# Bulletin of the Irish Mathematical Society

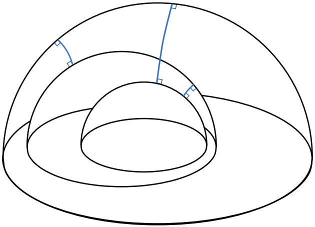
## Problems Page

We need your problems!

Please email contributions to imsproblems@gmail.com or give them to me in person.

# On a conjecture of Lorentzen and Ruscheweyh

Ian Short



Tuesday 18 June 2013

Joint work with Matthew Jacques

# Iterating analytic maps

**Definition.** Let $R$ be a Riemann surface and $f : R \to R$ an analytic map. The maps $f^n$, $n = 1, 2, \ldots$, are called the *iterates* of $f$.

Definition. Let $R$ be a Riemann surface and $f : R \to R$ an analytic map. The maps $f^n$, $n = 1, 2, \ldots$, are called the *iterates* of $f$.

You can iterate

- Möbius maps

**Definition.** Let $R$ be a Riemann surface and $f : R \to R$ an analytic map. The maps $f^n$, $n = 1, 2, \ldots$, are called the *iterates* of $f$.

You can iterate

- Möbius maps
- rational maps

Definition. Let $R$ be a Riemann surface and $f : R \to R$ an analytic map. The maps $f^n$, $n = 1, 2, \ldots$, are called the *iterates* of $f$.

You can iterate

- Möbius maps
- rational maps
- transcendental entire functions

Definition. Let $R$ be a Riemann surface and $f : R \to R$ an analytic map. The maps $f^n$, $n = 1, 2, \ldots$, are called the *iterates* of $f$.

You can iterate

- Möbius maps
- rational maps
- transcendental entire functions
- analytic maps of hyperbolic Riemann surfaces.

**Definition.** Let $f_i : R \to R$, $i = 1, 2, \ldots,$ be analytic maps.

## Repeated composition of analytic maps

**Definition.** Let $f_i : R \to R$, $i = 1, 2, \ldots$, be analytic maps. The associated *inner-composition sequence* is $F_n = f_1 \cdots f_n$ and the *outer-composition sequence* is $G_n = f_n \cdots f_1$.

Definition. Let $f_i : R \to R$, $i = 1, 2, \ldots,$ be analytic maps. The associated *inner-composition sequence* is $F_n = f_1 \cdots f_n$ and the *outer-composition sequence* is $G_n = f_n \cdots f_1$.

Contrast:

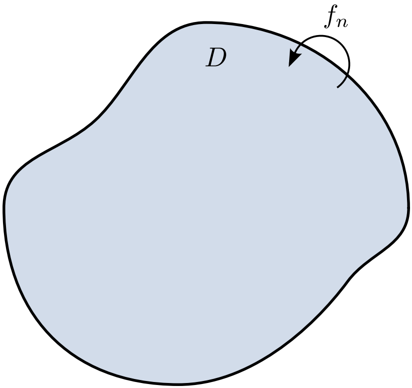
- $f_n \cdots f_1$ simpler for computations

**Definition.** Let $f_i : R \to R$, $i = 1, 2, \ldots,$ be analytic maps. The associated *inner-composition sequence* is $F_n = f_1 \cdots f_n$ and the *outer-composition sequence* is $G_n = f_n \cdots f_1$.

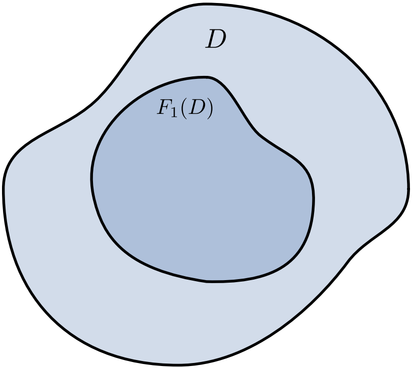
Contrast:

- $f_n \cdots f_1$ simpler for computations
- $f_1 \cdots f_n$ has useful properties.

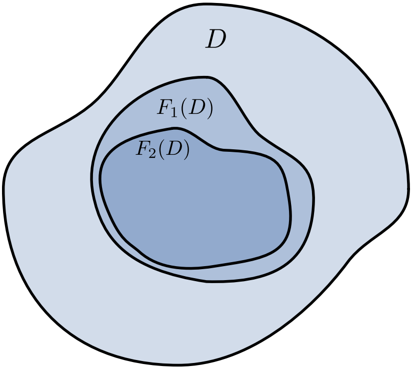
# Nested domains



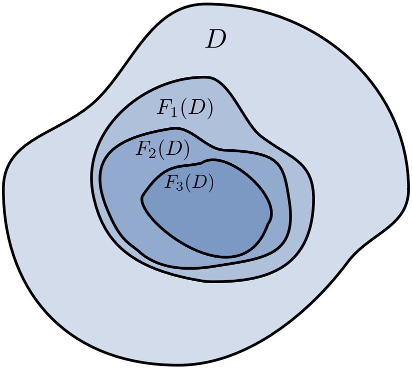
# Nested domains



# Nested domains



Nested domains



# The Schwarz–Pick lemma for hyperbolic Riemann surfaces

Theorem. Let $R$ be a hyperbolic Riemann surface with hyperbolic metric $\rho$ and let $f : R \to R$ be an analytic map.

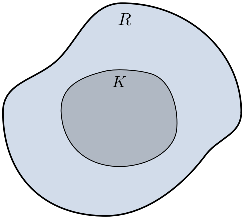
**Theorem.** Let $R$ be a hyperbolic Riemann surface with hyperbolic metric $\rho$ and let $f : R \to R$ be an analytic map. Then $\rho(f(z), f(w)) \leqslant \rho(z, w)$ for all points $z$ and $w$ in $R$.

# The Schwarz–Pick lemma for hyperbolic Riemann surfaces

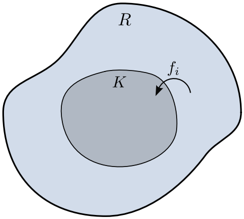
**Theorem.** Let $R$ be a hyperbolic Riemann surface with hyperbolic metric $\rho$ and let $f : R \to R$ be an analytic map. Then $\rho(f(z), f(w)) \leqslant \rho(z, w)$ for all points $z$ and $w$ in $R$.

**Corollary.** The collection of analytic maps from $R$ to itself is a normal family.

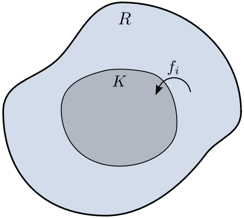
Theorem. Let $K$ be a compact subset of a hyperbolic Riemann surface $R$.



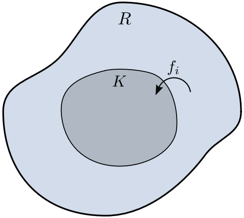
**Theorem.** Let $K$ be a compact subset of a hyperbolic Riemann surface $R$. Suppose that $f_i : R \to R$, $i = 1, 2, \ldots,$ are analytic maps such that $f_i(R) \subset K$.



**Theorem.** Let $K$ be a compact subset of a hyperbolic Riemann surface $R$. Suppose that $f_i : R \to R$, $i = 1, 2, \ldots$, are analytic maps such that $f_i(R) \subset K$. Then the sequence $F_n = f_1 \cdots f_n$ converges uniformly on compact subsets of $R$ to a constant.

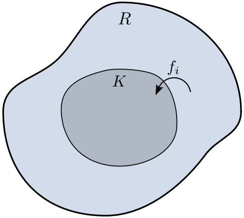


**Theorem.** Let $K$ be a compact subset of a hyperbolic Riemann surface $R$. Suppose that $f_i : R \to R$, $i = 1, 2, \ldots,$ are analytic maps such that $f_i(R) \subset K$. Then the sequence $F_n = f_1 \cdots f_n$ converges uniformly on compact subsets of $R$ to a constant.




$$\rho(f_i(z), f_i(w)) \leqslant k\rho(z, w) \qquad k < 1$$

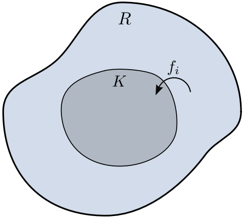
**Theorem.** Let $K$ be a compact subset of a hyperbolic Riemann surface $R$. Suppose that $f_i : R \to R$, $i = 1, 2, \ldots$, are analytic maps such that $f_i(R) \subset K$. Then the sequence $F_n = f_1 \cdots f_n$ converges uniformly on compact subsets of $R$ to a constant.




$$\rho(f_i(z), f_i(w)) \leqslant k\rho(z, w) \qquad k < 1$$

$$\rho(F_{n-1}(z), F_n(z)) \quad \leqslant \quad k\rho(f_2 \cdots f_{n-1}(z), f_2 \cdots f_n(z))$$

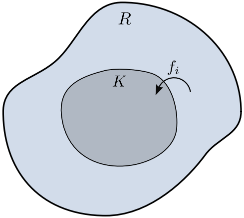
**Theorem.** Let $K$ be a compact subset of a hyperbolic Riemann surface $R$. Suppose that $f_i : R \to R$, $i = 1, 2, \ldots,$ are analytic maps such that $f_i(R) \subset K$. Then the sequence $F_n = f_1 \cdots f_n$ converges uniformly on compact subsets of $R$ to a constant.




$$\rho(f_i(z), f_i(w)) \leqslant k\rho(z, w) \qquad k < 1$$

$$\begin{aligned}\rho(F_{n-1}(z), F_n(z)) &\leqslant k\rho(f_2 \cdots f_{n-1}(z), f_2 \cdots f_n(z)) \\ &\leqslant k^2\rho(f_3 \cdots f_{n-1}(z), f_3 \cdots f_n(z))\end{aligned}$$

**Theorem.** Let $K$ be a compact subset of a hyperbolic Riemann surface $R$. Suppose that $f_i : R \to R$, $i = 1, 2, \ldots,$ are analytic maps such that $f_i(R) \subset K$. Then the sequence $F_n = f_1 \cdots f_n$ converges uniformly on compact subsets of $R$ to a constant.




$$\rho(f_i(z), f_i(w)) \leqslant k\rho(z, w) \qquad k < 1$$

$$\begin{aligned} \rho(F_{n-1}(z), F_n(z)) &\leqslant k\rho(f_2 \cdots f_{n-1}(z), f_2 \cdots f_n(z)) \\ &\leqslant k^2\rho(f_3 \cdots f_{n-1}(z), f_3 \cdots f_n(z)) \\ &\leqslant k^{n-1}\rho(z, f_n(z)) \end{aligned}$$

# Selected literature


Random iteration of Möbius transformations and Furstenberg's theorem
A. Ambroladze, H. Wallin
*Ergodic Theory Dynam. Systems, 2000*

Towers of exponents and other composite maps
I.N. Baker, P.J. Rippon
*Complex Variables Theory Appl., 1989*

Random iteration of analytic maps
A.F. Beardon, T.K. Carne, D. Minda, T.W. Ng
*Ergodic Theory Dynam. Systems, 2004*

Hyperbolic geometry from a local viewpoint
L. Keen, N. Lakic
*London Mathematical Society Student Texts, 68, 2007*

Semi-groups of analytic functions that contain the identity map
A. Kuznetsov
*Comput. Methods Funct. Theory, 2007*

## Möbius transformations

$$f(z) = \frac{az + b}{cz + d} \qquad ad - bc \neq 0$$

# Möbius transformations

$$f(z) = \frac{az+b}{cz+d} \qquad ad - bc \neq 0$$

Programme. Find the strongest results of the type described for inner-composition sequences of Möbius transformations.

# Möbius transformations

$$f(z) = \frac{az+b}{cz+d} \qquad ad - bc \neq 0$$

**Programme.** Find the strongest results of the type described for inner-composition sequences of Möbius transformations.

**Advantages.** Möbius transformations map discs to discs.

# Möbius transformations

$$f(z) = \frac{az + b}{cz + d} \qquad ad - bc \neq 0$$

**Programme.** Find the strongest results of the type described for inner-composition sequences of Möbius transformations.

**Advantages.** Möbius transformations map discs to discs. There are already strong theorems about inner-composition sequences of Möbius transformations acting on discs.

## Möbius transformations

$$f(z) = \frac{az+b}{cz+d} \qquad ad - bc \neq 0$$

Programme. Find the strongest results of the type described for inner-composition sequences of Möbius transformations.

Advantages. Möbius transformations map discs to discs. There are already strong theorems about inner-composition sequences of Möbius transformations acting on discs.

Disadvantages. Cannot use the Riemann mapping theorem.

# Möbius transformations

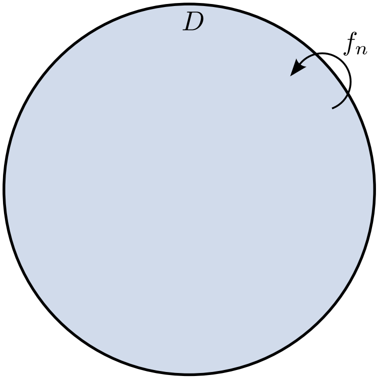
$$f(z) = \frac{az+b}{cz+d} \qquad ad - bc \neq 0$$

Programme. Find the strongest results of the type described for inner-composition sequences of Möbius transformations.

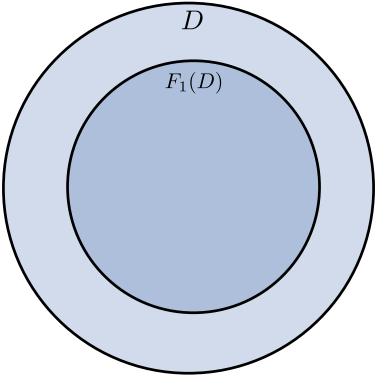
Advantages. Möbius transformations map discs to discs. There are already strong theorems about inner-composition sequences of Möbius transformations acting on discs.

Disadvantages. Cannot use the Riemann mapping theorem. There are few theorems about inner-composition sequences of Möbius transformations acting on more general domains than discs.

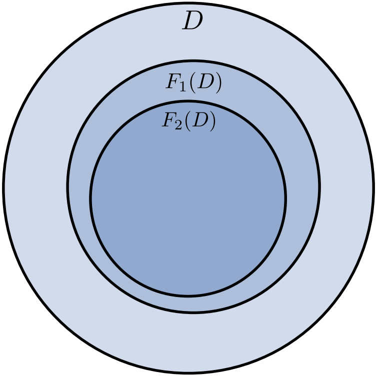
# Nested discs



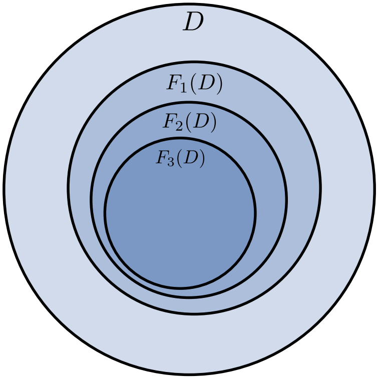
## Nested discs



# Nested discs



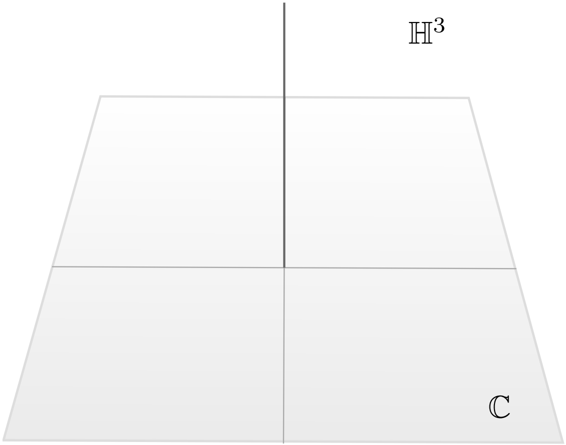
## Nested discs



## Poincaré extension (isometric action on $\mathbb{H}^3$)

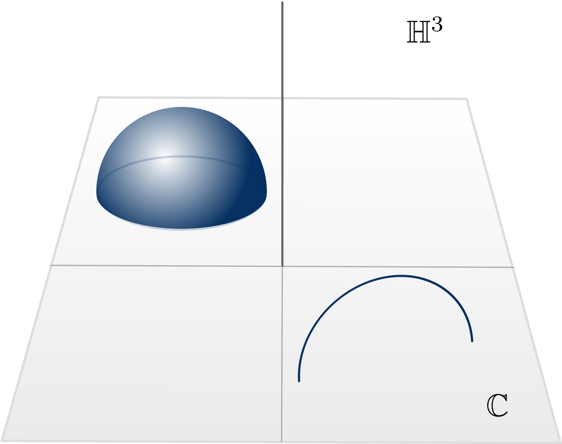

$$f(z) = \frac{az+b}{cz+d} \qquad ad - bc = 1$$

# Poincaré extension (isometric action on $\mathbb{H}^3$)



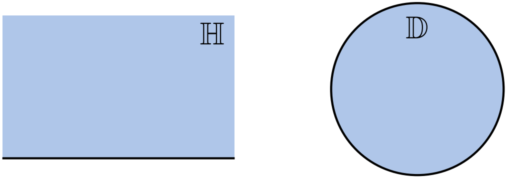

$$f(z,t) = \left( \frac{az+b}{cz+d} + \frac{|c|^2 t^2}{c(cz+d)(|cz+d|^2 + |c|^2 t^2)}, \frac{t}{|cz+d|^2 + |c|^2 t^2} \right)$$

# Poincaré extension (isometric action on $\mathbb{H}^3$)

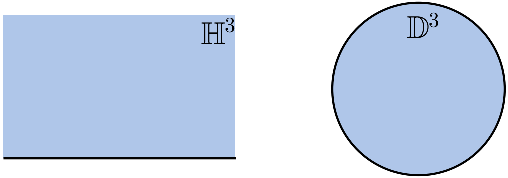


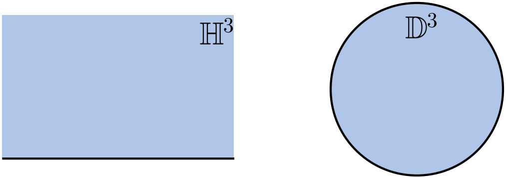

$$f(z,t) = \left( \frac{az+b}{cz+d} + \frac{|c|^2 t^2}{c(cz+d)(|cz+d|^2 + |c|^2 t^2)}, \frac{t}{|cz+d|^2 + |c|^2 t^2} \right)$$

# Hyperbolic geometry



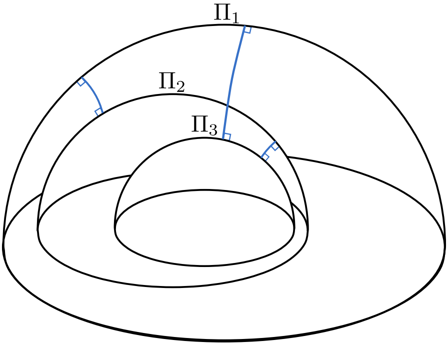
## Hyperbolic geometry





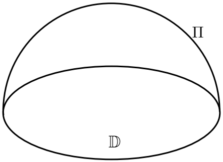

Geometric definition of completeness. A metric space is *complete* if every sequences of points of finite length converges.

## The reverse triangle inequality



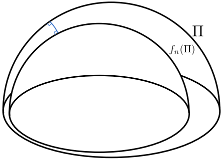

$$\rho(\Pi_1, \Pi_3) \geqslant \rho(\Pi_1, \Pi_2) + \rho(\Pi_2, \Pi_3)$$

**Theorem (Aebischer, 1990).** Let $\Pi$ be the hyperbolic plane with ideal boundary $\partial\mathbb{D}$.



**Theorem (Aebischer, 1990).** Let $\Pi$ be the hyperbolic plane with ideal boundary $\partial\mathbb{D}$. Suppose that Möbius transformations $f_1, f_2, \ldots$ satisfy $f_n(\mathbb{D}) \subset \mathbb{D}$ and

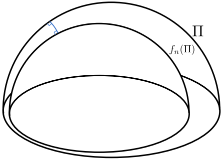
$$\sum_{n=1}^{\infty} \rho(\Pi, f_n(\Pi)) = +\infty.$$



**Theorem (Aebischer, 1990).** Let $\Pi$ be the hyperbolic plane with ideal boundary $\partial\mathbb{D}$. Suppose that Möbius transformations $f_1, f_2, \ldots$ satisfy $f_n(\mathbb{D}) \subset \mathbb{D}$ and

$$\sum_{n=1}^{\infty} \rho(\Pi, f_n(\Pi)) = +\infty.$$

Then $F_n = f_1 \cdots f_n$ converges uniformly on $\mathbb{D}$ to a constant.



**Conjecture.** Suppose that $D$ is a bounded domain in the plane, and the maps $t_n(z) = a_n/(1+z)$, $n = 1, 2, \ldots$, satisfy $t_n(D) \subset D$ for each positive integer $n$.

Conjecture. Suppose that $D$ is a bounded domain in the plane, and the maps $t_n(z) = a_n/(1+z)$, $n = 1, 2, \dots$, satisfy $t_n(D) \subset D$ for each positive integer $n$. Then $T_n = t_1 \cdots t_n$ converges uniformly on compact subsets of $D$ to a constant.

**Conjecture.** Suppose that $D$ is a bounded domain in the plane, and the maps $t_n(z) = a_n/(1+z)$, $n = 1, 2, \ldots$, satisfy $t_n(D) \subset D$ for each positive integer $n$. Then $T_n = t_1 \cdots t_n$ converges uniformly on compact subsets of $D$ to a constant.

**Conjecture (strong form).** Suppose that $D$ is a bounded domain in the plane, and the maps $t_n(z) = a_n/(1+z)$, $n = 1, 2, \ldots$, satisfy $t_n(D) \subset D$ for each positive integer $n$. Then $T_n = t_1 \cdots t_n$ converges uniformly on $D$ to a constant.

# References


Simple convergence sets for continued fractions $K(a_n/1)$
L. Lorentzen, S. Ruscheweyh
*J. Math. Anal. Appl., 1993*

Convergence criteria for continued fractions $K(a_n/1)$ based on value sets
L. Lorentzen
*Contemp. Math., 1998*

Convergence of compositions of self-mappings
L. Lorentzen
*Ann. Univ. Mariae Curie-Skłodowska Sect. A, 1999*

# Connection with continued fractions

$$\cfrac{a_1}{1+\cfrac{a_2}{1+\cfrac{a_3}{1+\cfrac{a_4}{1+\cdots}}}}$$

$$t_1(z) = \frac{a_1}{1+z}$$

# Connection with continued fractions

$$\cfrac{a_1}{1+\cfrac{a_2}{1+\cfrac{a_3}{1+\cfrac{a_4}{1+\cdots}}}}$$

$$t_1(z) = \tfrac{a_1}{1+z}$$

$$t_2(z) = \tfrac{a_2}{1+z}$$

# Connection with continued fractions

$$\cfrac{a_1}{1+\cfrac{a_2}{1+\cfrac{a_3}{1+\cfrac{a_4}{1+\cdots}}}}$$

$$t_1(z) = \tfrac{a_1}{1+z}$$
$$t_2(z) = \tfrac{a_2}{1+z}$$
$$t_3(z) = \tfrac{a_3}{1+z}$$

# Connection with continued fractions

$$\cfrac{a_1}{1+\cfrac{a_2}{1+\cfrac{a_3}{1+\cfrac{a_4}{1+\cdots}}}}$$

$$t_1(z) = \tfrac{a_1}{1+z}$$
$$t_2(z) = \tfrac{a_2}{1+z}$$
$$t_3(z) = \tfrac{a_3}{1+z}$$
$$t_4(z) = \tfrac{a_4}{1+z}$$

## Known results

**Hypotheses.** Suppose that $D$ is a bounded domain in the plane, and the maps $t_n(z) = a_n/(1+z)$, $n = 1, 2, \dots$, which are not all equal, satisfy $t_n(D) \subset D$ for each positive integer $n$. Let $T_n = t_1 \cdots t_n$.

# Known results

**Hypotheses.** Suppose that $D$ is a bounded domain in the plane, and the maps $t_n(z) = a_n/(1+z)$, $n = 1, 2, \ldots$, which are not all equal, satisfy $t_n(D) \subset D$ for each positive integer $n$. Let $T_n = t_1 \cdots t_n$.

**Theorem.** If $D$ a Euclidean disc, then $T_n$ converges uniformly on $D$ to a constant.

## Known results

**Hypotheses.** Suppose that $D$ is a bounded domain in the plane, and the maps $t_n(z) = a_n/(1+z)$, $n = 1, 2, \ldots,$ which are not all equal, satisfy $t_n(D) \subset D$ for each positive integer $n$. Let $T_n = t_1 \cdots t_n$.

**Theorem.** If $D$ a Euclidean disc, then $T_n$ converges uniformly on $D$ to a constant.

**Theorem.** For any two points $z$ and $w$ in $D$, $|T_n(z) - T_n(w)| \to 0$ as $n \to \infty$.

# Known results

**Hypotheses.** Suppose that $D$ is a bounded domain in the plane, and the maps $t_n(z) = a_n/(1+z)$, $n = 1, 2, \dots,$ which are not all equal, satisfy $t_n(D) \subset D$ for each positive integer $n$. Let $T_n = t_1 \cdots t_n$.

**Theorem.** If $D$ a Euclidean disc, then $T_n$ converges uniformly on $D$ to a constant.

**Theorem.** For any two points $z$ and $w$ in $D$, $|T_n(z) - T_n(w)| \to 0$ as $n \to \infty$.

**Theorem.** None of the maps $t_n$ fix $D$.

## Known results

**Hypotheses.** Suppose that $D$ is a bounded domain in the plane, and the maps $t_n(z) = a_n/(1+z)$, $n = 1, 2, \ldots$, which are not all equal, satisfy $t_n(D) \subset D$ for each positive integer $n$. Let $T_n = t_1 \cdots t_n$.

**Theorem.** If $D$ a Euclidean disc, then $T_n$ converges uniformly on $D$ to a constant.

**Theorem.** For any two points $z$ and $w$ in $D$, $|T_n(z) - T_n(w)| \to 0$ as $n \to \infty$.

**Theorem.** None of the maps $t_n$ fix $D$.

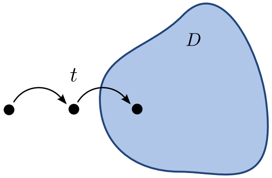
**Theorem.** If the sequence $t_n$ has at least one limit function $t$ that satisfies

$$t(\partial^* D) \cap \partial^* D = \emptyset, \quad \text{where} \quad \partial^* D = \partial D \cap (-1 - \partial D),$$

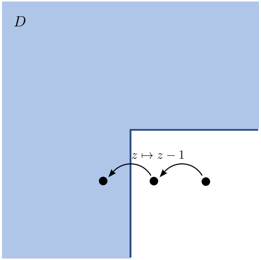
then $T_n$ converges uniformly on $D$ to a constant.

# The conjugation invariant viewpoint

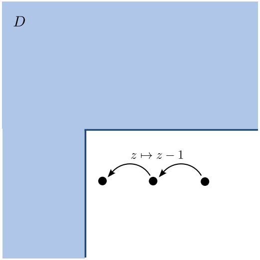
$$t(z) = \frac{a}{1+z} \qquad t(-1) = \infty \qquad t(\infty) = 0$$



**Conjecture (strong form).** Suppose that $D$ is a bounded domain in the plane, and the maps $t_n(z) = a_n/(1+z)$, $n = 1, 2, \ldots$, satisfy $t_n(D) \subset D$ for each positive integer $n$. Then $T_n = t_1 \cdots t_n$ converges uniformly on $D$ to a constant.

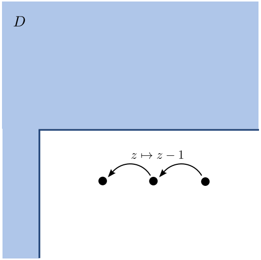


Conjecture (strong form). Suppose that $D$ is a bounded domain in the plane, and the maps $t_n(z) = a_n/(1+z)$, $n = 1, 2, \ldots,$ satisfy $t_n(D) \subset D$ for each positive integer $n$. Then $T_n = t_1 \cdots t_n$ converges uniformly on $D$ to a constant.



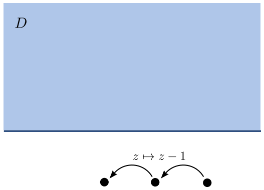
# A counterexample to the strong conjecture

**Conjecture (strong form).** Suppose that $D$ is a bounded domain in the plane, and the maps $t_n(z) = a_n/(1+z)$, $n = 1, 2, \ldots,$ satisfy $t_n(D) \subset D$ for each positive integer $n$. Then $T_n = t_1 \cdots t_n$ converges uniformly on $D$ to a constant.



## A counterexample to the strong conjecture

**Conjecture (strong form).** Suppose that $D$ is a bounded domain in the plane, and the maps $t_n(z) = a_n/(1+z)$, $n = 1, 2, \ldots$, satisfy $t_n(D) \subset D$ for each positive integer $n$. Then $T_n = t_1 \cdots t_n$ converges uniformly on $D$ to a constant.



# Digression on involutions

Let

$$t(z) = \frac{a}{1+z} \qquad \tau(z) = -1 - z.$$

# Digression on involutions

Let

$$t(z) = \frac{a}{1+z} \qquad \tau(z) = -1 - z.$$

Then

$$\tau t \tau = t^{-1}$$

# Digression on involutions

Let

$$t(z) = \frac{a}{1+z} \qquad \tau(z) = -1 - z.$$

Then

$$\tau t \tau = t^{-1}$$

Therefore

$$\tau t_1 \cdots t_n \tau = t_1^{-1} \cdots t_n^{-1}.$$

Let

$$t(z) = \frac{a}{1+z} \qquad \tau(z) = -1 - z.$$

Then

$$\tau t \tau = t^{-1}$$

Therefore

$$\tau t_1 \cdots t_n \tau = t_1^{-1} \cdots t_n^{-1}.$$

Likewise, if

$$s(z) = \frac{1}{b+z} \qquad \sigma(z) = -\frac{1}{z}$$

Let

$$t(z) = \frac{a}{1+z} \qquad \tau(z) = -1 - z.$$

Then

$$\tau t \tau = t^{-1}$$

Therefore

$$\tau t_1 \cdots t_n \tau = t_1^{-1} \cdots t_n^{-1}.$$

Likewise, if

$$s(z) = \frac{1}{b+z} \qquad \sigma(z) = -\frac{1}{z}$$

then

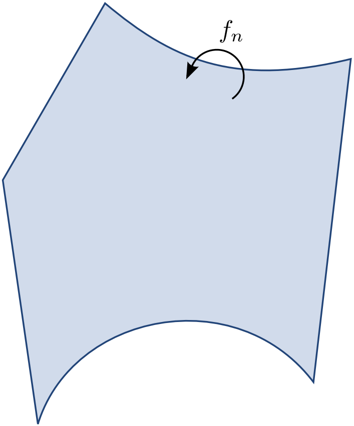
$$\sigma s \sigma = s^{-1}.$$

# Main result

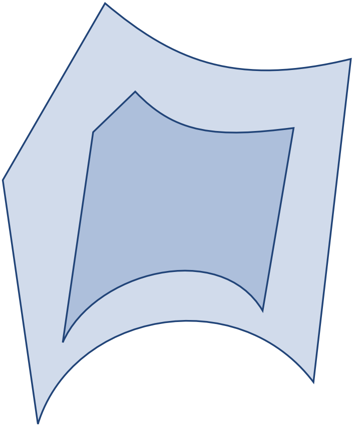
Theorem. Suppose that $D$ is a bounded *polygon* in the plane, and the maps $t_n(z) = a_n/(1+z)$, $n = 1, 2, \ldots$, satisfy $t_n(D) \subset D$ for each positive integer $n$.

Theorem. Suppose that $D$ is a bounded *polygon* in the plane, and the maps $t_n(z) = a_n/(1+z)$, $n = 1, 2, \ldots$, satisfy $t_n(D) \subset D$ for each positive integer $n$. Then $T_n = t_1 \cdots t_n$ converges uniformly on compact subsets of $D$ to a constant.

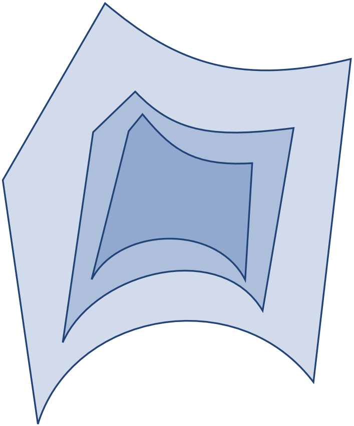
# Nested polygons



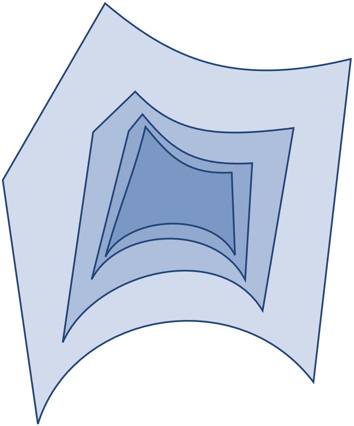
# Nested polygons



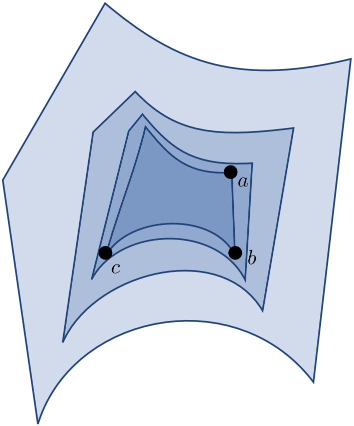
# Nested polygons



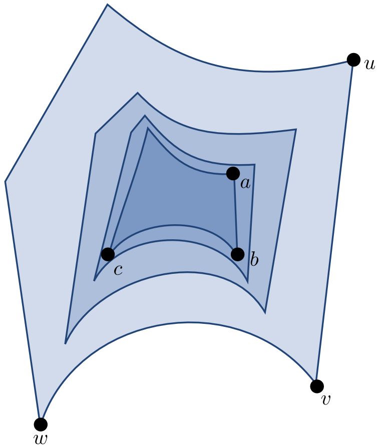
# Nested polygons



# Limit polygon



# Limit polygon



**Definition.** A sequence $F_1, F_2, \ldots$ of Möbius transformations is *restrained* if, for some point $\zeta$ in $\mathbb{D}^3$, the sequence $F_1(\zeta), F_2(\zeta), \ldots$ accumulates (in the Euclidean metric) only on the ideal boundary of $\mathbb{D}^3$.

Definition. A sequence $F_1, F_2, \ldots$ of Möbius transformations is *restrained* if, for some point $\zeta$ in $\mathbb{D}^3$, the sequence $F_1(\zeta), F_2(\zeta), \ldots$ accumulates (in the Euclidean metric) only on the ideal boundary of $\mathbb{D}^3$.

Equivalently, $\rho(F_n(\zeta), \zeta) \to \infty$ as $n \to \infty$.

**Definition.** A sequence $F_1, F_2, \ldots$ of Möbius transformations is *restrained* if, for some point $\zeta$ in $\mathbb{D}^3$, the sequence $F_1(\zeta), F_2(\zeta), \ldots$ accumulates (in the Euclidean metric) only on the ideal boundary of $\mathbb{D}^3$.

Equivalently, $\rho(F_n(\zeta), \zeta) \to \infty$ as $n \to \infty$.

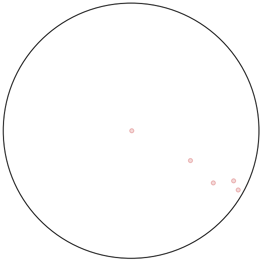
Equivalently, $\rho(\zeta, F_n^{-1}(\zeta)) \to \infty$ as $n \to \infty$.

Definition. A sequence $F_1, F_2, \ldots$ of Möbius transformations is of *convergence type* if, for some point $\zeta$ in $\mathbb{D}^3$,

$$\sum_{n=1}^{\infty} \exp[-\rho(F_n(\zeta), \zeta)] < +\infty.$$

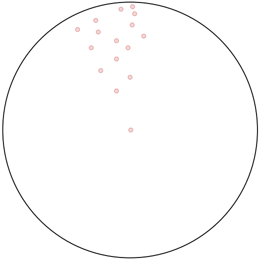
**Definition.** A sequence $F_1, F_2, \ldots$ of Möbius transformations is of *convergence type* if, for some point $\zeta$ in $\mathbb{D}^3$,

$$\sum_{n=1}^{\infty} \exp[-\rho(F_n(\zeta), \zeta)] < +\infty.$$



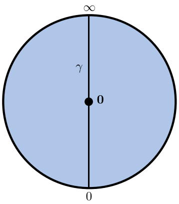
**Definition.** A sequence $F_1, F_2, \dots$ of Möbius transformations is of *convergence type* if, for some point $\zeta$ in $\mathbb{D}^3$,

$$\sum_{n=1}^{\infty} \exp[-\rho(F_n(\zeta), \zeta)] < +\infty.$$

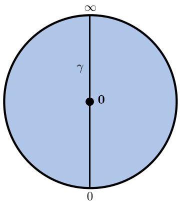


# Convergence type implies convergence

**Theorem.**

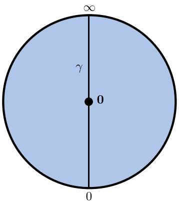


**Theorem.** Suppose that Möbius transformations $F_1, F_2, \ldots$ satisfy $F_n(\infty) = F_{n-1}(0)$ for $n = 1, 2, \ldots$.

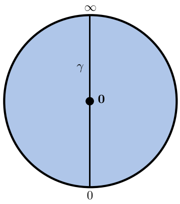


# Convergence type implies convergence

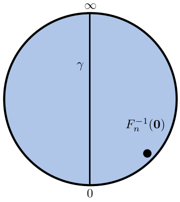
**Theorem.** Suppose that Möbius transformations $F_1, F_2, \dots$ satisfy $F_n(\infty) = F_{n-1}(0)$ for $n = 1, 2, \dots$. Suppose also that the sequence is of convergence type, and the backwards orbit $F_1^{-1}(\mathbf{0}), F_2^{-1}(\mathbf{0}), \dots$ doesn't accumulate at 0 or $\infty$.



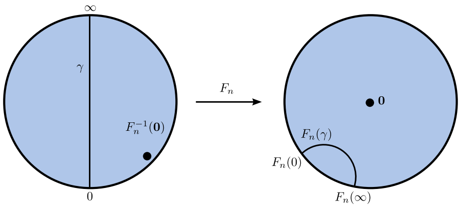
**Theorem.** Suppose that Möbius transformations $F_1, F_2, \dots$ satisfy $F_n(\infty) = F_{n-1}(0)$ for $n = 1, 2, \dots$. Suppose also that the sequence is of convergence type, and the backwards orbit $F_1^{-1}(\mathbf{0}), F_2^{-1}(\mathbf{0}), \dots$ doesn't accumulate at 0 or $\infty$. Then $F_1(0), F_2(0), \dots$ converges.



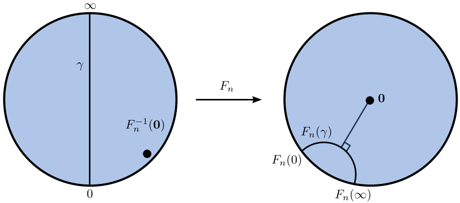
**Theorem.** Suppose that Möbius transformations $F_1, F_2, \ldots$ satisfy $F_n(\infty) = F_{n-1}(0)$ for $n = 1, 2, \ldots$. Suppose also that the sequence is of convergence type, and the backwards orbit $F_1^{-1}(\mathbf{0}), F_2^{-1}(\mathbf{0}), \ldots$ doesn't accumulate at 0 or $\infty$. Then $F_1(0), F_2(0), \ldots$ converges.



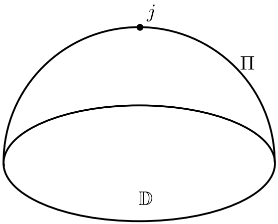
**Theorem.** Suppose that Möbius transformations $F_1, F_2, \ldots$ satisfy $F_n(\infty) = F_{n-1}(0)$ for $n = 1, 2, \ldots$. Suppose also that the sequence is of convergence type, and the backwards orbit $F_1^{-1}(\mathbf{0}), F_2^{-1}(\mathbf{0}), \ldots$ doesn't accumulate at 0 or $\infty$. Then $F_1(0), F_2(0), \ldots$ converges.



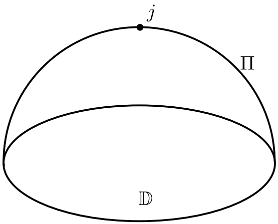
**Theorem.** Suppose that Möbius transformations $F_1, F_2, \dots$ satisfy $F_n(\infty) = F_{n-1}(0)$ for $n = 1, 2, \dots$. Suppose also that the sequence is of convergence type, and the backwards orbit $F_1^{-1}(\mathbf{0}), F_2^{-1}(\mathbf{0}), \dots$ doesn't accumulate at 0 or $\infty$. Then $F_1(0), F_2(0), \dots$ converges.



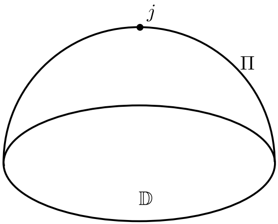
**Theorem.** Let $\Pi$ be the hyperbolic plane with ideal boundary $\partial\mathbb{D}$, and let $j = (0, 0, 1)$.



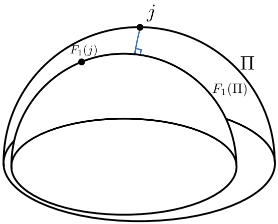
**Theorem.** Let $\Pi$ be the hyperbolic plane with ideal boundary $\partial\mathbb{D}$, and let $j = (0, 0, 1)$. Suppose that Möbius transformations $f_n$, $n = 1, 2, \ldots$, satisfy $f_n(\mathbb{D}) \subset \mathbb{D}$ and $\rho(f_n(\Pi), j) > \delta$, for some positive constant $\delta$.



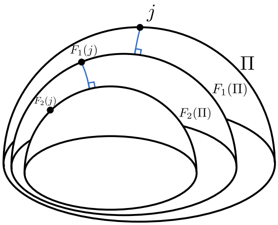
**Theorem.** Let $\Pi$ be the hyperbolic plane with ideal boundary $\partial\mathbb{D}$, and let $j = (0, 0, 1)$. Suppose that Möbius transformations $f_n$, $n = 1, 2, \ldots$, satisfy $f_n(\mathbb{D}) \subset \mathbb{D}$ and $\rho(f_n(\Pi), j) > \delta$, for some positive constant $\delta$. Then $F_n = f_1 \cdots f_n$ is of convergence type.



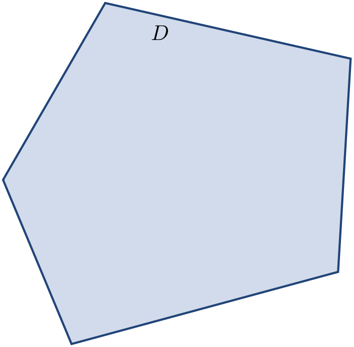
**Theorem.** Let $\Pi$ be the hyperbolic plane with ideal boundary $\partial\mathbb{D}$, and let $j = (0, 0, 1)$. Suppose that Möbius transformations $f_n$, $n = 1, 2, \ldots,$ satisfy $f_n(\mathbb{D}) \subset \mathbb{D}$ and $\rho(f_n(\Pi), j) > \delta$, for some positive constant $\delta$. Then $F_n = f_1 \cdots f_n$ is of convergence type.



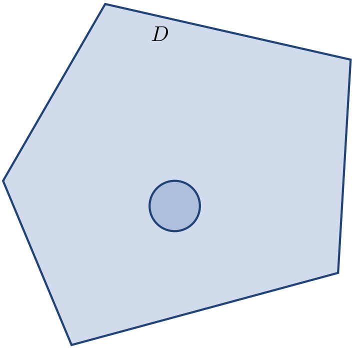
**Theorem.** Let $\Pi$ be the hyperbolic plane with ideal boundary $\partial\mathbb{D}$, and let $j = (0, 0, 1)$. Suppose that Möbius transformations $f_n$, $n = 1, 2, \ldots$, satisfy $f_n(\mathbb{D}) \subset \mathbb{D}$ and $\rho(f_n(\Pi), j) > \delta$, for some positive constant $\delta$. Then $F_n = f_1 \cdots f_n$ is of convergence type.



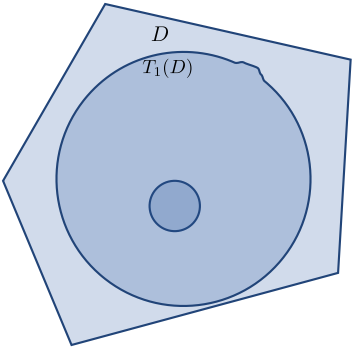
**Theorem.** Suppose that $D$ is a bounded polygon in the plane, and the maps $t_n(z) = a_n/(1+z)$, $n = 1, 2, \ldots$, satisfy $t_n(D) \subset D$ for each positive integer $n$. Then $T_n = t_1 \cdots t_n$ converges uniformly on compact subsets of $D$ to a constant.



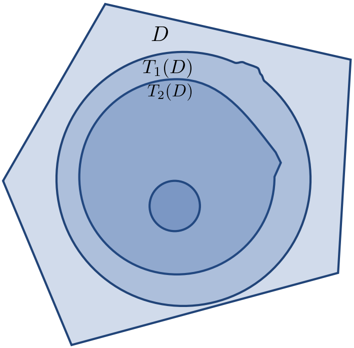
**Theorem.** Suppose that $D$ is a bounded polygon in the plane, and the maps $t_n(z) = a_n/(1+z)$, $n = 1, 2, \ldots$, satisfy $t_n(D) \subset D$ for each positive integer $n$. Then $T_n = t_1 \cdots t_n$ converges uniformly on compact subsets of $D$ to a constant.



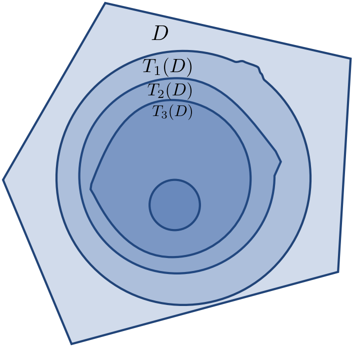
**Theorem.** Suppose that $D$ is a bounded polygon in the plane, and the maps $t_n(z) = a_n/(1+z)$, $n = 1, 2, \ldots$, satisfy $t_n(D) \subset D$ for each positive integer $n$. Then $T_n = t_1 \cdots t_n$ converges uniformly on compact subsets of $D$ to a constant.



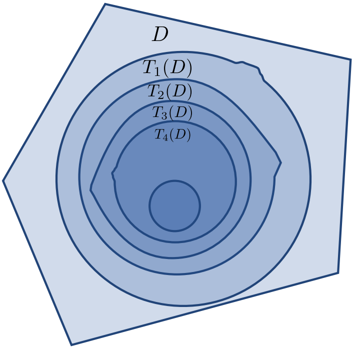
**Theorem.** Suppose that $D$ is a bounded polygon in the plane, and the maps $t_n(z) = a_n/(1+z)$, $n = 1, 2, \ldots$, satisfy $t_n(D) \subset D$ for each positive integer $n$. Then $T_n = t_1 \cdots t_n$ converges uniformly on compact subsets of $D$ to a constant.



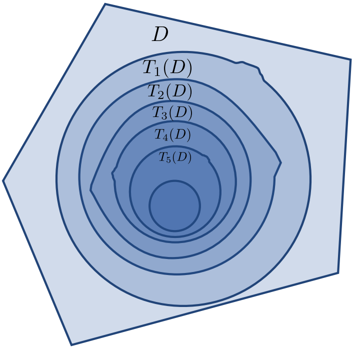
**Theorem.** Suppose that $D$ is a bounded polygon in the plane, and the maps $t_n(z) = a_n/(1+z)$, $n = 1, 2, \ldots$, satisfy $t_n(D) \subset D$ for each positive integer $n$. Then $T_n = t_1 \cdots t_n$ converges uniformly on compact subsets of $D$ to a constant.



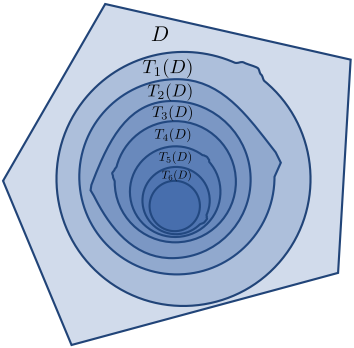
**Theorem.** Suppose that $D$ is a bounded polygon in the plane, and the maps $t_n(z) = a_n/(1+z)$, $n = 1, 2, \ldots,$ satisfy $t_n(D) \subset D$ for each positive integer $n$. Then $T_n = t_1 \cdots t_n$ converges uniformly on compact subsets of $D$ to a constant.



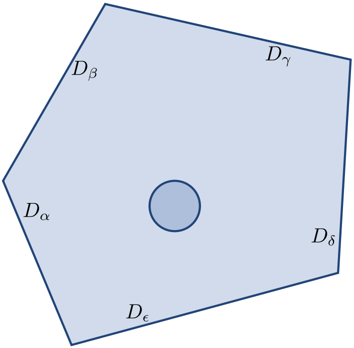
**Theorem.** Suppose that $D$ is a bounded polygon in the plane, and the maps $t_n(z) = a_n/(1+z)$, $n = 1, 2, \ldots,$ satisfy $t_n(D) \subset D$ for each positive integer $n$. Then $T_n = t_1 \cdots t_n$ converges uniformly on compact subsets of $D$ to a constant.



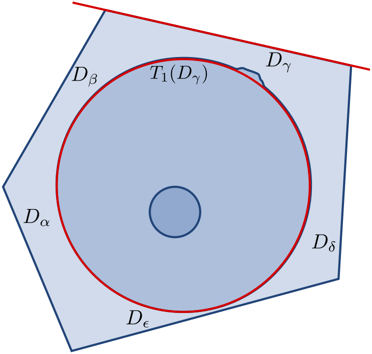
**Theorem.** Suppose that $D$ is a bounded polygon in the plane, and the maps $t_n(z) = a_n/(1+z)$, $n = 1, 2, \ldots,$ satisfy $t_n(D) \subset D$ for each positive integer $n$. Then $T_n = t_1 \cdots t_n$ converges uniformly on compact subsets of $D$ to a constant.



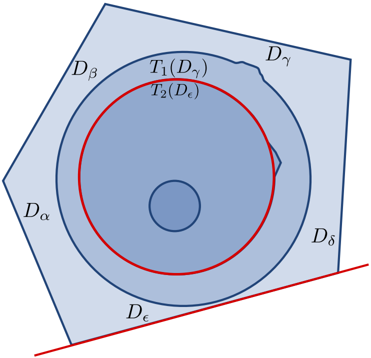
**Theorem.** Suppose that $D$ is a bounded polygon in the plane, and the maps $t_n(z) = a_n/(1+z)$, $n = 1, 2, \ldots,$ satisfy $t_n(D) \subset D$ for each positive integer $n$. Then $T_n = t_1 \cdots t_n$ converges uniformly on compact subsets of $D$ to a constant.



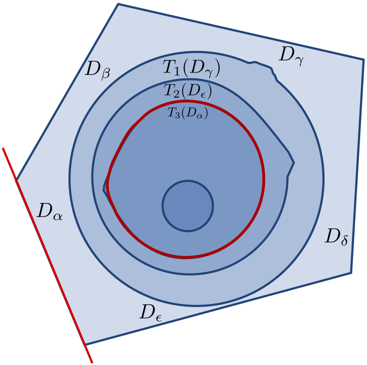
**Theorem.** Suppose that $D$ is a bounded polygon in the plane, and the maps $t_n(z) = a_n/(1+z)$, $n = 1, 2, \ldots$, satisfy $t_n(D) \subset D$ for each positive integer $n$. Then $T_n = t_1 \cdots t_n$ converges uniformly on compact subsets of $D$ to a constant.



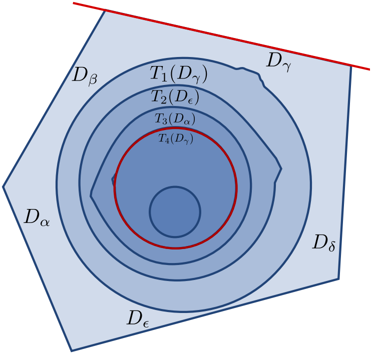
**Theorem.** Suppose that $D$ is a bounded polygon in the plane, and the maps $t_n(z) = a_n/(1+z)$, $n = 1, 2, \ldots,$ satisfy $t_n(D) \subset D$ for each positive integer $n$. Then $T_n = t_1 \cdots t_n$ converges uniformly on compact subsets of $D$ to a constant.



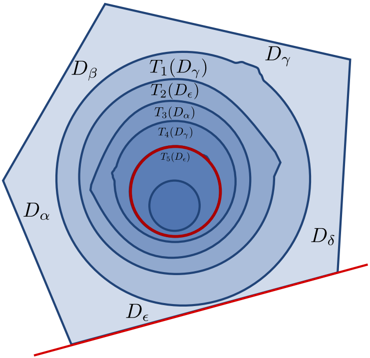
**Theorem.** Suppose that $D$ is a bounded polygon in the plane, and the maps $t_n(z) = a_n/(1+z)$, $n = 1, 2, \ldots,$ satisfy $t_n(D) \subset D$ for each positive integer $n$. Then $T_n = t_1 \cdots t_n$ converges uniformly on compact subsets of $D$ to a constant.



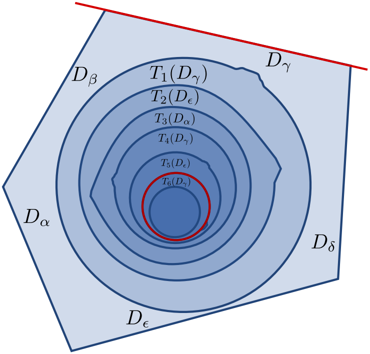
**Theorem.** Suppose that $D$ is a bounded polygon in the plane, and the maps $t_n(z) = a_n/(1+z)$, $n = 1, 2, \dots$, satisfy $t_n(D) \subset D$ for each positive integer $n$. Then $T_n = t_1 \cdots t_n$ converges uniformly on compact subsets of $D$ to a constant.



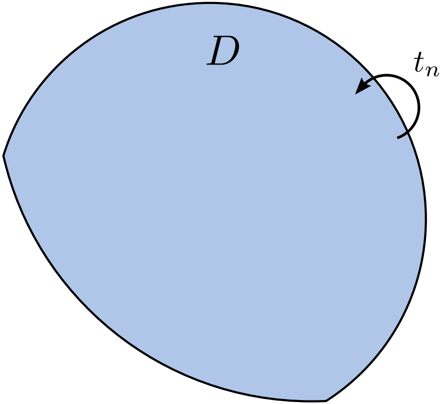
**Theorem.** Suppose that $D$ is a bounded polygon in the plane, and the maps $t_n(z) = a_n/(1+z)$, $n = 1, 2, \ldots$, satisfy $t_n(D) \subset D$ for each positive integer $n$. Then $T_n = t_1 \cdots t_n$ converges uniformly on compact subsets of $D$ to a constant.



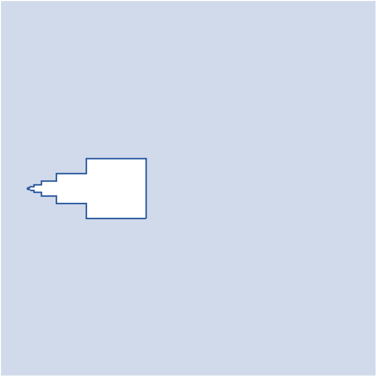
**Theorem.** Suppose that $D$ is a bounded polygon in the plane, and the maps $t_n(z) = a_n/(1+z)$, $n = 1, 2, \ldots$, satisfy $t_n(D) \subset D$ for each positive integer $n$. Then $T_n = t_1 \cdots t_n$ converges uniformly on compact subsets of $D$ to a constant.



**Example.** Theorem for polygons more general than the theorem for discs.



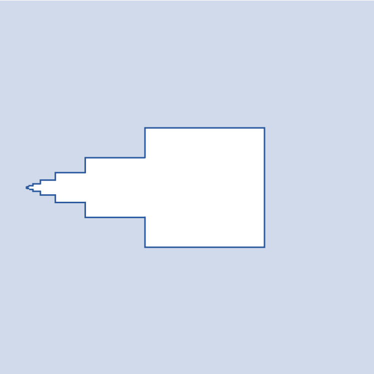
**Example.** Fail to get limit disc shape if domain insufficiently smooth.



$$f_n(z) = 2z \text{ for } n = 1, 2, \ldots$$

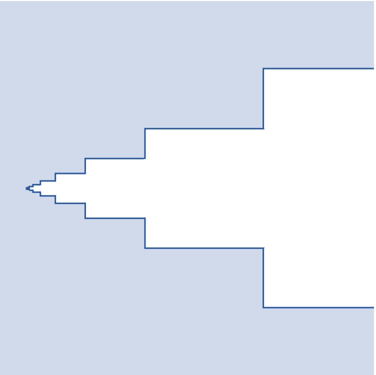
## Example – necessity of polygon

Example. Fail to get limit disc shape if domain insufficiently smooth.



$$f_n(z) = 2z \text{ for } n = 1, 2, \ldots$$

Example. Fail to get limit disc shape if domain insufficiently smooth.



$$f_n(z) = 2z \text{ for } n = 1, 2, \ldots$$

Problem 1. Does the main result hold for a more general class of maps than those of the form $t(z) = a/(1 + z)$?

Problem 2. Is the Lorentzen–Ruscheweyh conjecture true for more general domains?

Thank you for your attention.