PICARD THEOREMS

1879-2013

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Picard's Little Theorem (1879)

Let f be a meromorphic function on $\mathbb{C}$. Suppose $f(z) \neq a, b, c$ for $z \in \mathbb{C}$, where $a, b, c \in \hat{\mathbb{C}}$ are distinct. Then f is constant.

In other words, a nonconstant meromorphic function on $\mathbb{C}$ takes on every value in the extended complex plane with at most two exceptions.

Picard's Big Theorem (1880)

A transcendental (i.e., nonrational) meromorphic function on $\mathbb{C}$ takes on every value in $\hat{\mathbb{C}}$ infinitely often, with at most two exceptions

Examples e^z, $Q(z)e^{h(z)}$

Q rational, h entire

$$D \subset \mathbb{C}$$

$$f : (D, |\ |_{\mathbb{R}^2}) \longrightarrow (\hat{\mathbb{C}}, \chi)$$

$$\chi(z,w) = \frac{|z-w|}{\sqrt{1+|z|^2}\,\sqrt{1+|w|^2}} \qquad z, w \in \mathbb{C}$$

$$\chi(z,\infty) = \frac{1}{\sqrt{1+|z|^2}}$$

$$f^{\#}(z) = \lim_{h\to 0} \frac{\chi(f(z+h), f(z))}{|h|}$$

$$= \frac{|f'(z)|}{1+|f(z)|^2} \qquad (f(z) \neq \infty)$$

A family $\mathcal{F}$ of functions mer. on $D \subset \mathbb{C}$ is NORMAL on D if each sequence $\{f_n\} \subset \mathcal{F}$ has a subsequence which converges χ-uniformly on compact subsets of D.

<u>Remarks</u>. 1. Limit function is meromorphic on D (or ∞).

2. For families of <u>analytic</u> functions requirement is equivalent to existence of a subsequence which either converges uniformly on compacta or diverges uniformly on compacta.

3. Paradigmatic NONnormal family: $\mathcal{F} = \{nz\}$ on $\Delta = \{|z|<1\}$

Normality = (pre)compactness

= equicontinuity on compacta

= uniformly bounded "derivatives" (on compacta)

MARTY'S THEOREM

A family $\mathcal{F}$ of functions meromorphic on $D \subset \mathbb{C}$ is normal on D if and only if for each compact set $K \subset D$ there exists a constant $M = M(K)$ such that

$$f^{\#}(z) \leq M(K)$$

for all $z \in K$ and all $f \in \mathcal{F}$.

Montel's Theorem. The collection $\mathcal{F}$ of all meromorphic functions which omit three fixed values $a, b, c \in \hat{\mathbb{C}}$ on a domain $D \subset \mathbb{C}$ is a normal family on D.

Montel's Theorem

$\Downarrow$

Big Picard Theorem

$\Downarrow$

Little Picard Theorem

BLOCH'S PRINCIPLE

Let P be a property which forces a meromorphic function on $\mathbb{C}$ to be constant. Then any family of functions meromorphic on a common domain D, all of which have P on D, is [apt to be] normal on D.

Montel's Theorem. The collection $\mathcal{F}$ of all meromorphic functions which omit three fixed values $a, b, c \in \hat{\mathbb{C}}$ on a domain $D \subset \mathbb{C}$ is a normal family on D.

$$\begin{tikzcd}
\text{Montel's Theorem} \arrow[d, Rightarrow] \\
\text{Big Picard Theorem} \arrow[d, Rightarrow] \\
\text{Little Picard Theorem} \arrow[uu, bend left, "\text{BLOCH}"]
\end{tikzcd}$$

HAYMAN'S ALTERNATIVE (1959)

Theorem A. Let f be a transcendental meromorphic function on $\mathbb{C}$. Then either

(i) f assumes each value $a \in \mathbb{C}$ infinitely often, or

(ii) f' assumes each value $b \in \mathbb{C} \setminus \{0\}$ infinitely often.

Remark. Considering $g(z) = [f(z) - a]/b$ shows that one can take $a = 0$ and $b = 1$.

Examples: e^z, $z + e^z$

Remark. In fact, (ii) can be strengthened to

(ii') $f^{(k)}$ assumes each value $b \in \mathbb{C} \setminus \{0\}$ infinitely often for each $k = 1, 2, 3, \dots$.

__Theorem__ (Hayman 1959)

Let f be a meromorphic function on $\mathbb{C}$ such that

(1) $f(z) \neq 0$

(2) $f'(z) \neq 1$

for $z \in \mathbb{C}$. Then f is constant.

__Theorem__ (Gu 1979)

Let $\mathcal{F}$ be a family of meromor functions on $D \subset \mathbb{C}$. Suppose that

(1) $f(z) \neq 0$

(2) $f'(z) \neq 1$

for all $f \in \mathcal{F}$ and all $z \in D$. Then $\mathcal{F}$ is a normal family on D.

Theorem (Wang-Fang 1998)

Let f be a meromorphic function on $\mathbb{C}$ such that

(1) all zeros of f have multiplicity ≥ 3

(2) $f'(z) \neq 1$

for $z \in \mathbb{C}$. Then f is constant.

Theorem (Wang-Fang 1998)

Let $\mathcal{F}$ be a family of meromor. functions on $D \subset \mathbb{C}$. Suppose that

(1) all zeros of f have multiplicity ≥ 3

(2) $f'(z) \neq 1$

for all $f \in \mathcal{F}$ and all $z \in D$. Then $\mathcal{F}$ is a normal family on D.

Theorem (Wang-Fang 1998)
Let f be a transcendental meromorphic function on $\mathbb{C}$ all of whose zeros have multiplicity ≥ 3. Then f' takes on every nonzero complex value infinitely often.

Question: Can one replace 3 in the above theorems by 2?

<u>Example</u> 1. Let

$$f(z) = \frac{(z-a)^2}{z-b} \qquad a \neq b$$

$$= z + (b-2a) + \frac{(a-b)^2}{z-b}$$

Then

$$f'(z) = 1 - \frac{(a-b)^2}{(z-b)^2} \neq 1$$

Thus a meromorphic function on $\mathbb{C}$, all of whose zeros are multiple, which satisfies $f'(z) \neq 1$ for all $z \in \mathbb{C}$ need NOT be constant.

<u>Example</u> 2. Let $\mathcal{F} = \{f_\alpha\}$, $\alpha \in \mathbb{C} \setminus \{0\}$, where

$$f_\alpha(z) = \frac{(z-\alpha)^2}{(z-2\alpha)}.$$

Then $$f_\alpha(z) = z + \frac{\alpha^2}{z-2\alpha}$$

so $f_\alpha'(z) \neq 1$ for $z \in \mathbb{C}$.

But the family $\mathcal{F}$ is clearly not equicontinuous at 0, so $\mathcal{F}$ is not normal on any domain containing 0.

Thus one canNOT replace 3 by 2 in the Wang-Fang generalization of Gu's Theorem.

The order of an entire function f is

$$\rho = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r},$$

where $M(r) = \max_{|z|\le r} |f(z)|$

The order of a meromorphic function f is

$$\rho = \limsup_{r\to\infty} \frac{\log T_0(r)}{\log r},$$

where $T_0(r) = \int_0^r \frac{S(t)}{t}\,dt$

and $S(t) = \frac{1}{\pi} \iint_{|z|<t} [f^\#(z)]^2\,dx\,dy$

If $f^\#$ is bounded on $\mathbb{C}$, then f has order $\rho \le 2$.

Theorem Let f be a transcendental meromorphic function on $\mathbb{C}$ of finite order, all of whose zeros are multiple. Then f' takes on every nonzero value in $\mathbb{C}$ infinitely often.

This follows from

Theorem (Bergweiler-Eremenko) Let f be a transcendental meromorphic function on $\mathbb{C}$ of finite order which has an infinite number of multiple zeros. Then f' takes on every nonzero value in $\mathbb{C}$ infinitely often.

However, the Bergweiler-Eremenko theorem is NOT true for functions of infinite order.

Example. Let

$$f(z) = z + a\int_0^z e^{be^{\zeta}-\zeta}\, d\zeta,$$

where $1+ab=0$ and $1+ae^b=0$ (choose b such that $b=e^b$ and $a=-1/b$). Set $w=e^{\zeta}$ so that $d\zeta = dw/w$. Then

$$f(z+2\pi i)-f(z) = 2\pi i + a\int_{\Gamma} \frac{e^{bw}}{w^2}\, dw$$

$$= 2\pi i\,(1+ab) = 0,$$

so f has period $2\pi i$. Clearly, $f(0)=0$. Since $f'(z) = 1 + a\exp(be^z - z)$, $f'(0) = 1+ae^b = 0$. Thus f has multiple zeros at $2\pi i k$, $k\in\mathbb{Z}$. But clearly $f'(z)\neq 1$.

How to prove the Theorem for multiplicity ≥ 3 (but not for multiplicity ≥ 2) assuming the result on normal families

Suppose all zeros of f have multiplicity ≥ 3 but f' takes on some nonzero value, say 1, only finitely often. Then f has infinite order. So $f^{\#}(z)$ is unbounded on $\mathbb{C}$. Thus there exist $z_n \to \infty$ such that $f^{\#}(z_n) \to \infty$.

Consider $\mathcal{F} = \{f_n\}$ on the unit disc Δ, where $f_n(z) = f(z+z_n)$. Since $f_n'(z) = f'(z+z_n) \neq 1$ on Δ for $n \geq N$ and all zeros of f_n have multiplicity ≥ 3, $\mathcal{F}$ is normal on Δ. But $f_n^{\#}(0) = f^{\#}(z_n) \to \infty$, which contradicts Marty's Theorem.

<u>Remark</u>. This proof fails utterly if we assume only multiplicity ≥ 2, as a family of meromorphic functions all of whose zeros are multiple and such that $f'(z) \neq 1$ for each function in the family and each point in the domain is NOT necessarily normal.

Quasinormality

A family $\mathcal{F}$ of functions meromorphic on a plane domain $D \subset \mathbb{C}$ is <u>quasinormal</u> on D if from each sequence $\{f_n\} \subset \mathcal{F}$ one can extract a subsequence $\{f_{n_k}\}$ which converges locally uniformly on $D \setminus E$ with respect to the spherical metric, where $E \subset D$ (which may depend on $\{f_{n_k}\}$) has no accumulation point in D.
If E can always be chosen to contain no more than K points, $\mathcal{F}$ is said to be quasinormal of order K on D.

<u>Example</u> $\{nz\}$ is not normal on $\{|z|<1\}$, but it is quasinormal of order 1 there.

Theorem (Nevo-Pang-Z) Let $\mathcal{F}$ be a family of meromorphic functions on D, all of whose zeros are multiple. If for each $f \in \mathcal{F}$, $f'(z) \neq 1$ for all $z \in D$, then $\mathcal{F}$ is quasinormal of order 1 on D.

Example. Let $D = \{z : 2 < |z| < 4\}$ and let $\mathcal{F} = \{f_n\}$, where $f_n(z) = z^n - 3^n$. Then $f_n'(z) = nz^{n-1} \neq 1$ for $z \in D$. No subsequence of $\{f_n\}$ can converge uniformly on a neighborhood of any point of $\{z : |z| = 3\}$. Thus $\mathcal{F}$ is not quasinormal on D. This shows that Theorem 2 is not true without the hypothesis of multiple zeros.

<u>Theorem</u> 1 (Nevo-Pang-Z) The derivative of a transcendental meromorphic function on $\mathbb{C}$, all but finitely many of whose zeros are multiple, takes on every nonzero complex value infinitely often.

<u>Corollary</u> If f is a transcendental meromorphic function on $\mathbb{C}$, then $f'f^n$ takes on every nonzero complex value infinitely often for each $n \in \mathbb{N}$.

<u>Proof</u>. $(f^{n+1})' = (n+1)f'f^n$.

Theorem Let f be a transcendental meromorphic function on $\mathbb{C}$, all of whose zeros are multiple. Then f' takes on every nonzero value in $\mathbb{C}$

Proof. Suppose not, say $f'(z) \neq 1$, $z \in \mathbb{C}$
Then certainly f has infinite order, s
there exist $z_n \to \infty$ such that $f^{\#}(z_n) \to \infty$
Consider the family of functions $\mathcal{F} = \{f_n$
where $f_n(z) = f(z_n z)/z_n$, on $\mathbb{C}$. Then f_n has only multiple zeros on $\mathbb{C}$ and $f_n'(z) = f'(z_n z) \neq 1$ on $\mathbb{C}$. Thus $\mathcal{F}$ is quasinormal of order 1 on $\mathbb{C}$.

On the other hand, for $|z_n| \geq 1$,

$$f_n^{\#}(1) = \frac{|f'(z_n)|}{1 + \left|\frac{f(z_n)}{z_n}\right|^2} \geq \frac{|f'(z_n)|}{1 + |f(z_n)|^2} = f^{\#}(z_n) \to \infty,$$

so, by Marty's Theorem, no subsequence of $\mathcal{F}$ can be normal at $z = 1$.

Similarly, for each $\epsilon > 0$ and $|z_n| \geq 1$,

$$\sup_{|z| \leq \epsilon} f_n^{\#}(z) = \sup_{|z| \leq \epsilon} \frac{|f'(z_n z)|}{1 + \left| \frac{f(z_n z)}{z_n} \right|^2}$$

$$\geq \sup_{|z| \leq \epsilon} \frac{|f'(z_n z)|}{1 + |f(z_n z)|^2}$$

$$= \sup_{|w| \leq \epsilon |z_n|} f^{\#}(w) \longrightarrow \infty,$$

so that no subsequence of $\mathcal{F}$ can be normal at $z = 0$. Thus, the family $\mathcal{F}$ cannot be quasinormal of order 1 on any domain containing the points 0 and 1, a contradiction.

<u>Theorem</u> (Nevo - Pang - Z) Let f be a transcendental meromorphic function, all but finitely many of whose zeros are multiple, and let $R \not\equiv 0$ be a rational functio[n]. Then $f' - R$ vanishes infinitely often on $\mathbb{C}$.

A Natural Question

Let f be a transcendental meromorphic function on $\mathbb{C}$ all (but finitely many) of whose zeros are multiple. Must $f^{(k)}$ take on every nonzero complex value infinitely often for $k = 2, 3, \ldots$?

THE EVIDENCE

① If all zeros of f have multiplicity ≥ 3, then $f^{(k)}$ takes on each nonzero complex value infinitely often for $k \geq 2$.

HOWEVER,

② The proof of the previous statement fails completely when the multiplicity is only ≥ 2.

MOREOVER,

③ A family $\mathcal{F}$ of functions meromorphic on $D \subset \mathbb{C}$, all of whose zeros are multiple and such that $f'' \neq 1$ on D for each $f \in \mathcal{F}$ need <u>not</u> be quasinormal on D of any finite order.

ON THE OTHER HAND, . . .

Theorem. If f is a transcendental meromorphic function of finite order, all of whose zeros are multiple, and $k \geq 2$, then $f^{(k)}$ takes on each nonzero complex value infinitely often.

Proof. Suppose that $f^{(k)}(z) = 1$ only finitely often. If f has only finitely many poles, then

$$f^{(k)}(z) = 1 + R(z) e^{P(z)},$$

where R is a rational function and P is a polynomial. In this case, an elementary calculation leads to a contradiction.

Otherwise, f has infinitely many poles. However, according to a theorem of Langley, a meromorphic function g of finite order such that $g^{(k)}$ has only finitely many zeros for some $k \geq 2$ has only finitely many poles. Applying this result to $g(z) = f(z) - \frac{1}{k!} z^k$ yields a contradiction. (Note that the hypothesis of multiple zeros was not used here.)

Unfortunately, Langley's result does NOT hold for meromorphic functions of infinite order.

Thus, the evidence seems equivocal, at best. Nonetheless, it turns out that the ayes have it:

Theorem. Let f be a transcendental meromorphic function, all but finitely many of whose zeros are multiple. Then $f^{(k)}$ takes on each nonzero complex number infinitely often for $k = 1, 2, 3, \cdots$.

This has been shown by Mingliang Fang and Yufei Wang, who derive the result in fairly straightforward fashion from an inequality proved in the breakthrough paper of Yamanoi, in which he proves the Gol'dberg Conjecture.

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Mingliang Fang and Yufei Wang, A note on the conjectures of Hayman, Mues and Gol'dberg, *Computational Methods and Function Theory*, to appear.