Approximate identities in Banach function algebras

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19 June 2013

In honour of the retirement of
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Work in progress with Ali Ülger, Istanbul

Definitions

Let K be a locally compact space. Then $C_0(K)$ is the space of all complex-valued, continuous functions on K that vanish at infinity. This is a commutative Banach algebra with respect to the uniform norm $|\cdot|_K$.

A **function algebra** on K is a subalgebra of $C_0(K)$ such that, for $x, y \in K$ with $x \neq y$, there is $f \in A$ with $f(x) \neq f(y)$, and, for each $x \in K$, there is $f \in A$ with $f(x) \neq 0$.

A **Banach function algebra** on K is a function algebra A that is a Banach algebra for a norm $\|\cdot\|$, so that $\|fg\| \leq \|f\| \, \|g\|$ for $f, g \in A$.

Necessarily, $\|f\| \geq |f|_K$ for $f \in A$.

The algebra A is a **uniform algebra** if it is closed in $C_0(K)$.

Natural Banach function algebras

A Banach function algebra A on K is **natural** if every character on A has the form $f \mapsto f(x) = \varepsilon_x(f)$ for some $x \in K$. Equivalently, every maximal modular ideal has the form

$$M_x = \{f \in A : f(x) = 0\}$$

for some $x \in K$.

Every commutative, semisimple Banach algebra is a Banach function algebra on its character space.

Approximate identities

Let $(A, \|\cdot\|)$ be a natural Banach function algebra on K.

An **approximate identity** is a net (e_α) in A such that $\|fe_\alpha - f\| \to 0$ for each $f \in A$.

A **pointwise approximate identity** is a net (e_α) in A such that $e_\alpha \to 1$ pointwise on K.

Clearly, each approximate identity is a pointwise approximate identity.

These approximate identities (e_α) are **bounded** if $\sup \|e_\alpha\| < \infty$ (and the sup is the **bound**); they are **contractive** if the bound is 1. Say BAI and BPAI, CAI and CPAI.

The algebra A is **(pointwise) contractive** if every M_x has a C(P)AI.

We are interested in the best bound of BAIs.

Factorization

Let A be an algebra. Then

$$A^{[2]} = \{ab : a, b \in A\}, \quad A^2 = \mathrm{lin}\, A^{[2]}\,.$$

Cohen's factorization theorem Let A be a commutative Banach algebra. Then the following are equivalent:

(a) A has a BAI of bound m;

(b) for each $\varepsilon > 0$, each $a \in A$ can be written as $a = bc$, where $\|b\| \leq m$ and $\|a - c\| < \varepsilon$ (and so $A = A^{[2]}$, and A **factors**). $\square$

[Much more is true.]

Examples

(I) Take K compact and $A = C(K)$. Then every M_x has a CAI.

Are there any more (pointwise) contractive uniform algebras?

(II) Let G be a LCA group (e.g., $G = \mathbb{Z}$ or $G = \mathbb{R}$), and let A be the group algebra $(L^1(G), \star)$. The Fourier transform maps $L^1(G)$ onto the **Fourier algebra** $A(\Gamma)$, where Γ is the dual group to G.

Now $A(\Gamma)$ is a natural Banach function algebra on Γ. The algebra $A(\Gamma)$ always has a CAI, and all the maximal modular ideals M_γ have a BAI, of bound 2 (very standard); further, '2' is the best bound of a BAI in M_γ (less well-known; Derighetti).

More examples

Look at natural Banach function algebras on $\mathbb{I} = [0, 1]$.

(III) Take $A = \text{Lip}_{\alpha}(\mathbb{I})$, where $\alpha > 0$, so that

$$\|f\|_{\alpha} = |f|_{\mathbb{I}} + \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|^{\alpha}} \right\}$$

for $f \in A$. If $f(0) = 0$ and $f(1/n) = 1$, then $\|f\| \sim n^{\alpha}$, so there are no BPAI in M_0.

(IV) Look at $BVC(\mathbb{I})$, the algebra of continuous functions of bounded variation on $\mathbb{I}$. Here

$$\|f\|_{var} = |f|_{\mathbb{I}} + \text{var}(f)$$

for $f \in BVC(\mathbb{I})$. There is an obvious BAI in M_0 of bound 2.

The Choquet boundary

Let A be a Banach function algebra on K.

A closed subset F of K is a **peak set** if there exists a function $f \in A$ with $f(x) = 1 \ (x \in F)$ and $|f(y)| < 1 \ (y \in K \setminus F)$; in this case, f **peaks** on F; a point $x \in K$ is a **peak point** if $\{x\}$ is a peak set, and a p-**point** if $\{x\}$ is an intersection of peak sets. The set of p-points of A is $\Gamma_0(A)$, the **Choquet boundary** of A. Its closure is the **Šilov boundary**.

Theorem Let A be a Banach function algebra on K. Then M_x has a BAI $\Rightarrow x \in \Gamma_0(A)$. □

Example Let A be the disc algebra on $\overline{\mathbb{D}}$. Then $\{f \in A : f(0) = 0\}$ does not have a BAI. This is obvious anyway. □

Uniform algebras

Theorem Let A be a uniform algebra on compact K. Then the following are equivalent:

(a) $x \in \Gamma_0(A)$;

(b) M_x has a BAI;

(c) ε_x is an extreme point of the state space;

(d) M_x has a CAI.

The implication (a) $\Rightarrow$ (d): Take f that peaks at x. Then $(1_K - f^n : n \in \mathbb{N})$ is a BAI of bound 2 for M_x. One can modify the functions f^n into functions $g_n \in A$ such that $(1_K - g_n : n \in \mathbb{N})$ is a CAI for M_x. $\square$

Definition A natural uniform algebra on a compact space K is a **Cole algebra** if $\Gamma_0(A) = K$.

Important fact There are Cole algebras other than $C(K)$.

First question

Question What is the relation between the existence of a CPAI and a CAI in Banach function algebras?

Remark Suppose that there is a bounded net (e_α) in A such that $fe_\alpha \to f$ weakly for each $f \in A$. Then A has a BAI, with the same bound.

Original example – Jones and Lahr, 1977 Let $S = (\mathbb{Q}^{+\bullet}, +)$ be the semigroup of strictly positive rational numbers. The semigroup algebra $A = (\ell^1(S), \star)$ is a natural Banach function algebra on $\widehat{S}$, the collection of semi-characters on S. Then there is a sequence (n_d) in $\mathbb{N}$ such that (δ_{1/n_d}) is a CPAI for A. However, A does not have any approximate identity.

This example is not pointwise contractive. $\square$

First question - more examples

Example Let G be a LCA group that is not compact. Then the minimum bound of a BAI in a maximal ideal of $A(\Gamma)$ is 2. We can show that the minimum bound of a BPAI in a maximal ideal is

$$\frac{1}{2}(1+\sqrt{2}) > 1\,,$$

so $L^1(G)$ is not pointwise contractive. We do not know if this constant is best-possible. □

Example Let G be a LCA group that is not discrete. Let

$$A = \left\{ f \in L^1(G) : \widehat{f} \in L^1(\Gamma) \right\}\,,$$

with $\|f\| = \max\{\|f\|_1\,, \left\|\widehat{f}\right\|_1\}$. Then A is a natural Banach function algebra on Γ. It has a CPAI, but no BAI. □

First question - more examples

Example Let $B = (L^1(\mathbb{R}), \star)$. Then there is a singular measure μ_0 on $\mathbb{R}$ such that $\mu_0 \star \mu_0 \in B$ (Hewitt). Look at $A = B \oplus \mathbb{C}\mu_0$ as a closed subalgebra of $(M(\mathbb{R}), \star)$. Then A has CPAI, but no approximate identity at all because $A^2 = B$, which is not dense in A. (Hence CPAI $\not\Rightarrow$ factorization.) $\square$

Example Let A be a Banach function algebra that is reflexive as a Banach space. For example, take the set of sequences $\alpha = (\alpha_n)$ on $\mathbb{Z}^+$ such that

$$\|\alpha\| = \left(\sum_{n=0}^{\infty} |\alpha_n|^2 (1+n)^2 \right)^{1/2} < \infty \,,$$

with convolution product, giving a Banach function algebra on $\overline{\mathbb{D}}$.

Suppose that M_x has a BPAI. Then A contains the characteristic function of $K \setminus \{x\}$, and so x is isolated. Thus maximal ideals of our example do not have BPAIs. $\square$

Uniform algebras

So far we have not given a pointwise contractive Banach function algebra that is not contractive.

Fact Let A be a natural uniform algebra on K. Suppose that A has a BPAI (respectively, CPAI) that is a sequence. Then A has a BAI (respectively, CAI).

Proof Suppose that (f_n) is bounded and $f_n \to 1$ pointwise on K. Then $f_n \to 1$ weakly by the dominated convergence theorem. An earlier remark now shows that A has a BAI, with the same bound. $\square$

Feinstein's example

Example Joel Feinstein has an amazing example of a natural uniform algebra A on a compact, metrizable space K such that there is a point $x_0 \in K$ with $\Gamma_0(K) = K \setminus \{x_0\}$.

The construction is a modification of Cole's original construction.

Take $M = M_{x_0}$. Each finite F disjoint from x_0 is a peak set, and so there exists $f_F \in M$ that peaks on F. The net $\{f_F\}$ is a CPAI for M. All the other M_x have a CAI, so A is pointwise contractive. But M does not have a BAI because x_0 is not a peak point, and so A is not contractive. (However M factors.) $\square$

Second question

Question Let A be a pointwise contractive, natural Banach function algebra. Is A necessarily a uniform algebra?

Let A be a contractive, natural Banach function algebra. Is A necessarily a Cole algebra?

A special case

Definition Let S be a non-empty set. A **Banach sequence algebra** on S is a Banach function algebra A on S such that $c_{00}(S) \subset A$.

Theorem Let A be a pointwise contractive, natural Banach sequence algebra on a set S. Then $A = c_0(S)$.

This follows from a classical theorem of Bade and Curtis:

Theorem Let A be a Banach function algebra on a compact K. Suppose that there is $m > 0$ such that for disjoint, closed F and G in S, there is $f \in A$ with $|f|_F < 1/2$, $|1 - f|_G < 1/2$, and with $\|f\| \leq m$. Then $A = C(K)$. $\square$

The BSE norm

Definition Let A be a natural Banach function algebra on K. Then $L(A)$ is the closed linear span of $\{\varepsilon_x : x \in K\}$ as a subset of A', and

$$\|f\|_{BSE} = \sup\{|\langle f, \lambda\rangle| : \lambda \in L(A)_{[1]}\} \quad (f \in A).$$

The abbreviation 'BSE' stands for Bochner–Schoenberg–Eberlein because of their theorem that characterizes the Fourier–Stieltjes transforms of the bounded Borel measures on LCA groups. The BSE-norm was introduced by Takahasi and Hatori in 1990.

Clearly

$$|f|_K \leq \|f\|_{BSE} \leq \|f\| \quad (f \in A).$$

Definition Let A be a natural Banach function algebra on K. Then A **has a BSE norm** if there is a constant $C > 0$ such that

$$\|f\| \leq C\,\|f\|_{BSE} \quad (f \in A).$$

Examples of BSE norms

(1) Trivially every uniform algebra has a BSE-norm.

(2) Let G be a LCA group and $A = L^1(G)$. Then $\| \cdot \|_1 = \| \cdot \|_{BSE}$ for A.

(3) Take $\alpha > 0$, and consider $A = \mathrm{Lip}_\alpha(\mathbb{I})$ and $A = \mathrm{lip}_\alpha(\mathbb{I})$. Then $\| \cdot \|_\alpha = \| \cdot \|_{BSE}$ for A.

(4) Suppose that A is an ideal in $(A'', \Box)$ and A has a BPAI. Then A has a BSE norm. This covers Banach function algebras which are reflexive as Banach spaces, and all Banach sequence algebras on S with $c_{00}(S)$ dense.

A Banach sequence algebra with BSE norm

Here is another example of Feinstein.

Example For $\alpha = (\alpha_k) \in \mathbb{C}^{\mathbb{N}}$, say $\alpha \in A$ if

$$\|\alpha\| = |\alpha|_{\mathbb{N}} + \sup_{n \in \mathbb{N}} \frac{1}{n} \sum_{k=1}^{n} k \left|\alpha_{k+1} - \alpha_k\right| < \infty \,.$$

Then A is a natural Banach sequence algebra on $\mathbb{N}$.

Each maximal modular ideal of A has a BPAI of bound 4. However A^2 is a closed subspace of infinite codimension in A, and so A does not have any approximate identity.

Clearly $c_{00}(S)$ is not dense in A. However again $\| \cdot \|_{BSE} = \| \cdot \|$.

We note that the Banach algebra A is not separable. □

Query Do all natural Banach sequence algebras have a BSE norm?

Embarrassing fact

So far we have not actually found a Banach function algebra that does not have a BSE norm.

Specific open questions

Query Let Γ be a locally compact group, not necessarily abelian. There is a **Fourier algebra** $A(\Gamma)$. It is a natural, strongly regular Banach function algebra on Γ.

The following are equivalent: (a) Γ is amenable; (b) $A(\Gamma)$ has a BPAI; (c) $A(\Gamma)$ has a BAI; (d) $A(\Gamma)$ has a CAI. In these cases, $A(\Gamma)$ has a BSE norm.

If Γ is not amenable, does $A(\Gamma)$ have a BSE norm?

Query Let K and L be locally compact spaces, and set $V(K,L) = C_0(K)\widehat{\otimes}C_0(L)$, the **Varopoulos algebra**.

Here $\|F\|_\pi = \inf \sum_{j=1}^n \left|f_j\right|_K \left|g_j\right|_L$ taken over all representations $F = \sum_{j=1}^n f_j \otimes g_j$. This is the projective tensor norm.

We know that $c_0\widehat{\otimes}c_0$ has a BSE norm. Does $V(K,L)$ always have a BSE norm?

A theorem

Proposition Suppose that A is pointwise contractive. Then $\|f\|_{BSE} \leq 4\sqrt{2}\,|f|_K \quad (f \in A)$. $\square$

Theorem Let A be a natural Banach function algebra on K such that A has a BSE-norm.

Suppose that A is pointwise contractive. Then A is a uniform algebra.

Suppose that A is contractive. Then A is a Cole algebra.

Proof Combine some earlier results. $\square$