Conference for Tony O'Farrell, June 2013

SWISS CHEESE

&

CHAMPAGNE BUBBLES

Ivan Netuka

Mathematical Institute, Faculty of Mathematics and Physics, Charles University in Prague

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A **Swiss cheese** is a *nowhere dense* compact set $K \subset \mathbb{C}$ formed by removing from the closure $\overline{B}$ of the open unit disc B a collection of open discs $B_n := B(x_n, r_n)$ such that $B_n \subset B$, B_n are pairwise disjoint, and $\sum_{n=1}^{\infty} r_n < \infty$.

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For a compact set X in $\mathbb{C}$, $R(X)$ is the uniform closure in $C(X)$ of the algebra $R_0(X)$ of rational functions with poles outside X.

Theorem (F. Hartogs and A. Rosenthal).

If X is a compact subset of $\mathbb{C}$ and $\lambda_2(X) = 0$, then $R(X) = C(X)$.

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[An extension of the **Weierstrass Approximation Theorem**.]

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[An extension of the **Weierstrass Approximation Theorem**.]

Proof.

For a complex Radon measure ν on X, the *Cauchy transform* is defined by

$$\hat{\nu}(\zeta) = \int \frac{d\nu(z)}{z - \zeta} \qquad , \ \zeta \in \mathbb{C}.$$

Facts:

- $\hat{\nu} = 0$ on X^c if and only if $\nu \perp R(X)$
- $\hat{\nu} = 0$ λ_2 - a.e. implies $\nu = 0$

□

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Theorem (A. Roth; S. N. Mergelyan).

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Proof.

Let μ be the measure on $\partial B \cup \bigcup_{n=1}^{\infty} \partial B_n$ which coincides with dz on ∂B and $-dz$ on $\bigcup_{n=1}^{\infty} \partial B_n$. Since $\sum_{n=1}^{\infty} r_n < \infty$, μ is a finite measure, and obviously $\mu \neq 0$. **Cauchy's theorem** gives

$$\int_K r \, d\mu = 0 \qquad , \; r \in R_0(K).$$

Thus

$$\int_K f \, d\mu = 0 \qquad , \; f \in R(K).$$

Hence $R(K) \neq C(K)$ by the **Hahn-Banach Theorem**. □

Summary

- $R(X) = C(X)$ for every compact set X with $\lambda_2(X) = 0$
- $R(K) \neq C(K)$ for every Swiss cheese K

Corollary.

$\lambda_2(K) > 0$ *for every Swiss cheese* K.

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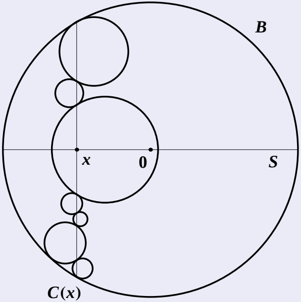
A direct proof (O. Wesler).



Take $x \in S$, the chord $C(x)$, and denote $Z_1 := \{x \in S : \lambda_1(C(x) \cap K) > 0\}$. *Suppose that* $\lambda_2(K) = 0$. **Fubini's theorem:** $\lambda_1(Z_1) = 0$. Put $Z_2 := \{x \in S \backslash Z_1 : C(x)$ meets only *finitely* many discs $B_n\}$. For $x \in Z_2$, $C(x)$ must pass, in this finite string of discs, through **points of tangency**. There is countably many points of tangency among B and the B_n. Hence Z_2 is countable, thus $\lambda_1(Z_1 \cup Z_2) = 0$.

Let I_n be the projection of B_n onto S. Then I_n is an open interval of length $2\,r_n$. Almost all points of S belong to **infinitely** many I_n. **Borel-Cantelli lemma** (the easy half): $\sum_{n=1}^{\infty} r_n = \infty$, *a contradiction.* □

Swiss cheese origin remark. It is often stated that **Swiss cheese** is *basically due to Mergelyan* overlooking the work of Swiss mathematician Alice Roth (Commentarii Mathematici **Helvetici**, 1938).

Approximationseigenschaften und Strahlengrenzwerte meromorpher und ganzer Funktionen

Von Alice Roth, Zollikon (Zürich)

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Approximationseigenschaften und Strahlengrenzwerte meromorpher und ganzer Funktionen

Von Alice Roth, Zollikon (Zürich)

In the article *Alice in Switzerland: the life and mathematics of Alice Roth* (Mathematical Intelligencer, 2005), the authors say:
. . . her Swiss cheese has been modified (*to an entire variety of cheeses*).

TONY'S FINE CHEESE

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A **point derivation** on $R(X)$ at $x \in X$ is a linear functional D on $R(X)$ such that

$$D(fg) = f(x)Dg + g(x)Df, \qquad f,\ g \in R(X).$$

D is called a **bounded point derivation** at x, if D is continuous.

Equivalent to: there exists $\alpha \in \mathbb{R}$ such that

$$|r'(x)| \leq \alpha \, ||r||_X, \qquad r \in R_0(X).$$

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Facts:

- There exists a Swiss cheese K such that $R(K)$ admits **no** bounded point derivations. (J. Wermer)

- There exists a Swiss cheese K (*Tony's fine cheese*) with the property that $R(K)$ admits a bounded point derivation at **exactly one point**.

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PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 39, Number 3, August 1973

AN ISOLATED BOUNDED POINT DERIVATION[1]

ANTHONY G. O'FARRELL

ABSTRACT. For a compact subset X of the plane, $R(X)$ denotes the class of uniform limits on X of rational functions with poles off X. $R(X)$ is a function algebra on X. An example X is constructed such that $R(X)$ admits a bounded point derivation at exactly one point of X.

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This result stands in marked contrast to the following result.

- For every compact $X \subset \mathbb{C}$, the set of points of X at which $R(X)$ admits a (not necessarily bounded) point derivation has **no isolated points**. (A. Browder)

ANOTHER CHEESE

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Theorem.

There exists a sequence $\{x_n\}$ *of distinct points in* $B := B(0,1) \subset \mathbb{R}^2$ *and a sequence* $\{a_n\} \in l^1$ *such that, for every bounded* ***harmonic function*** h *on* B,

$$h(0) = \sum_{n=1}^{\infty} a_n\, h(x_n).$$

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$$h(0) = \sum_{n=1}^{\infty} a_n \, h(x_n).$$

Proof.

Choose pairwise disjoint closed discs $\overline{B_n} := \overline{B}(x_n, r_n)$ in B such that $\lambda_2\big(B \setminus \bigcup_{n=1}^{\infty} \overline{B_n}\big) = 0$ (the **Vitali Covering Theorem**). Then, for $a_n := r_n^2$,

$$h(0) = (1/\pi) \int_B h \, d\lambda_2 = \sum_{n=1}^{\infty} a_n \cdot (1/\pi r_n^2) \cdot \int_{B_n} h \, d\lambda_2 = \sum_{n=1}^{\infty} a_n \, h(x_n).$$

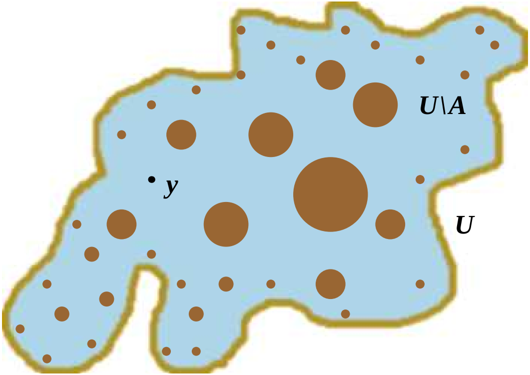
□

Remark

- a quadrature formula $\int_B h\, d\lambda_2 = \sum_{n=1}^{\infty} \pi a_n\, h(x_n)$
- an obvious modification for $\mathbb{R}^d,\ d > 2$

UNAVOIDABLE SETS

A relatively closed subset A of a connected open set $U \neq \emptyset$ in $\mathbb{R}^d$, $d \geq 2$, is **unavoidable**, if **Brownian motion** (B_t), starting in $U \setminus A$ and killed when leaving U, hits A almost surely. Equivalently, denoting $\mu_y^{U \setminus A}$ the harmonic measure at y with respect to $U \setminus A$, $\mu_y^{U \setminus A}(A) = 1$ for every $y \in U \setminus A$.



CHAMPAGNE BUBBLES

Let X be a countable subset of U having no accumulation point in U, and let $r_x > 0,\ x \in X$, be such that the closed balls $\overline{B}(x, r_x)$, the **bubbles**, are pairwise disjoint and

$$\sup_{x \in X} \big(r_x/\operatorname{dist}(x, \partial U)\big) < 1.$$

Then the union A of all $\overline{B}(x, r_x)$ is relatively closed in U.

The (connected) open set $U \setminus A$ is called **champagne subdomain** of U.

The set of centers of all bubbles forming A will be denoted by X_A.

Rosé champagne

UNAVOIDABLE SETS OF BUBBLES

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Taking $T_F(\omega) := \inf\{t \geq 0 : B_t(\omega) \in F\}$, we have: $A := \bigcup_{x \in X} \overline{B}(x, r_x)$ unavoidable if

$$\begin{aligned} P^y[T_A < T_{U^c}] &= 1 \quad \text{for some (all)} \quad y \in U \backslash A, \\ \mu_y^{U \backslash A}(A) &= 1 \quad \text{for some (all)} \quad y \in U \backslash A. \end{aligned}$$

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Question

How **small** may an unavoidable set A of bubbles be?

Investigated by J. R. Akeroyd (2002); J. Ortega-Cerdà and K. Seip (2004); T. Carroll and J. Ortega-Cerdà (2007); S. J. Gardiner and M. Ghergu (2010), J. O'Donovan (2010), J. Pres (2012). Also W. Hansen and N. Nadirashvili (1994); T. Lundh (2001); T. Carroll, J. O'Donovan and J. Ortega-Cerdà (2012).

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Let us define the (*capacity*) function $\varphi(t) := 1/\log\frac{1}{t}$, $d = 2$, $\varphi(t) := t^{d-2}$, $d > 2$, $t \in (0,1)$.

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Theorem (S. J. Gardiner and M. Ghergu; J. Pres).

If $d \geq 2$, then, for all $\varepsilon > 0$ and $\delta > 0$, there exists a champagne subdomain $B(0,1)\setminus A$ such that A is unavoidable and

$$\sum_{x \in X_A} \varphi(r_x)^{1+\varepsilon} < \delta.$$

Now we fix an **arbitrary** function $h : (0,1) \to (0,1)$ such that $\liminf_{t\to 0} h(t) = 0$.

Examples

$h(t) = \varphi^\varepsilon(t)$, $t \in (0,1)$, $\varepsilon > 0$; $h(t) = \big(\log\log \ldots \log(1/\varphi(t))\big)^{-\varepsilon}$, t small

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$h(t) = \varphi^\varepsilon(t),\ t \in (0,1),\ \varepsilon > 0;\ h(t) = \big(\log\log\ldots\log(1/\varphi(t))\big)^{-\varepsilon}$, t small

Theorem (W. Hansen and I. Netuka).

Let $U \neq \emptyset$ be a connected open set in $\mathbb{R}^d$, $d \geq 2$. Then, for every $\delta > 0$, there exists a champagne subdomain $U \setminus A$ such that A is unavoidable and

$$\sum_{x \in X_A} \varphi(r_x) h(r_x) < \delta.$$

Remark (by Gardiner/Ghergu and Pres)

The result is **optimal**. Indeed, if A is an unavoidable set of bubbles, then $\sum_{x \in X_A} \varphi(r_x) = \infty$, hence the factor $h(r_x)$ cannot be omitted.

THE KEY RESULT FOR $B(0,1)$

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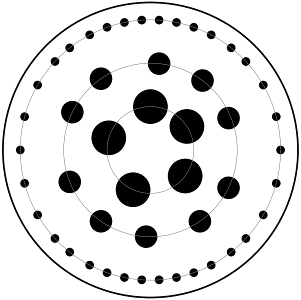
Let $d \geq 2$, $\delta > 0$, and (R_k) be a sequence in $(0,1)$ which is strictly increasing to 1. Then there exist finite sets X_k in $\partial B(0, R_k)$ and $r_k > 0$ such that, $\sum_{k=1}^{\infty} \# X_k \, \varphi(r_k) \, h(r_k) < \delta$ and, taking



$$A := \bigcup_{x \in X_k, \; k \in \mathbb{N}} \overline{B}(x, r_k),$$

the set $B(0,1) \backslash A$ is a champagne subdomain, A is unavoidable and

$$\sum_{x \in X_A} \varphi(r_x) \, h(r_x) < \delta.$$

A GENERAL CRITERION FOR UNAVOIDABLE SETS

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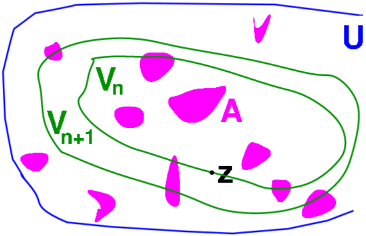
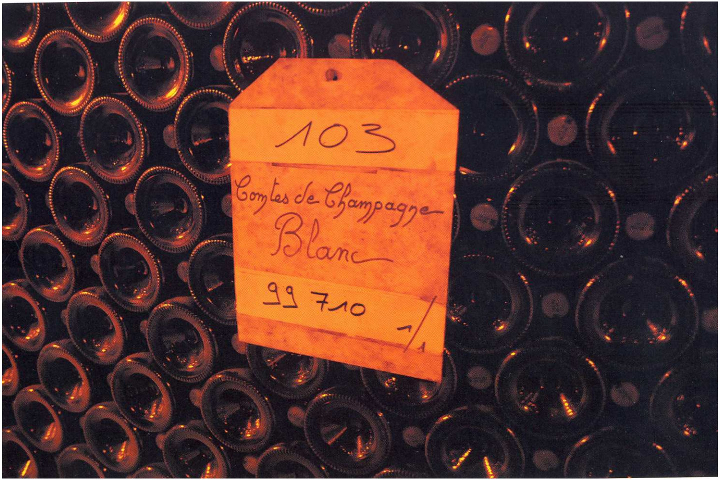
Let $A \subset U$ be relatively closed and let (V_n) be an exhaustion of U by bounded open sets ($V_n \nearrow U$, $\overline{V}_n \subset V_{n+1}$). Suppose that $\alpha_n \geq 0$ and, for all $n \in \mathbb{N}$ and $z \in \partial V_n$, $P^z[T_A < T_{V^c_{n+1}}] \geq \alpha_n$.



Then, for all $m > n$ and $z \in \overline{V_n}$, $P^z[T_A < T_{U^c}] \geq 1 - \prod_{n \leq j < m}(1 - \alpha_j)$.

In particular, A is unavoidable provided $\sum_{n=1}^{\infty} \alpha_n = \infty$.



THANK YOU!