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Embeddings, Carleson measures and weights

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Notation

- $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$
- $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.
- $\mathcal{H}(\mathbb{D})$ = space of all holomorphic functions in $\mathbb{D}$.
- $X \subset \mathcal{H}(\mathbb{D})$, normally a Banach space or F-space.
- μ = positive Borel measure on $\mathbb{D}$.

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General problem

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General problem Given a space X, describe those measures μ such that $X \hookrightarrow L^p(\mu)$, $p > 0$, i.e., when is there a constant $k > 0$ such that

$$\int_{\mathbb{D}} |f|^p d\mu \leq k \|f\|_X^p, \quad f \in X\,?$$

Hardy spaces

For $p > 0$ and f analytic in $\mathbb{D}$, the integral mean of order p are

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{1/p} .$$

In the case

$$\|f\|_{H^p} := \lim_{r\to 1^+} M_p(r,f) < \infty$$

we will say that $f \in \mathrm{H}^p(\mathbb{D})$, the Hardy space.

For $p = \infty$

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For $p = \infty$

$$M_\infty(r,f) = \sup_{|z|=r} |f(z)| .$$

For $p \geq 1$, $H^p(\mathbb{D})$ is a Banach space and, for $0 < p < 1$, is a F- space.

Classic Carleson measures

Let $I \subset \mathbb{T}$ be an arc of the unit circle with normalized Lebesgue measure $|I| \le 1$.
Consider the Carleson box

$$S(I) = \{z = re^{i\theta}\ : 1 - |I| \le r < 1\,,\ e^{i\theta} \in I\}\,.$$

Theorem (Carleson, 1960)

Let μ be a Borel measure on $\mathbb{D}$ and $p > 0$. The following assertions are equivalent (from now on, TFAE):

(a) $H^p \hookrightarrow L^p(\mu)$.

(b)
$$\sup_I \frac{\mu(S(I))}{|I|} < \infty\,.$$

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¡¡¡¡¡¡ The same measures for any p !!!!

Measures μ satisfying the statement in (b) are called (classic) Carleson measures.

As a consequence of Carleson theorem, Carleson measures are conformally invariant:

Corollary

Let μ be a positive Borel measure on $\mathbb{D}$. μ is a Carleson measure if and only if

$$\sup_{z\in\mathbb{D}} \int_{\mathbb{D}} \frac{1-|z|^2}{|1-z\overline{w}|^2}\, d\mu(w) < \infty\,.$$

A positive Borel measure μ on $\mathbb{D}$ is a compact Carleson measure or (vanishing Carleson measure) si

$$\lim_{|I|\to 0} \frac{\mu(S(I))}{|I|} = 0\,.$$

It is easy to see that the following assertions are equivalent:

1. μ is a compact Carleson measure.
2. The embedding $H^p \hookrightarrow L^p(\mu)$ is compact.
3. $$\lim_{|z|\to 1^-} \int_{\mathbb{D}} \frac{1-|z|^2}{|1-z\overline{w}|^2}\, d\mu(w) = 0\,.$$

The Carleson theorem was extended by P. Duren (1969) as follows:

Theorem (Duren, 1969)

Let μ be a positive Borel measure on $\mathbb{D}$ and $0 < p \leq q < \infty$. TFAE:

(a) $H^p \hookrightarrow L^q(\mu)$.

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Some authors call these measures *q/p- Carleson measures.*

We prefer to use the more precise definition of e *q-Carleson measure for $H^p(\mathbb{D})$.*

Bergman spaces

For $p > 0$, a function f, holomorphic in $\mathbb{D}$, is said to belong to the Bergman space A^p if

$$\|f\|_{A^p} := \left(\int_{\mathbb{D}} |f(z)|^p dA(z) \right)^{\frac{1}{p}} < \infty \, .$$

dA is the normalized area measure on $\mathbb{D}$, namely,

$$dA(z) = \frac{dxdy}{\pi}.$$

V. Oleinik (1974) and W.Hasting (1975) extended the Carleson result to the Bergman spaces A^p .

Theorem (Hasting-Oleinik, 1974-75)

Let μ be a positive Borel measure on $\mathbb{D}$ and $p > 0$. TFAE :

(a) $A^p \hookrightarrow L^p(\mu)$.

(b)

$$\sup_I \frac{\mu(S(I))}{|I|^2} < \infty \, .$$

D. Luecking extended the result above in several directions. One of them is using pseudo-hyperbolic discs.
Recall that if $a \in \mathbb{D}$,

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z}\,, \qquad z \in \mathbb{D}\,.$$

is the analytic automorphism on $\mathbb{D}$ that interchanges 0 with a.
For $z\,, w \in \mathbb{D}$ the pseudo-hyperbolic metric is defined as

$$\rho(z, w) = |\varphi_z(w)|$$

and for $\alpha \in \mathbb{D}$ and $0 < r < 1$ the pseudo-hyperbolic disc of (pseudo-hyperbolic) center α and (pseudo-hyperbolic) radius r is

$$\Delta(\alpha, r) = \{z \in \mathbb{D} \,:\, \rho(z, \alpha) < r\}\,.$$

Theorem (Luecking, 1982)

Let μ be a positive Borel measure on $\mathbb{D}$ and $p > 0$. TFAE:

(a) $A^p \hookrightarrow L^p(\mu)$, *i.e., there is a constant $K > 0$ such that* $\int_{\mathbb{D}} |f(z)|^p d\mu \leq K\|f\|^p_{A^p}$, *for any function $f \in \mathcal{H}(\mathbb{D})$.*

(b)

$$\sup_I \frac{\mu(S(I))}{|I|^2} < \infty .$$

(c) **For all** $r \in (0,1)$ *there is a constant C, depending only on r, such that*

$$\mu(\Delta(\alpha, r)) \leq C|\Delta(\alpha, r)| , \qquad \alpha \in \mathbb{D} .$$

(d) **There are some** $r \in (0,1)$ *and a constant C such that*

$$\mu(\Delta(\alpha, r)) \leq C|\Delta(\alpha, r)| , \qquad \alpha \in \mathbb{D} .$$

A function $f \in \mathcal{H}(\mathbb{D})$ belongs to the (standard) weighted Bergman space A^p_α si $\|f\|_{A^p_\alpha} < \infty$ where

$$\|f\|_{A^p_\alpha} = \left(\int_{\mathbb{D}} |f(z)|^p dA_\alpha(z) \right)^{1/p}$$

and $dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z)$

Theorem (Luecking, 1985)

Let $q \geq p > 0$, $\alpha > -1$ *and* μ *be a positive Borel measure on* $\mathbb{D}$. *TFAE:*

(a) $A^p_\alpha \hookrightarrow L^q(\mu)$, *i.e., there is a constant* $K > 0$ *such that*

$$\left(\int_{\mathbb{D}} |f(z)|^q d\mu \right)^{1/q} \leq K \|f\|_{A^p_\alpha}, \qquad f \in \mathcal{H}(\mathbb{D}).$$

(b)

$$\sup_I \frac{\mu(S(I))}{|I|^{(2+\alpha)\frac{q}{p}}} < \infty .$$

Theorem (Luecking, 1985)

Let $q \geq p > 0$, $\alpha > -1$, $n \in \mathbb{N}$ *and* μ *a positive Borel measure on* $\mathbb{D}$. *TFAE:*

(a) *There is a constant* $K > 0$ *such that*

$$\left(\int_{\mathbb{D}} |f^{(n)}(z)|^q d\mu \right)^{1/q} \leq K \|f\|_{A^p_\alpha}, \qquad f \in A^p_\alpha(\mathbb{D}).$$

(b)

$$\sup_I \frac{\mu(S(I))}{|I|^{(2+\alpha)\frac{q}{p}+nq}} < \infty.$$

(c) *There exists a constant* $C > 0$ *such that*

$$\mu(\Delta(a,1/2)) \leq C(1-|a|)^{(2+\alpha)\frac{q}{p}+nq}$$

for all $a \in \mathbb{D}$.

El caso $0 < q < p$

Theorem (Luecking, 1991)

Let $0 < q < p$, $\alpha > -1$ and μ be a positive Borel measure on $\mathbb{D}$. TFAE:

1. *There is a constant $C > 0$ such that $f \in A^p_\alpha$,*

$$\left(\int_{\mathbb{D}} |f^{(n)}(z)|^q d\mu(z) \right)^{1/q} \leq C \|f\|_{A^p_\alpha}, \qquad f \in A^p_\alpha$$

2. *For some $\delta_0 > 0$, the function*

$$z \mapsto \frac{\mu(\Delta(z, \delta_0))}{(1 - |z|^2)^{nq} A_\alpha(\Delta(z, \delta_0))}$$

is in $L^{\frac{p}{p-q}}(dA)$, the dual space of $L^{\frac{p}{q}}(dA)$.

Dirichlet type spaces

For $0 < p < \infty$ and $\alpha > -1$, the Dirichlet type space $\mathcal{D}^p_\alpha$ consists of all functions $f \in \mathcal{H}(\mathbb{D})$ such that $f' \in A^p_\alpha$. The norm is given by

$$\|f\|_{\mathcal{D}^p_\alpha} = |f(0)|^p + \|f'\|_{A^p_\alpha}.$$

- $p < \alpha + 1 \Rightarrow \mathcal{D}^p_\alpha = A^p_{\alpha - p}$.
- $\mathcal{D}^2_1 = H^2$.
- $\mathcal{D}^2_0 = \mathcal{D}$, the classic Dirichlet space.

Theorem (Wu, 1999)

Let $\alpha > -1$, $p \geq \alpha + 1$ and μ be a positive Borel measure on $\mathbb{D}$. μ es p-Carleson measure for $\mathcal{D}^p_\alpha$ if

(a) *(For $p \geq \alpha + 2$) and only if μ is finite.*

(b) *(For $\alpha + 1 \leq p \leq 1$) and only if*

$$\sup_I \frac{\mu(S(I))}{|I|^{\alpha+2-p}} < \infty .$$

(c) *(For $1 < p = \alpha + 1 \leq 2$) and only if*

$$\sup_I \frac{\mu(S(I))}{|I|} < \infty .$$

Corollary

Let $0 < p \leq 2$ and μ be a positive Borel measure on $\mathbb{D}$. μ is p-Carleson measure for $\mathcal{D}^p_{p-1}$ if and only if it is a classic Carleson measure.

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Wu conjectured that this corollary also holds for $p > 2$.

Corollary

Let $0 < p \leq 2$ and μ be a positive Borel measure on $\mathbb{D}$. μ is p-Carleson measure for $\mathcal{D}^p_{p-1}$ if and only if it is a classic Carleson measure.

Wu conjectured that this corollary also holds for $p > 2$.

However, D. Girela y J. A. Peláez showed that there is a function $g \in \mathcal{H}(\mathbb{D})$ such that the measure $d\mu(z) = (1 - |z|^2)^{p-1}|g'(z)|^p dA(z)$ **is not** a p-Carleson measure for $\mathcal{D}^p_{p-1}$ but **it is** a classic Carleson measure.

The embedding $\mathcal{D}^p_\alpha \subset L^q(\mu)$

Theorem (Girela-Peláez, 2006)

Suppose that $0 < p < q < \infty$, $\alpha > -1$ *and let* μ *be a positive Borel measure on* $\mathbb{D}$. *Then,* μ *is a q-Carleson measure for* $\mathcal{D}^p_\alpha$ *if*

(a) *(For* $p < \alpha + 2$*) and only if there is a constant* k *such that*

$$\mu(S(I)) \leq k|I|^{(\alpha-p+2)\frac{q}{p}}, \quad \text{for any interval } I \subset \mathbb{T}.$$

(b) *(For* $p = \alpha + 2$*) and only if there is a constant* $k > 0$ *such that*

$$\mu(S(I)) \leq k\left(\left(\log\frac{1}{|I|}\right)^{\left(\frac{1}{p}-1\right)q}\right), \quad \text{for any interval } I \subset \mathbb{T}.$$

(c) *(For* $p > \alpha + 2$*) and only if* μ *is a finite measure.*

Multipliers of Dirichlet spaces

A function $g \in \mathcal{H}(\mathbb{D})$ is a multiplier of the space $\mathcal{D}_\alpha^p$ if $g\mathcal{D}_\alpha^p \subset \mathcal{D}_\alpha^p$, i.e., if $fg \in \mathcal{D}_\alpha^p$ for all $f \in \mathcal{D}_\alpha^p$.

By closed graph theorem, g is a multiplier of $\mathcal{D}_\alpha^p$ if and only if there is a constant $C > 0$ such that

$$\|fg\|_{\mathcal{D}_\alpha^p} \leq C\|f\|_{\mathcal{D}_\alpha^p}, \qquad f \in \mathcal{D}_\alpha^p.$$

Theorem (Wu, 1999)

Let $0 < p < \infty$, $\alpha > -1$ *and* $g \in \mathcal{H}(\mathbb{D})$.

1. *If* $p > \alpha + 2$, g *is a multiplier of* $\mathcal{D}^p_\alpha$ *if and only if* $g \in \mathcal{D}^p_\alpha$.
2. *If* $p \leq \alpha + 2$, g *is a multiplier of* $\mathcal{D}^p_\alpha$ *if and only if*
   - (a) $g \in H^\infty$.
   - (b) *The measure* $d\mu_{g,p,\alpha}(z) = (1-|z|)^\alpha |g'(z)|^p dA(z)$ *is a* $p-$ *Carleson measure for* $\mathcal{D}^p_\alpha$.

Theorem (Girela and Peláez, 2006)

Suppose $g \in \mathcal{H}(\mathbb{D})$, $0 < p < q$ *and* $\beta > -1$. *If* $p - 2 < \alpha < p$, *then, TFAE:*

1. *The multiplication operator* M_g *is bounded from* $\mathcal{D}^p_\alpha$ *to* $\mathcal{D}^q_\beta$.
2. *The measure* $d\mu(z) := d\mu_{g,q,\beta}(z) = (1 - |z|^2)^\beta |g'(z)|^q dA(z)$ *satisfies that there is a constant* $k > 0$ *such that*

$$\mu(S(I)) \leq k|I|^{(\alpha-p+2)\frac{q}{p}}, \qquad \text{for any } I \subset \mathbb{T}.$$

The integration operator

For $g \in \mathcal{H}(\mathbb{D})$ we define the integration operator

$$T_g(f)(z) = \int_0^z f(s)g'(s)ds\,, \qquad z \in \mathbb{D}\,.$$

- When $g(z) = z$, T_g is the classic integration operator.
- Para $g(z) = \log \frac{1}{1-z}$, T_g is the Cesàro operator.

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General problem:

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General problem: Given two spaces X and Y, to find conditions on the symbol g so that T_g maps X in Y boundedly or compactly.

Theorem

Suppose that $g \in \mathcal{H}(\mathbb{D})$, $0 < p < q$ *and* $\beta > -1$. *If* $p - 2 < \alpha$, *then, TFAE: .*

1. *The operator* T_g *is bounded from* $\mathcal{D}^p_\alpha$ *in* $\mathcal{D}^q_\beta$.
2. *The measure* $d\mu(z) := d\mu_{g,q,\beta}(z) = (1-|z|^2)^\beta |g'(z)|^q dA(z)$ *satisfies that there is a constant* $k > 0$ *such that*

$$\mu(S(I)) \leq k|I|^{(\alpha - p + 2)\frac{q}{p}}, \quad \text{for any interval } I \subset \mathbb{T}.$$

The Bloch space

The Bloch space $\mathcal{B}$ consists of all those functions $f \in \mathcal{H}(\mathbb{D})$ such that

$$\rho_{\mathcal{B}}(z) = \sup_{z\in\mathbb{D}}(1 - |z|^2)|f'(z)| < \infty \,.$$

Endowed with the norm $\|f\|_{\mathcal{B}} \stackrel{\text{def}}{=} |f(0)| + \rho_{\mathcal{B}}(f)$, $\mathcal{B}$ is a Banach space.

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It can be seen that

$$\mathcal{B} \subset A^p_\alpha\,, \qquad p>0, \quad \alpha > -1\,.$$

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$$\mathcal{B} \subset A^p_\alpha\,, \qquad p>0, \quad \alpha > -1\,.$$

A p-Carleson measure for $\mathcal{B}$ will be a measure such that

$$\int_{\mathbb{D}} |f(z)|^p d\mu(z) \le C\|f\|^p_{\mathcal{B}}\,, \qquad f \in \mathcal{B}\,.$$

If $f \in \mathcal{B}$ then

$$|f(z)| \lesssim \log \frac{1}{1-|z|}, \qquad r \to 1^-.$$

This estimate leads us to introduce the space $H^\infty_{\log}$ of all functions $f \in \mathcal{H}(\mathbb{D})$ such that

$$M_\infty(r,f) = O\left(\log \frac{1}{1-r}\right).$$

Endowed with the norm

$$\|f\|_{H^\infty_{\log}} := \sup_{z\in\mathbb{D}} \frac{|f(z)|}{\log \frac{e}{1-|z|}},$$

$H^\infty_{\log}$ is a Banach space containing $\mathcal{B}$.

It is not difficult to see that if $0 < p < \infty$ and μ is a positive Borel measure on $\mathbb{D}$ which satisfies

$$\int_{\mathbb{D}} \left(\log \frac{1}{1-|z|} \right)^p d\mu(z) < \infty, \tag{5.1}$$

then μ is a $p-$ Carleson measure for $H^\infty_{\log}$.

Thus, such μ is a p-Carleson measure for the Bloch space $\mathcal{B}$.

Theorem (Girela, Peláez, P-G, Rättyä, 2008)

Suppose $0 < p < \infty$ *and let* μ *be a positive Borel measure on* $\mathbb{D}$.

1. *If*
$$\int_{\mathbb{D}} \left(\log \frac{1}{1-|z|}\right)^p d\mu(z) < \infty, \tag{5.2}$$
then μ *is a p-Carleson measure for* $\mathcal{B}$.
2. *If* μ *is a p-Carleson measure for* $\mathcal{B}$ *then*
$$\int_{\mathbb{D}} \left(\log \frac{1}{1-|z|}\right)^{\frac{p}{2}} d\mu(z) < \infty. \tag{5.3}$$

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$$\int_{\mathbb{D}} \left(\log \frac{1}{1-|z|}\right)^{\frac{p}{2}} d\mu(z) < \infty. \tag{5.3}$$

The natural questions are whether or not the converse of statements (5.2) and (5.3) hold.

W. Ramey and D. Ullrich proved in 1991 that there are two functions $f_1\,, f_2 \in \mathcal{B}$ such that

$$|f_1'(z)| + |f_2'(z)| \geq \frac{1}{1-|z|^2}\,, \qquad z \in \mathbb{D}\,.$$

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If we could find two functions $f_1, f_2 \in \mathcal{B}$ so that

$$|f_1(z)| + |f_2(z)| \geq \log\left(\frac{1}{1-|z|}\right), \qquad z \in \mathbb{D}.$$

then the converse of (5.2) would hold.

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$$|f_1(z)| + |f_2(z)| \geq \log\left(\frac{1}{1-|z|}\right)\,, \qquad z \in \mathbb{D}\,.$$

then the converse of (5.2) would hold.
However, this is not possible since it would get into contradiction with the estimate

$$M_p(r,f) = O\left(\left(\log\frac{1}{1-r}\right)^{1/2}\right)\,, \quad r \to 1\,.$$

which holds for any $f \in \mathcal{B}$ (Clunie-MacGregor (1984), Makarov (1985)).

Carleson measures for $H^\infty_{\log}$

Theorem (Girela, Peláez, P-G, Rättyä, 2008)

There are two functions $f_1, f_2 \in H^\infty_{\log}$ *such that*

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Theorem (Girela, Peláez, P-G, Rättyä, 2008)

Let $0 < p < \infty$ *and* μ *be a positive Borel measure on* $\mathbb{D}$. μ *is a p-Carleson measure for* $H^\infty_{\log}$ *if and only if*

$$\int_{\mathbb{D}} \left(\log \frac{1}{1-|z|}\right)^p d\mu(z) < \infty.$$

Theorem (Girela, Peláez, P-G, Rättyä, 2008)

Suppose that $0 < p < \infty$, $0 < r < 1$ *and* μ *is a positive Borel measure on* $\mathbb{D}$ *with the property of that there is a constant* $A > 0$ *such that*

$$\mu(\Delta(\lambda_1, r)) \leq A\mu(\Delta(\lambda_2, r))$$

for any pair of points λ_1 *and* $\lambda_2 \in \mathbb{D}$ *with* $|\lambda_1| = |\lambda_2|$. *Then,* μ *is a p-Carleson measure for* $\mathcal{B}$ *if and only if*

$$\int_{\mathbb{D}} \left(\log \frac{1}{1-|z|}\right)^{\frac{p}{2}} d\mu(z) < \infty.$$

Theorem (Girela, Peláez, P-G, Rättyä, 2008)

Let $0 < p < \infty$ *and* $\{a_k\}_{k=1}^{\infty}$ *be a sequence of positive numbers. Set*

$$\mu = \sum_{k=1}^{\infty} a_k \delta_{1-e^{-k}} ,$$

where δ_{z_k} *represents the point mass concentrated at* z_k*. TFAE*

1. $\sum_{k=1}^{\infty} a_k k^p < \infty$.
2. $\int_{\mathbb{D}} \left(\log \frac{1}{1-|z|}\right)^p d\mu(z) < \infty$.
3. μ *is a p-Carleson measure for the Bloch space* $\mathcal{B}$*.*

For the Hilbert space $L^2(\mathbb{D})$, the orthogonal projection on the subspace $A^2(\mathbb{D})$ is

$$(Pf)(z) = \int_{\mathbb{D}} \frac{f(\zeta)}{(1-\overline{\zeta}z)^2} dA(\zeta) \,.$$

This operator is linear and bounded with reproducing kernel

$$K_z(\zeta) = \frac{1}{(1-\overline{\zeta}z)^2} \,, \qquad \zeta \in \mathbb{D} \,.$$

For the standard weighted Bergman spaces the situation is similar. Indeed, if $\eta > -1$, the operator

$$P_\eta f(z) = \int_{\mathbb{D}} \frac{f(\zeta)}{(1-\overline{\zeta} z)^{2+\eta}} \, dA_\eta(\zeta) \,,$$

is the Bergman projection of $L^p_\eta(\mathbb{D})$ on A^p_η, which is bounded.

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The framework, natural for the Hilbert setting, also holds for $p > 1$.

Still more: ! The projection also holds for other weights.

A weight ω in $\mathbb{D}$ is a function $\omega : \mathbb{D} \to (0, \infty)$ that is integrable, that is, $\omega \in L^1(\mathbb{D})$.

Definition

Let $1 < p_0, p_0' < \infty$ such that $\frac{1}{p_0} + \frac{1}{p_0'} = 1$, and let $\eta > -1$. A weight ω satisfies the $B_{p_0}(\eta)$ condition of Bekollé-Bonami, denoted by $\omega \in B_{p_0}(\eta)$, if there is a constant $C = C(p_0, \eta, \omega) > 0$ such that

$$\left(\int_{S(I)} \omega(z) dA_\eta(z)\right) \left(\int_{S(I)} \omega(z)^{-\frac{p_0'}{p_0}} dA_\eta(z)\right)^{\frac{p_0}{p_0'}} \leq C A_\eta(S(I))^{p_0}$$

for any interval $I \subset \mathbb{T}$.

Theorem (Bekollé-Bonami, 1981/82)

Let ω be a weight on $\mathbb{D}$, $p_0 > -1$ and $\eta > -1$. TFAE:

1. *P_η is a bounded linear operator from $L^{p_0}(\omega)$ on $A^{p_0}_\omega$.*
2. *The sublinear operator*

$$\tilde{P}_\eta f(z) = \int_{\mathbb{D}} \frac{|f(\zeta)|}{|1-\bar{\zeta}z|^{2+\eta}} dA_\eta(z)$$

   is bounded from $L^{p_0}(\omega)$ on $A^{p_0}_\omega$.

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Bekollé-Bonami weights play in Bergman space theory a role similar to that the Muckenhoupt $A(p)$ classes do in harmonic analysis.

For a Bekollé-Bonami weight ω, and given $0 < p, q < \infty$ and $n = 0, 1, 2, \ldots$, Olivia Constantin (2010) has described the finite positive Borel measures on $\mathbb{D}$ such that

$$\left(\int_{\mathbb{D}} |f^{(n)}|^q d\mu\right)^{1/q} \leq k\|f\|_{A^p_\omega},$$

where

$$\|f\|^p_{A^p_\omega} = \int_{\mathbb{D}} |f(z)|^p \omega(z) dA(z).$$

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Before results, notation: If $\lambda \in \mathbb{D}$ and $\alpha \in (0, 1)$, then

$$D_{\lambda,\alpha} = \{z \in \mathbb{D} : |z - \lambda| < \alpha(1 - |\lambda|)\} .$$

Theorem (O. Constantin, 2010 $(q \geq p)$)

Let ω be a weight such that $\frac{\omega(z)}{(1-|z|^2)^\eta} \in B_{p_0}(\eta)$ for some $p_0 > 0$ and some $\eta > -1$. Let μ be a finite, positive Borel measure on $\mathbb{D}$ and suppose that $q \geq p > 0$ and $n \in \mathbb{N}$. Then, there is a constant $C > 0$ such that

$$\left(\int_{\mathbb{D}} |f^{(n)}(z) d\mu(z) \right)^{1/q} \leq C \|f\|^p_{A^p_\omega} , \qquad f \in A^p_\omega ,$$

if and only if μ satisfies

$$\mu(D_{\lambda,\alpha}) \leq C'(1 - |\lambda|^2)^{nq} \left(\omega(D_{\lambda,\alpha})\right)^{q/p} ,$$

where C' is a positive constant that does not depend on λ and $\alpha \in (0,1)$. As usual, $\int_{D_{\lambda,\alpha}} \omega(z) dA(z) = \omega(D_{\lambda,\alpha})$.

Theorem (O. Constantin, 2010 $(q < p)$)

Let ω be a weight such that $\frac{\omega(z)}{(1-|z|^2)^\eta} \in B_{p_0}(\eta)$ for some $p_0 > 0$ and some $\eta > -1$. Let μ be a finite, positive Borel measure on $\mathbb{D}$ and suppose that $q \geq p > 0$ and $n \in \mathbb{N}$. Then, there is a constant $C > 0$ such that

$$\left(\int_{\mathbb{D}} |f^{(n)}(z) d\mu(z) \right)^{1/q} \leq C \|f\|_{A^p_\omega}^p , \qquad f \in A^p_\omega ,$$

if and only if the function

$$\lambda \in \mathbb{D} \mapsto \frac{\mu(D_{\lambda,\alpha})}{(1-|\lambda|^2)^{nq}\omega(D_{\lambda,\alpha})}$$

belongs to $L^{\frac{p}{p-q}}(\omega)$ for some $\alpha \in (0,1)$.

Peláez-Rättyä weights

Suppose that $\omega : [0,1) \to (0,\infty)$ is a radial weight. Its distortion function is

$$\psi_\omega(r) = \frac{1}{\omega(r)} \int_r^1 \omega(s)ds\,, \qquad 0 \le r < 1\,.$$

ω is said to be regular if

$$\psi_\omega(r) \asymp (1-r)\,, \qquad 0 \le r < 1\,.$$

The class of regular weights is denoted by $\mathcal{R}$.

If $\omega \in \mathcal{R}$, then for any $s \in [0,1)$ there is a constant $C = C(s,\omega) > 1$ such that

$$\frac{1}{C}\omega(t) \le \omega(r) \le C\omega(t)\,, \qquad 0 \le r \le t \le r + s(1-r) < 1\,. \tag{7.1}$$

This implies that

$$\psi_\omega(r) \ge C(1-r)\,, \qquad 0 \le r < 1\,, \tag{7.2}$$

for some constant $C = C(\omega) > 0$.
But (7.1) does not imply that there is a constant such that

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The regularity of a radial weight is jointly characterized by (7.1) and (7.3).

Examples:

- $\omega(z) = (1 - |z|^2)^\alpha\,, \quad \alpha > -1.$
- $\omega(r) = (1-r)^\alpha \left(\log \frac{e}{1-r}\right)^\beta\,, \qquad \alpha > -1\,, \quad \beta \in \mathbb{R}.$
- $\omega(r) = \left(\beta \left(\log \frac{e}{1-r}\right)^\alpha\right)\,, \qquad 0 < \alpha \leq 1\,, \quad \beta > 0.$
- $\omega(r) = \left(\log \log \frac{e}{1-r}\right)^\alpha\,, \qquad \alpha > 0.$

Proposition

If $\omega \in \mathcal{R}$ then for each $p_0 > 1$ there is a $\eta = \eta(\omega, p_0) > -1$ such that $\frac{\omega(z)}{(1-|z|)^\eta} \in B_{p_0}(\eta)$.

Rapidly increasing weights

A radial weight ω is rapidly increasing $\omega \in \mathcal{I}$, si

$$\lim_{r\to 1^-} \frac{\psi_\omega(r)}{1-r} = \infty\,.$$

Typical example:

$$v_\alpha(r) = \left((1-r)\left(\log\frac{e}{1-r}\right)^\alpha\right)^{-1}, \qquad \alpha > 1\,.$$

Proposition

Si $\omega \in \mathcal{I}$ entonces $A^p_\omega \subset A^p_\beta$ para todo $\beta > -1$.

Theorem (Peláez - Rättyä, 2013)

Let $0 < p \leq q < \infty$, $\omega \in \mathcal{I} \cup \mathcal{R}$ *and* μ *a positive Borel measure on* $\mathbb{D}$.

1. μ *is a* $q-$ *Carleson measure for* A^p_ω *if and only if*

$$\sup_{I \subset \mathbb{T}} \frac{\mu(S(I))}{(\omega(S(I)))^{q/p}} < \infty .$$

2. *If* μ *is a* $q-$ *Carleson measure for* A^p_ω *then the identity operator* $I_d : A^p_\omega \to L^q(\mu)$ *satisfies that*

$$\|I_d\|_{(A^p_\omega, L^q(\mu))} = \sup_{I \subset \mathbb{T}} \frac{\mu(S(I))}{(\omega(S(I)))^{q/p}} .$$

3. *The operator* $I_d : A^p_\omega \to L^q(\mu)$ *is compact if and only if*

$$\lim_{|I| \to 0} \frac{\mu(S(I)}{(\omega(S(I)))^{q/p}} = 0 .$$

The proof of this result is strongly based on properties of the Hörmander -type maximal function

$$M_\omega(\varphi)(z) = \sup_{\{I : z \in S(I)\}} \frac{1}{\omega(S(I))} \int_{S(I)} |\varphi(\zeta)|\omega(\zeta)dA(\zeta)\,, \qquad z \in \mathbb{D}\,.$$

Function φ needs to belong to $L^1(\omega)$ and should be 2π-periodic with respect to θ for any $r \in (0,1)$.

As said at the beginning for the Hardy space $H^2(\mathbb{D})$ are equivalent:

1. $$\|f\|_{L^2(\mu)} \lesssim \|f\|_{H^2(\mathbb{D})}, \qquad f \in H^2(\mathbb{D}). \tag{8.1}$$

2. $$\sup_I \frac{\mu(S(I))}{|I|} < \infty.$$

So estimate (8.1) says that the embedding

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is continuous.
Recently, Lefèvre, Li, Queffélec,Rodriguez-Piazza (2012) studied when such embedding has closed range. Equivalently, when

$$\|f\|_{H^2(\mathbb{D})} \lesssim \|f\|_{L^2(\mu)}, \qquad f \in H^2(\mathbb{D})?$$

Theorem (Lefèvre, Li, Queffélec y Rodríguez-Piazza, 2012)

The norms $\|\cdot\|_{H^2(\mathbb{D})}$ *and* $\|\cdot\|_{L^2(\mu)}$ *are equivalent in* $H^2(\mathbb{D})$ *if and only if*

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For most spaces, it is an open problem.

To Prof. Anthony G. O'Farrell

with my high regard and gratitude