Classicalisation of Swiss Cheeses

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Three of my students (PhD/MSc) have made significant contributions to this (ongoing) work:

Matthew Heath, Jonathan Mason, and Hongfei Yang

As an exercise, you may wish to think about the following problem.

Problem

Does there exist a pair of sequences (λ_n), (a_n) of non-zero complex numbers such that

(i) no two of the a_n are equal,

(ii) $\sum_{n=1}^{\infty} |\lambda_n| < \infty$,

(iii) $|a_n| < 2$ for all $n \in \mathbb{N}$, and yet,

(iv) for all $z \in \mathbb{C}$,

$$\sum_{n=1}^{\infty} \lambda_n \exp(a_n z) = 0?$$

J. Wolff gave a solution to this problem in 1921!

Abstract

In this talk we will discuss various types of Swiss cheese set, and their applications.

Here we use the term Swiss cheese set in a rather general sense in order to include a wide class of examples: by a **Swiss cheese set** we simply mean a compact plane set obtained by deleting the union of some suitable sequence of open discs from some initial closed disc.

Of course, without some additional conditions on the discs, this would mean that every compact plane set was a Swiss cheese set!

In practice we place requirements on the positions and/or the radii of the deleted discs to ensure that the resulting set has desirable properties.

We discuss a few of the standard applications of Swiss cheese sets from the literature, including an example of O'Farrell (1979) of a regular uniform algebra with a continuous point derivation of infinite order.

We then describe the process that we call the **classicalisation** of Swiss cheeses, which enables us to modify a Swiss cheese set X in order to improve its topological properties, while attempting to retain desired properties of $R(X)$.

One direct application of this classicalisation procedure is to produce examples of essential, regular uniform algebras on locally connected topological spaces.

However, rather more care is required in order to classicalise the example of O'Farrell. We discuss some of the issues, and how they can be overcome.

Some uniform algebras on compact plane sets

Let X be a non-empty, compact Hausdorff space.

We denote by $C(X)$ the algebra of continuous, complex-valued functions on X.

We give $C(X)$ the uniform norm on X: this makes $C(X)$ into a Banach algebra.

Definition

A **uniform algebra** on X is a closed subalgebra, A, of $C(X)$ such that A contains the constant functions and A **separates the points** of X.

We now look at some very well-known uniform algebras on (non-empty) **compact plane sets**.

Let X be a compact plane set. Consider the following subalgebras of $C(X)$:

- $A(X)$ is the set of those functions in $C(X)$ which are analytic (holomorphic) on the interior of X;
- $P_0(X)$ is the set of restrictions to X of polynomial functions with complex coefficients;
- $R_0(X)$ is the set of restrictions to X of rational functions with complex coefficients whose poles (if any) lie off X.

It is easy to see that these subalgebras contain the constant functions and separate the points of X.

The algebra $A(X)$ is closed in $C(X)$, and so $A(X)$ is a uniform algebra on X.

The algebras $P_0(X)$ and $R_0(X)$ are usually not closed in $C(X)$.

We may obtain uniform algebras on X by taking the (uniform) closures of $P_0(X)$ and $R_0(X)$ in $C(X)$; $P(X)$ is the closure of $P_0(X)$, and $R(X)$ is the closure of $R_0(X)$.

- The functions in $P(X)$ are those which may be **uniformly approximated** on X by polynomials,
- and the functions in $R(X)$ are those which may be **uniformly approximated** on X by rational functions with no poles in X.

Clearly we have $P(X) \subseteq R(X) \subseteq A(X) \subseteq C(X)$.

It is also clear that $A(X) = C(X)$ if and only if $\operatorname{int} X = \emptyset$.

In this talk we will mostly look at the uniform algebra $R(X)$ and its properties.

The uniform algebra $R(X)$

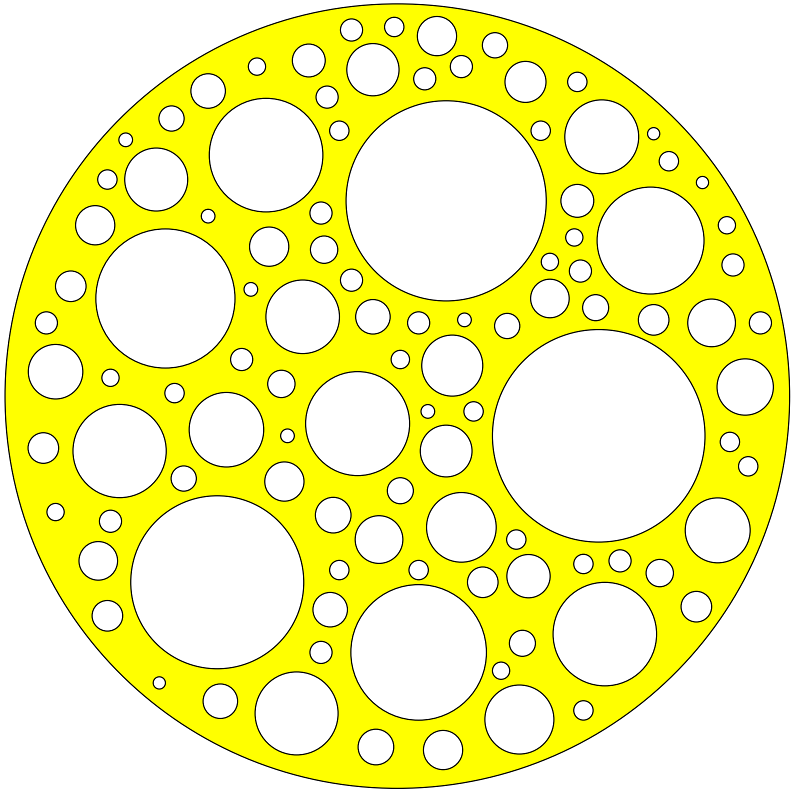
Let X be a compact plane set. When is it true that $R(X) = C(X)$?

- By above, $R(X) \neq C(X)$ whenever $\operatorname{int} X \neq \emptyset$.
- On the other hand (**Hartogs–Rosenthal theorem**, 1931), whenever the Lebesgue area measure of X is 0, we have $R(X) = C(X)$.

So this only leaves the question of compact plane sets which have empty interior, but which have positive area. (Both answers are still possible in this setting.)

The first example of a compact plane set X with $\operatorname{int} X = \emptyset$ but with $R(X) \neq C(X)$ was found by the Swiss mathematician **Alice Roth** in 1938. Her example is what we may describe as a 'classical' Swiss cheese.

Here is the picture that we (probably) think of when we talk about Swiss cheeses.



(1) In these 'classical' Swiss cheeses, we usually insist that the closures of the small discs are subsets of the interior of the larger cheese, and are pairwise disjoint.

(2) We also usually require that the interior of the resulting set is empty, and that the sum of the radii of the small discs is finite.

Alice Roth's Swiss cheese from 1938 fulfilled both conditions (1) and (2) above, and the resulting set X has

$$R(X) \neq A(X) = C(X)\,.$$

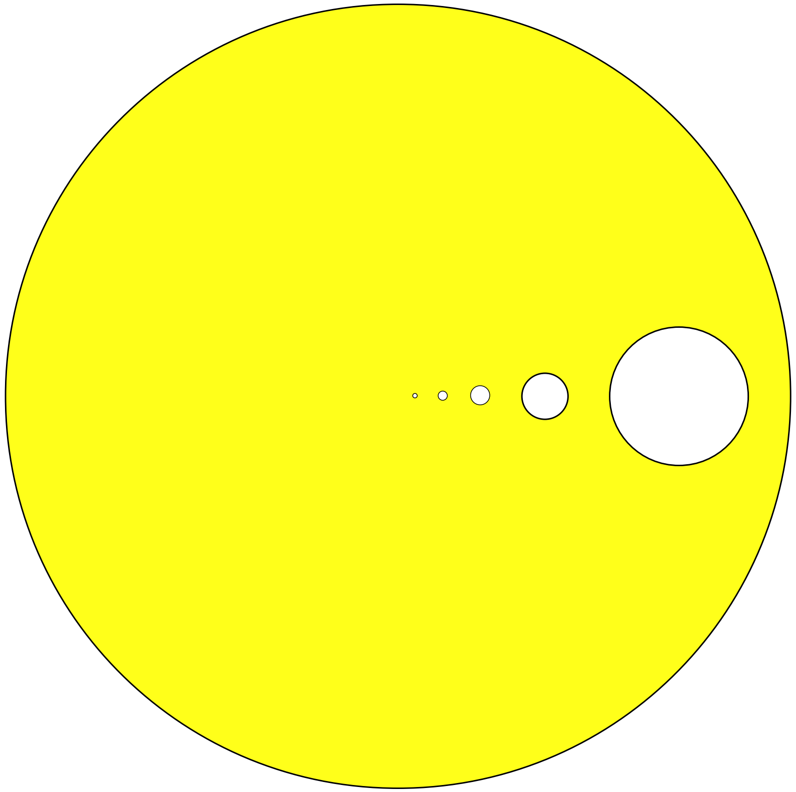
Roth's example was apparently forgotten for many years, and the example was rediscovered independently by **Mergelyan** in 1954.

There are many other such classical Swiss cheeses in the literature. For example:

- **Steen's cheese**(1966), where $R(X)$ is not **antisymmetric**, i.e., there is a non-constant, real-valued function in $R(X)$.
- A Swiss cheese X constructed (in 1967) by **John Wermer** such that $R(X)$ has **no non-zero, bounded point derivations:** for all $z \in X$, the map $f \mapsto f'(z)$ is discontinuous on $R_0(X)$.

A look at Gamelin's book on uniform algebras reveals many other useful examples of compact plane sets with slightly different properties, and with other names including: roadrunner sets; the stitched disc; the string of beads.

These examples each have dense interior, not empty interior.



A roadrunner set

In some important applications (see, for example, Stout's book on uniform algebras), we may need to allow the discs to overlap each other and/or to meet the boundary or the complement of the large disc.

The resulting sets need not be connected or locally connected, and can have isolated points.

We generally need to ensure that the resulting set has positive area (two-dimensional Lebesgue measure).
One way to achieve this is to insist that the sum of the radii of the small discs is strictly less than the radius of the big disc.

McKissick's example (1963) of a **non-trivial, normal uniform algebra** was the algebra $R(X)$ for a Swiss cheese X which is (probably) of this type, as was O'Farrell's example (1979) of a regular uniform algebra with a continuous point derivation of infinite order.

Essential uniform algebras

Definition

Let A be a uniform algebra on a compact space X. We say that A is **essential** if, for every non-empty open subset U of X, there is a function $f \in C(X)$ whose closed support is contained in U, but such that $f \notin A$

One advantage of classical Swiss cheese sets is that the resulting uniform algebra $R(X)$ is always essential.

In view of this, and the good topological properties of classical Swiss cheese sets, we may wish to find a way to 'classicalise' some of the non-classical examples in the literature.

We now describe the methods we have used, and the results we have so far.

Suppose that we are given a non-classical Swiss cheese set X.

When can we find a classical Swiss cheese set $Y \subseteq X$?

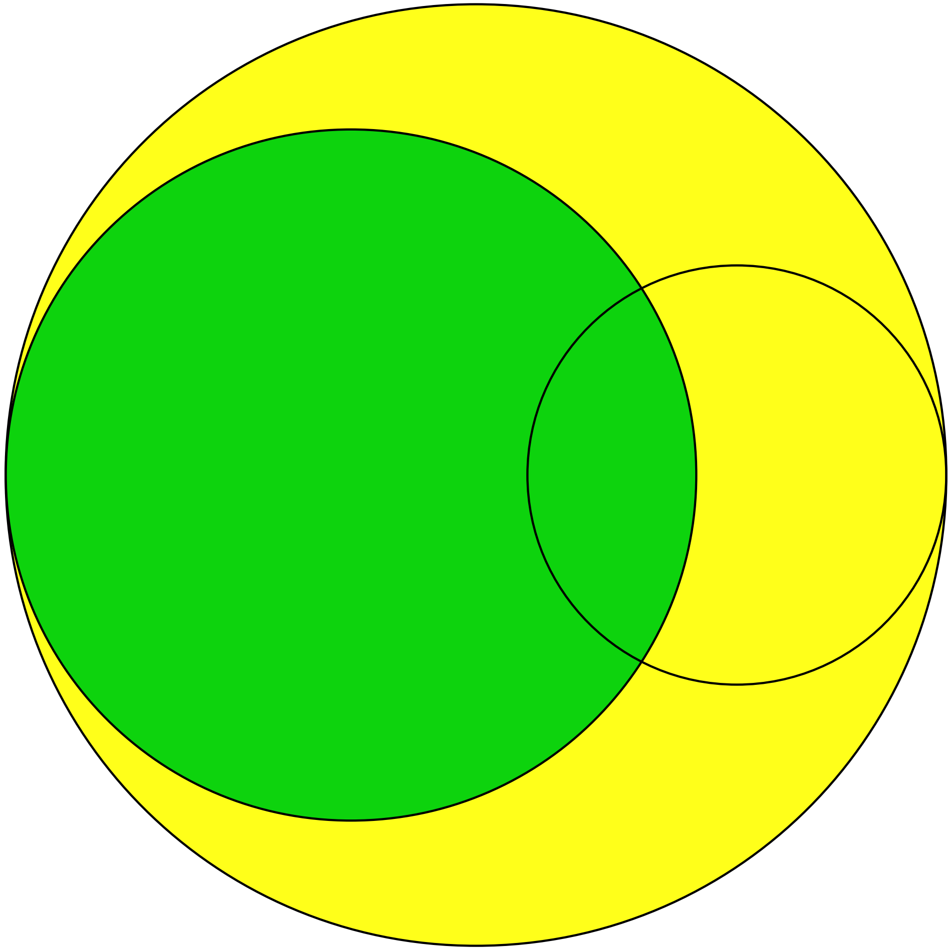
Our main result here is the following.

Theorem (F.–Heath, 2010)

Let X be a compact plane set obtained by deleting from a closed disc the union of a sequence of open discs such that the sum of the radii of the open discs is less than the radius of the closed disc.
Then there is a classical Swiss cheese set Y with $Y \subseteq X$.

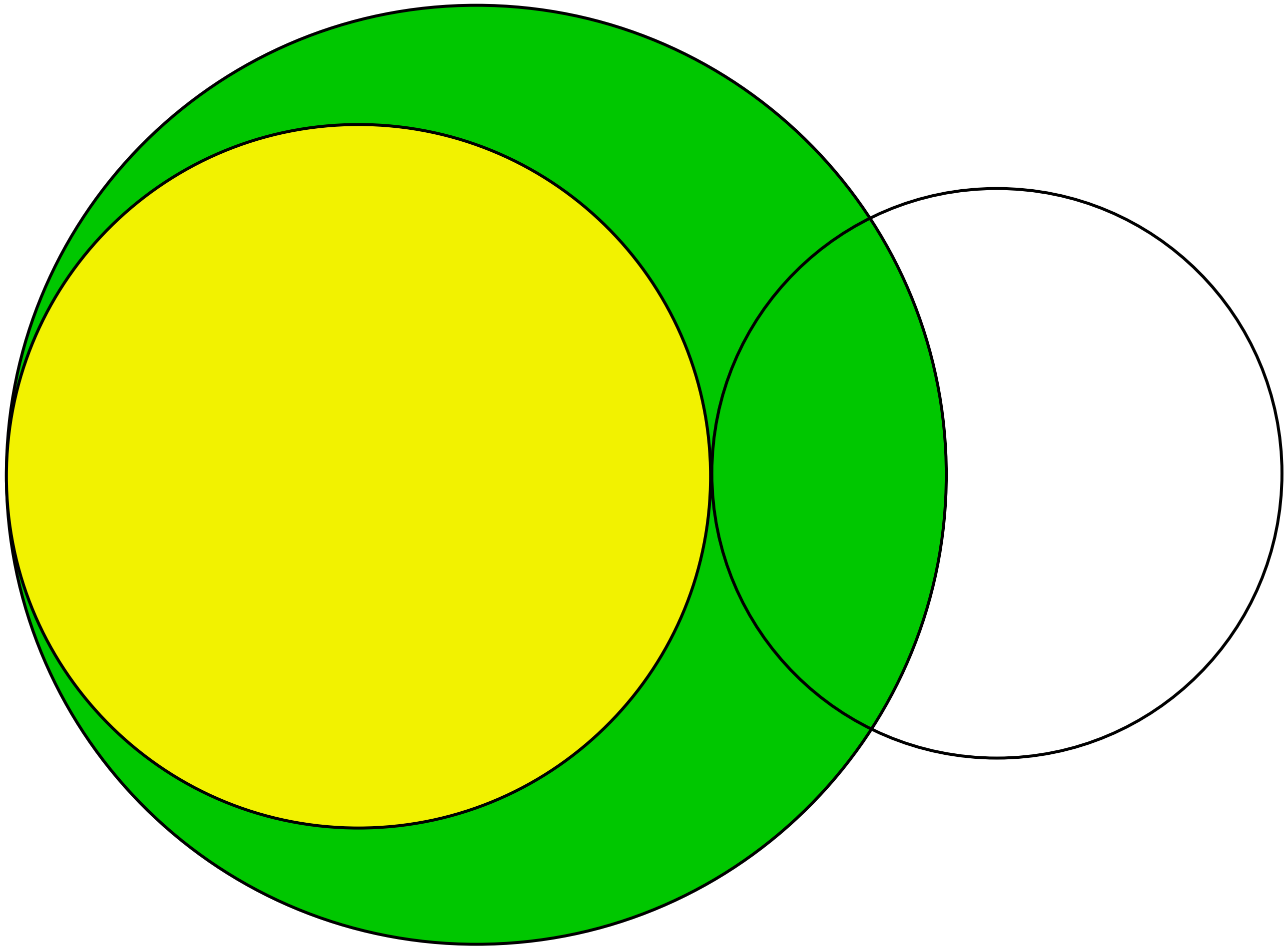
This result can be proved by Zorn's lemma or by transfinite induction, based on the following pair of elementary geometrical constructions.

Replacing two touching or overlapping open discs with one larger open disc



Note that the radius of the new disc is no greater than the sum of the radii of the two smaller discs.

Contracting the closed disc to eliminate an open disc whose closure is not a subset of the interior of the closed disc



Note that the decrease in radius of the closed disc is no greater than the radius of the eliminated open disc.

When we pass from a compact plane set X to a subset Y, many properties of $R(X)$ are inherited by $R(Y)$.

For example, if $R(X)$ is regular, then so is $R(Y)$.

Thus, applying the classicalisation theorem to McKissick's cheese, we obtain a classical Swiss cheese Y such that $R(Y)$ is regular.

This gives us examples of essential, regular uniform algebras on locally connected spaces.

However, more care is required in order to classicalise O'Farrell's example of a regular uniform algebra with a (non-zero) continuous point derivation of infinite order.

One approach is to look at classicalisation in an annulus.

Classicalisation in an annulus

When shrinking our large closed disc, we may prefer not to allow the centre of the disc to change.

We **can** preserve the centre, but the cost is that the decrease in radius may now be up to **twice** the radius of the smaller disc.

For this type of classicalisation, we should ensure that the radius of the closed disc is greater than **twice** the sum of the radii of the open discs.

Similarly, if we wish to delete a sequence of open discs from a closed annulus, we will want to keep the inner and outer circles concentric.

For this type of classicalisation, we should ensure that the difference between the radii of the inner and outer circles of the annulus is greater than twice the sum of the radii of the open discs.

Combining the methods above, we obtain the following variant of the classicalised McKissick cheese.

Theorem

Let C be a closed annulus in the plane, and let $\varepsilon > 0$.
Then there exists a sequence of open discs D_n with the following properties:

- *the closures of the discs D_n are pairwise disjoint subsets of $\operatorname{int} C$;*
- *the sum of the radii of the discs D_n is less than ε;*
- *the classical swiss cheese $X := C \setminus \bigcup_{n=1}^{\infty} D_n$ is such that $R(X)$ is regular.*

Developing these methods (with care!) finally allows us to produce a **classical** Swiss cheese Y such that $R(Y)$ is regular and has a continuous point derivation of infinite order.