Approximate identities in Banach function algebras

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Definitions

Let $K$ be a locally compact space. Then $C_0(K)$ is the space of all complex-valued, continuous functions on $K$ that vanish at infinity. This is a commutative Banach algebra with respect to the uniform norm $|\cdot|_K$.

A function algebra on $K$ is a subalgebra of $C_0(K)$ such that, for $x, y \in K$ with $x \neq y$, there is $f \in A$ with $f(x) \neq f(y)$, and, for each $x \in K$, there is $f \in A$ with $f(x) \neq 0$.

A Banach function algebra on $K$ is a function algebra $A$ that is a Banach algebra for a norm $\| \cdot \|$, so that $\|fg\| \leq \|f\| \|g\|$ for $f, g \in A$.

Necessarily, $\|f\| \geq |f|_K$ for $f \in A$.

The algebra $A$ is a uniform algebra if it is closed in $C_0(K)$.
Natural Banach function algebras

A Banach function algebra $A$ on $K$ is **natural** if every character on $A$ has the form $f \mapsto f(x) = \varepsilon_x(f)$ for some $x \in K$. Equivalently, every maximal modular ideal has the form

$$M_x = \{ f \in A : f(x) = 0 \}$$

for some $x \in K$.

Every commutative, semisimple Banach algebra is a Banach function algebra on its character space.
Approximate identities

Let \((A, \| \cdot \|)\) be a natural Banach function algebra on \(K\).

An approximate identity is a net \((e_\alpha)\) in \(A\) such that \(\|fe_\alpha - f\| \to 0\) for each \(f \in A\).

A pointwise approximate identity is a net \((e_\alpha)\) in \(A\) such that \(e_\alpha \to 1\) pointwise on \(K\).

Clearly, each approximate identity is a pointwise approximate identity.

These approximate identities \((e_\alpha)\) are bounded if \(\sup \|e_\alpha\| < \infty\) (and the sup is the bound); they are contractive if the bound is 1. Say BAI and BPAI, CAI and CPAI.

The algebra \(A\) is (pointwise) contractive if every \(M_x\) has a C(P)AI.

We are interested in the best bound of BAI's.
Factorization

Let $A$ be an algebra. Then

$$A^2 = \{ab : a, b \in A\}, \quad A^2 = \text{lin } A^2.$$  

Cohen's factorization theorem Let $A$ be a commutative Banach algebra. Then the following are equivalent:

(a) $A$ has a BAI of bound $m$;

(b) for each $\varepsilon > 0$, each $a \in A$ can be written as $a = bc$, where $\|b\| \leq m$ and $\|a - c\| < \varepsilon$ (and so $A = A^2$, and $A$ factors).

[Much more is true.]
Examples

(I) Take $K$ compact and $A = C(K)$. Then every $M_x$ has a CAI.

Are there any more (pointwise) contractive uniform algebras?

(II) Let $G$ be a LCA group (e.g., $G = \mathbb{Z}$ or $G = \mathbb{R}$), and let $A$ be the group algebra $(L^1(G), \star)$. The Fourier transform maps $L^1(G)$ onto the Fourier algebra $A(\Gamma)$, where $\Gamma$ is the dual group to $G$.

Now $A(\Gamma)$ is a natural Banach function algebra on $\Gamma$. The algebra $A(\Gamma)$ always has a CAI, and all the maximal modular ideals $M_\gamma$ have a BAI, of bound 2 (very standard); further, ‘2’ is the best bound of a BAI in $M_\gamma$ (less well-known; Derighetti).
More examples

Look at natural Banach function algebras on $I = [0, 1]$.

(III) Take $A = \text{Lip}_\alpha(I)$, where $\alpha > 0$, so that

$$\|f\|_\alpha = |f|_I + \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|^\alpha} \right\}$$

for $f \in A$. If $f(0) = 0$ and $f(1/n) = 1$, then $\|f\| \sim n^\alpha$, so there are no BPAI in $M_0$.

(IV) Look at $BVC'(I)$, the algebra of continuous functions of bounded variation on $I$. Here

$$\|f\|_{var} = |f|_I + \text{var}(f)$$

for $f \in BVC'(I)$. There is an obvious BAI in $M_0$ of bound 2.
The Choquet boundary

Let $A$ be a Banach function algebra on $K$.

A closed subset $F$ of $K$ is a **peak set** if there exists a function $f \in A$ with $f(x) = 1$ ($x \in F$) and $|f(y)| < 1$ ($y \in K \setminus F$); in this case, $f$ **peaks** on $F$; a point $x \in K$ is a **peak point** if \{x\} is a peak set, and a **p-point** if \{x\} is an intersection of peak sets. The set of $p$-points of $A$ is $\Gamma_0(A)$, the **Choquet boundary** of $A$. Its closure is the **Šilov boundary**.

**Theorem** Let $A$ be a Banach function algebra on $K$. Then $M_x$ has a BAI $\Rightarrow x \in \Gamma_0(A)$. $\square$

**Example** Let $A$ be the disc algebra on $\overline{D}$. Then \{f \in A : f(0) = 0\} does not have a BAI. This is obvious anyway. $\square$
Uniform algebras

**Theorem** Let $A$ be a uniform algebra on compact $K$. Then the following are equivalent:

(a) $x \in \Gamma_0(A)$;

(b) $M_x$ has a BAI;

(c) $\varepsilon_x$ is an extreme point of the state space;

(d) $M_x$ has a CAI.

The implication (a) $\Rightarrow$ (d): Take $f$ that peaks at $x$. Then $(1_K - f^n : n \in \mathbb{N})$ is a BAI of bound 2 for $M_x$. One can modify the functions $f^n$ into functions $g_n \in A$ such that $(1_K - g_n : n \in \mathbb{N})$ is a CAI for $M_x$. $\square$

**Definition** A natural uniform algebra on a compact space $K$ is a **Cole algebra** if $\Gamma_0(A) = K$.

**Important fact** There are Cole algebras other than $C(K)$. 

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First question

**Question** What is the relation between the existence of a CPAI and a CAI in Banach function algebras?

**Remark** Suppose that there is a bounded net \((e_\alpha)\) in \(A\) such that \(fe_\alpha \to f\) weakly for each \(f \in A\). Then \(A\) has a BAI, with the same bound.

**Original example – Jones and Lahr, 1977**

Let \(S = (\mathbb{Q}^+, +)\) be the semigroup of strictly positive rational numbers. The semigroup algebra \(A = (\ell^1(S), \star)\) is a natural Banach function algebra on \(\hat{S}\), the collection of semi-characters on \(S\). Then there is a sequence \((n_d)\) in \(\mathbb{N}\) such that \((\delta_1/n_d)\) is a CPAI for \(A\). However, \(A\) does not have any approximate identity.

This example is not pointwise contractive. \(\square\)
First question - more examples

Example Let $G$ be a LCA group that is not compact. Then the minimum bound of a BAI in a maximal ideal of $A(\Gamma)$ is 2. We can show that the minimum bound of a BPAI in a maximal ideal is

$$\frac{1}{2}(1 + \sqrt{2}) > 1,$$

so $L^1(G)$ is not pointwise contractive. We do not know if this constant is best-possible. □

Example Let $G$ be a LCA group that is not discrete. Let $A = \left\{ f \in L^1(G) : \hat{f} \in L^1(\Gamma) \right\}$, with $\|f\| = \max\{\|f\|_1, \|\hat{f}\|_1\}$. Then $A$ is a natural Banach function algebra on $\Gamma$. It has a CPAI, but no BAI. □
First question - more examples

**Example** Let $B = (L^1(\mathbb{R}), \ast)$. Then there is a singular measure $\mu_0$ on $\mathbb{R}$ such that $\mu_0 \ast \mu_0 \in B$ (Hewitt). Look at $A = B \oplus \mathbb{C} \mu_0$ as a closed subalgebra of $(M(\mathbb{R}), \ast)$. Then $A$ has CPAI, but no approximate identity at all because $A^2 = B$, which is not dense in $A$. (Hence CPAI $\nRightarrow$ factorization.)

**Example** Let $A$ be a Banach function algebra that is reflexive as a Banach space. For example, take the set of sequences $\alpha = (\alpha_n)$ on $\mathbb{Z}^+$ such that

$$
\|\alpha\| = \left( \sum_{n=0}^{\infty} |\alpha_n|^2 (1 + n)^2 \right)^{1/2} < \infty,
$$

with convolution product, giving a Banach function algebra on $\mathbb{D}$.

Suppose that $M_x$ has a BPAI. Then $A$ contains the characteristic function of $K \setminus \{x\}$, and so $x$ is isolated. Thus maximal ideals of our example do not have BPAIs. □
Uniform algebras

So far we have not given a pointwise contractive Banach function algebra that is not contractive.

**Fact** Let $A$ be a natural uniform algebra on $K$. Suppose that $A$ has a BPAI (respectively, CPAI) that is a sequence. Then $A$ has a BAI (respectively, CAI).

**Proof** Suppose that $(f_n)$ is bounded and $f_n \to 1$ pointwise on $K$. Then $f_n \to 1$ weakly by the dominated convergence theorem. An earlier remark now shows that $A$ has a BAI, with the same bound. 

\[\square\]
Feinstein’s example

**Example** Joel Feinstein has an amazing example of a natural uniform algebra $A$ on a compact, metrizable space $K$ such that there is a point $x_0 \in K$ with $\Gamma_0(K) = K \setminus \{x_0\}$.

The construction is a modification of Cole’s original construction.

Take $M = M_{x_0}$. Each finite $F$ disjoint from $x_0$ is a peak set, and so there exists $f_F \in M$ that peaks on $F$. The net $\{f_F\}$ is a CPAI for $M$. All the other $M_x$ have a CAI, so $A$ is pointwise contractive. But $M$ does not have a BAI because $x_0$ is not a peak point, and so $A$ is not contractive. (However $M$ factors.)
Second question

Question Let $A$ be a pointwise contractive, natural Banach function algebra. Is $A$ necessarily a uniform algebra?

Let $A$ be a contractive, natural Banach function algebra. Is $A$ necessarily a Cole algebra?
A special case

**Definition** Let $S$ be a non-empty set. A **Banach sequence algebra** on $S$ is a Banach function algebra $A$ on $S$ such that $c_{00}(S) \subset A$.

**Theorem** Let $A$ be a pointwise contractive, natural Banach sequence algebra on a set $S$. Then $A = c_0(S)$.

This follows from a classical theorem of Bade and Curtis:

**Theorem** Let $A$ be a Banach function algebra on a compact $K$. Suppose that there is $m > 0$ such that for disjoint, closed $F$ and $G$ in $S$, there is $f \in A$ with $|f|_F < 1/2$, $|1 - f|_G < 1/2$, and with $\|f\| \leq m$. Then $A = C(K)$.  \[\Box\]
The BSE norm

**Definition** Let $A$ be a natural Banach function algebra on $K$. Then $L(A)$ is the closed linear span of $\{\varepsilon_x : x \in K\}$ as a subset of $A'$, and

$$\|f\|_{BSE} = \sup \{|\langle f, \lambda \rangle| : \lambda \in L(A)\} \quad (f \in A).$$

The abbreviation ‘BSE’ stands for Bochner–Schoenberg–Eberlein because of their theorem that characterizes the Fourier–Stieltjes transforms of the bounded Borel measures on LCA groups. The BSE-norm was introduced by Takahasi and Hatori in 1990.

Clearly

$$|f|_K \leq \|f\|_{BSE} \leq \|f\| \quad (f \in A).$$

**Definition** Let $A$ be a natural Banach function algebra on $K$. Then $A$ has a **BSE norm** if there is a constant $C > 0$ such that

$$\|f\| \leq C \|f\|_{BSE} \quad (f \in A).$$
Examples of BSE norms

(1) Trivially every uniform algebra has a BSE-norm.

(2) Let \( G \) be a LCA group and \( A = L^1(G) \). Then \( \| \cdot \|_1 = \| \cdot \|_{BSE} \) for \( A \).

(3) Take \( \alpha > 0 \), and consider \( A = \text{Lip}_\alpha(\mathbb{I}) \) and \( A = \text{lip}_\alpha(\mathbb{I}) \). Then \( \| \cdot \|_\alpha = \| \cdot \|_{BSE} \) for \( A \).

(4) Suppose that \( A \) is an ideal in \( (A'', \Box) \) and \( A \) has a BPAI. Then \( A \) has a BSE norm. This covers Banach function algebras which are reflexive as Banach spaces, and all Banach sequence algebras on \( S \) with \( c_{00}(S') \) dense.
A Banach sequence algebra with BSE norm

Here is another example of Feinstein.

Example  For $\alpha = (\alpha_k) \in \mathbb{C}^\mathbb{N}$, say $\alpha \in A$ if

$$
\|\alpha\| = |\alpha|_\mathbb{N} + \sup_{n \in \mathbb{N}} \frac{1}{n} \sum_{k=1}^{n} k |\alpha_{k+1} - \alpha_k| < \infty .
$$

Then $A$ is a natural Banach sequence algebra on $\mathbb{N}$.

Each maximal modular ideal of $A$ has a BPAI of bound 4. However $A^2$ is a closed subspace of infinite codimension in $A$, and so $A$ does not have any approximate identity.

Clearly $c_{00}(S)$ is not dense in $A$. However again $\| \cdot \|_{BSE} = \| \cdot \|$.

We note that the Banach algebra $A$ is not separable.

Query Do all natural Banach sequence algebras have a BSE norm?
Embarrassing fact

So far we have not actually found a Banach function algebra that does not have a BSE norm.
Specific open questions

Query Let $\Gamma$ be a locally compact group, not necessarily abelian. There is a Fourier algebra $A(\Gamma)$. It is a natural, strongly regular Banach function algebra on $\Gamma$.

The following are equivalent: (a) $\Gamma$ is amenable; (b) $A(\Gamma)$ has a BPAI; (c) $A(\Gamma)$ has a BAI; (d) $A(\Gamma)$ has a CAI. In these cases, $A(\Gamma)$ has a BSE norm.

If $\Gamma$ is not amenable, does $A(\Gamma)$ have a BSE norm?

Query Let $K$ and $L$ be locally compact spaces, and set $V(K, L) = C_0(K) \hat{\otimes} C_0(L)$, the Varopoulos algebra.

Here $\|F\|_\pi = \inf \sum_{j=1}^n |f_j|_K |g_j|_L$ taken over all representations $F = \sum_{j=1}^n f_j \otimes g_j$. This is the projective tensor norm.

We know that $c_0 \hat{\otimes} c_0$ has a BSE norm. Does $V(K, L)$ always have a BSE norm?
A theorem

Proposition Suppose that $A$ is pointwise contractive. Then $\|f\|_{BSE} \leq 4\sqrt{2}|f|_K$ ($f \in A$). \qed

Theorem Let $A$ be a natural Banach function algebra on $K$ such that $A$ has a BSE-norm.

Suppose that $A$ is pointwise contractive. Then $A$ is a uniform algebra.

Suppose that $A$ is contractive. Then $A$ is a Cole algebra.

Proof Combine some earlier results. \qed