2(U)$.