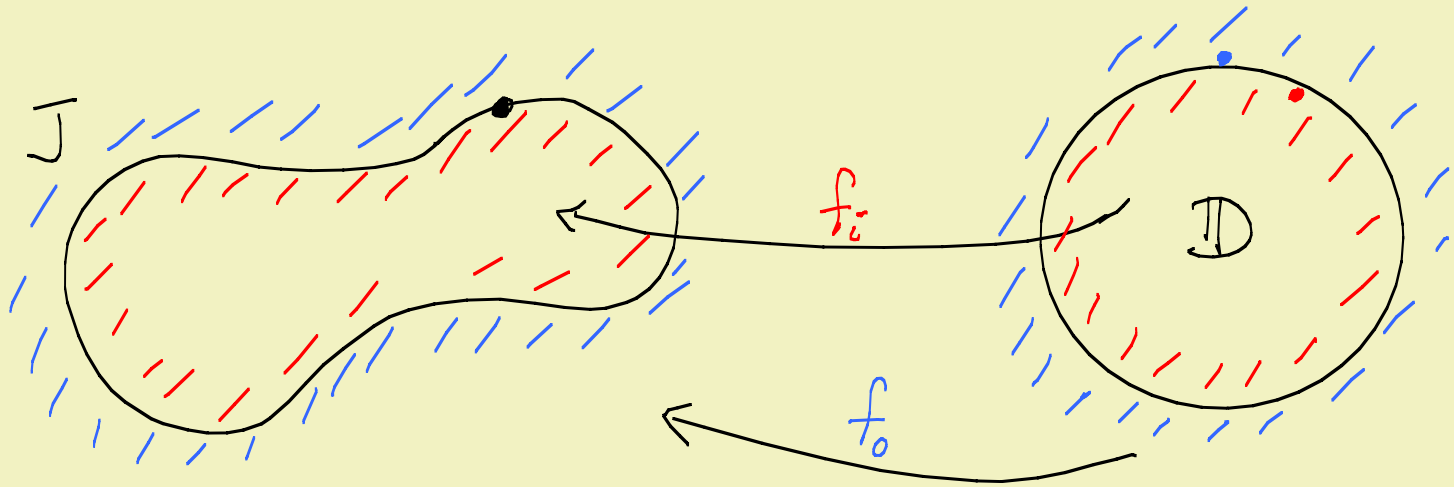
Conformal Welding

Maynooth June 18, 2013

CONFORMAL WELDING

D. MARSHALL & S. ROHDE



$$h = f_o^{-1} \circ f_i : \partial\mathbb{D} \to \partial\mathbb{D}$$

CONTINUOUS INCREASING HOMEOMORPHISM

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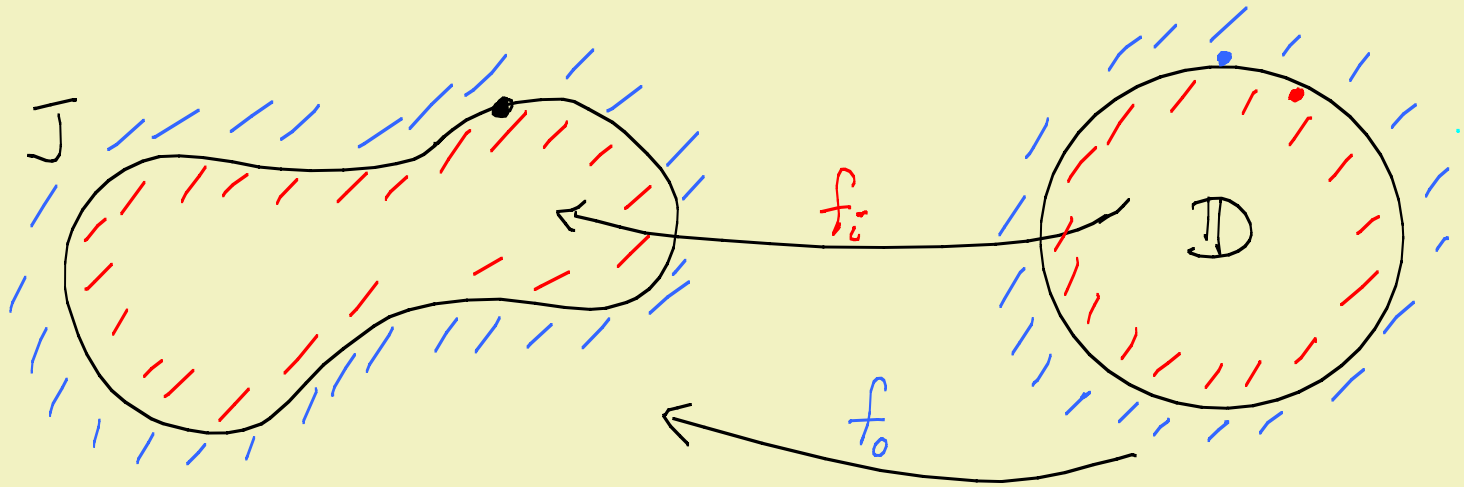

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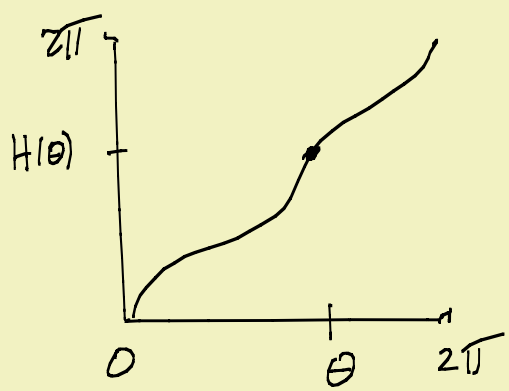
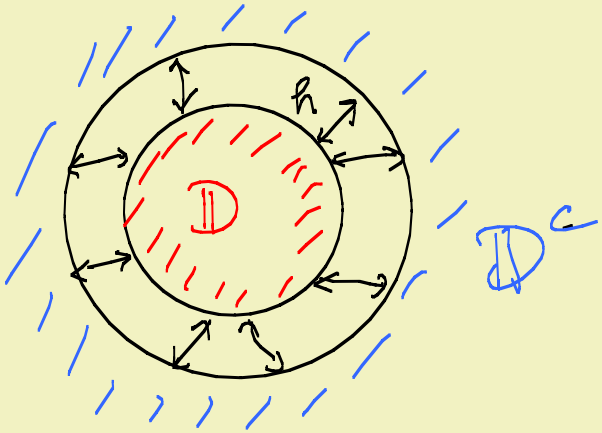
CONTINUOUS INCREASING HOMEOMORPHISM



$$h(e^{i\theta}) = e^{iH(\theta)}$$

VIEWPOINT: h "WELDS" INSIDE & OUTSIDE CIRCLES (WELDING MAP)



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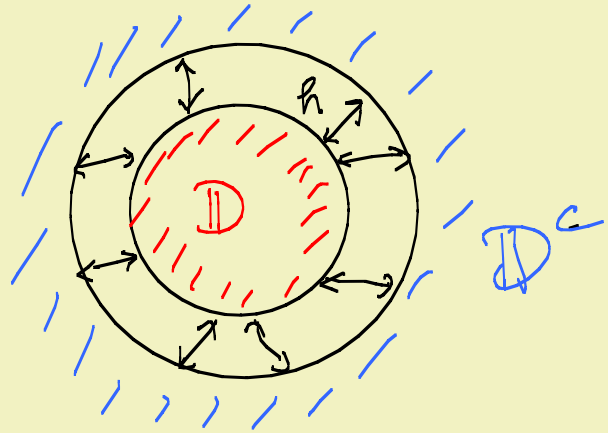
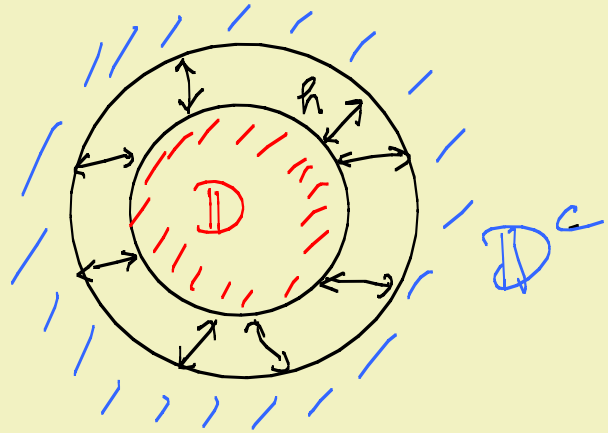


$\boxed{\text{WELDING PROBLEM:}}$ GIVEN INCREASING HOMEOMORPHISM h
FIND f_i & f_o & J

VIEWPOINT: h "WELDS" INSIDE & OUTSIDE CIRCLES (WELDING MAP)



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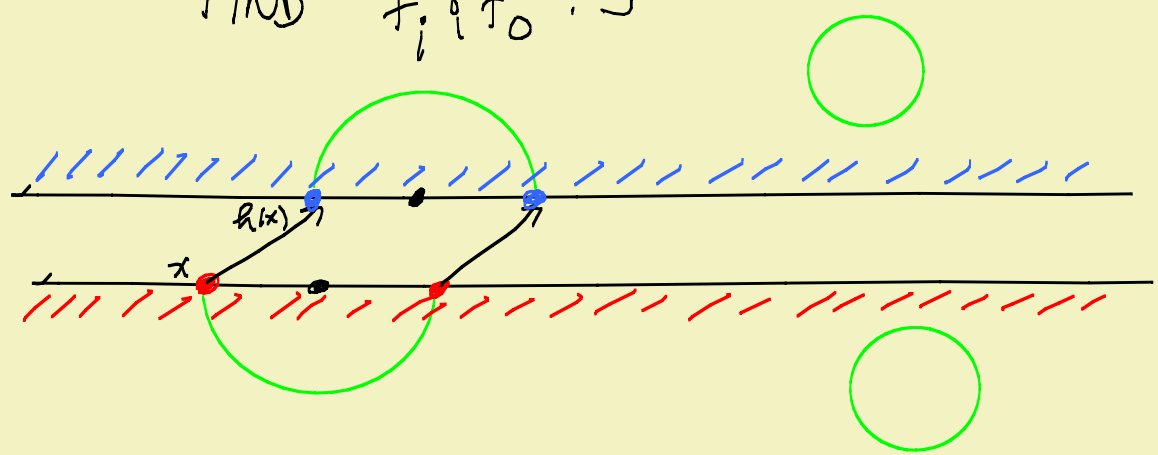
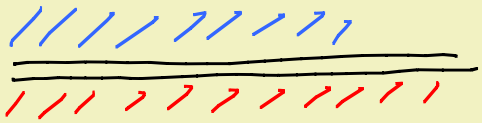
FIND f_i & f_o & J

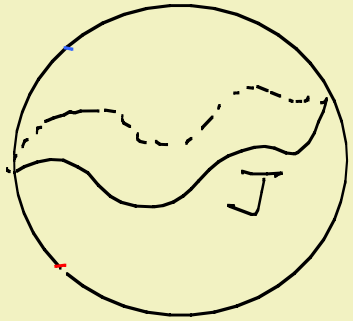
ON $\mathbb{R}$

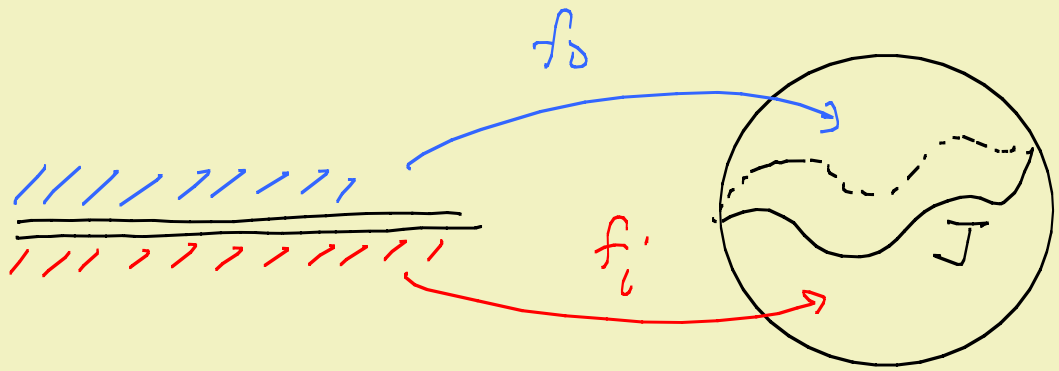


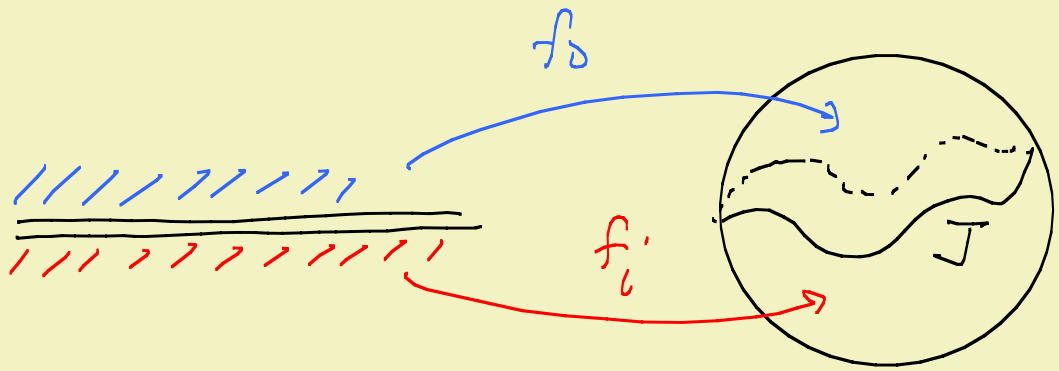
FORM RIEMANN SURFACE (GREEN "CHARTS")

CONFORMALLY EQUIVALENT TO SPHERE

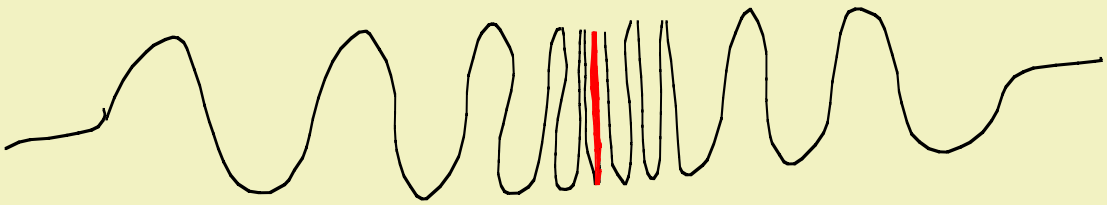



J


f_o
f_i
J

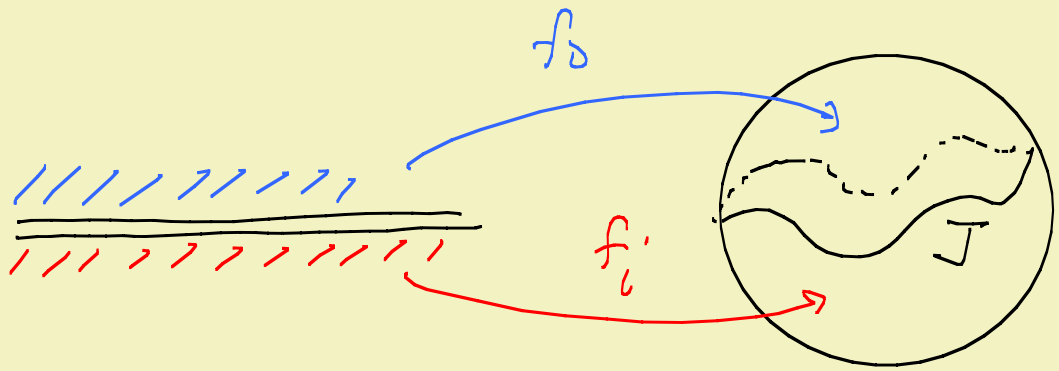


THE SET J IS NOT ALWAYS A CURVE



$f_o^{-1} \circ f_i$ IS CONTINUOUS

BUT f_i & f_o ARE NOT.



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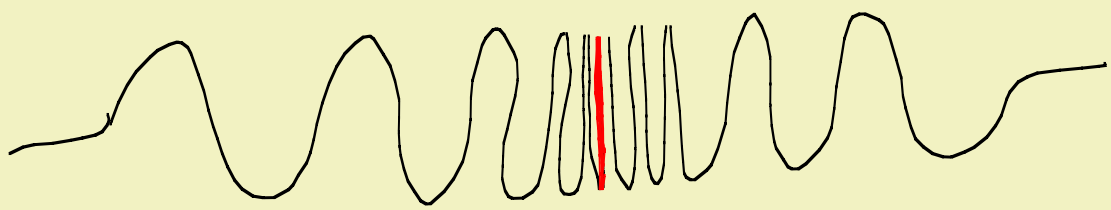

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BUT f_i & f_o ARE NOT.

J NOT NEC. UNIQUE, EVEN IF IT IS A CURVE

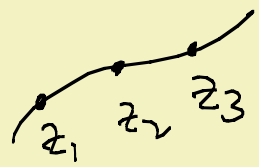
E.G. IF AREA(J) > 0, THERE EXISTS A HOMEOMORPHISM F OF $\mathbb{S}^2$ WHICH IS ANALYTIC OFF J AND NOT MÖBIUS.

J & f(J) HAVE SAME WELDING MAP h.

PROPER CONTEXT:

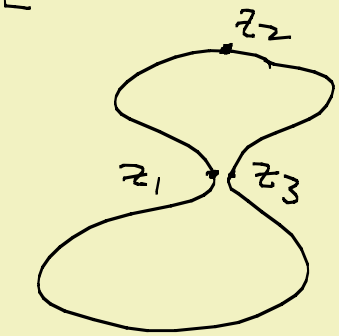
$J \in QC$:

CLOSED JORDAN CURVE



$$|z_1 - z_3| \geq C\left[|z_1 - z_2| + |z_2 - z_3|\right]$$

NO BOTTLENECKS:



EG. VON KOCH SNOWFLAKE FRACTAL QC

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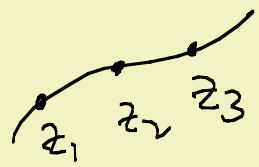

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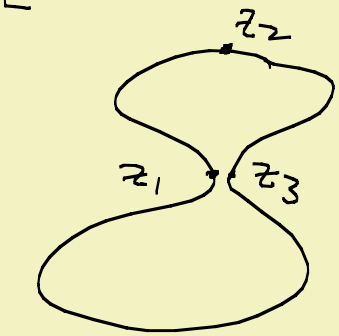

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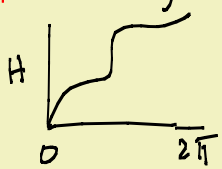

STEEP OR FLAT IS OK, QUICK CHANGE IN SLOPE IS NOT

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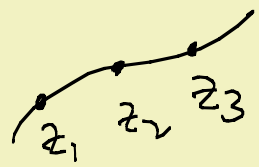

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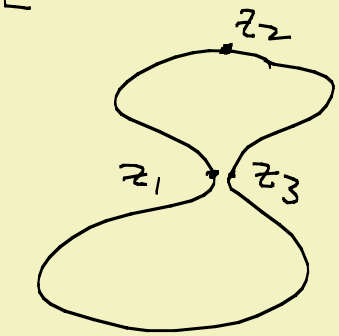

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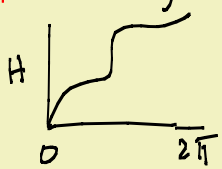


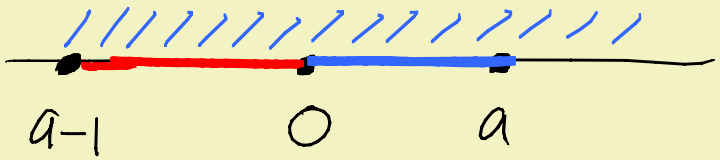
THEOREM (PFLUGER) WITH PROPER NORMALIZATION:

$$J \in QC \Longleftrightarrow h \in QS$$

one-one

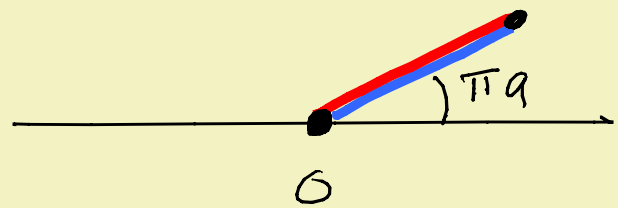
ZIPPER CONFORMAL MAPPING METHOD

$$f(z) = C(z-a)^{a}(z+1-a)^{1-a}$$



$a, a-1 \longrightarrow 0$

$0 \longrightarrow \text{TIP}$



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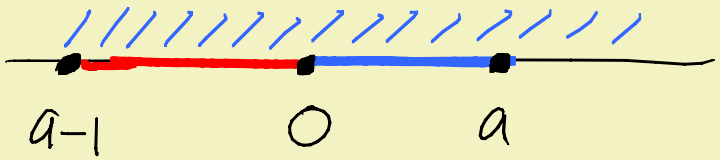

$a, a-1 \to 0$

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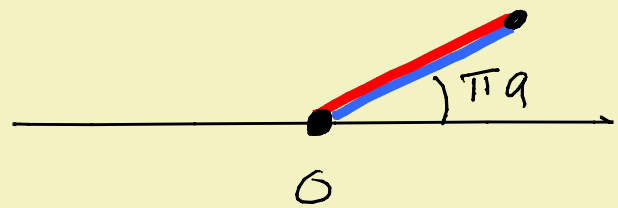
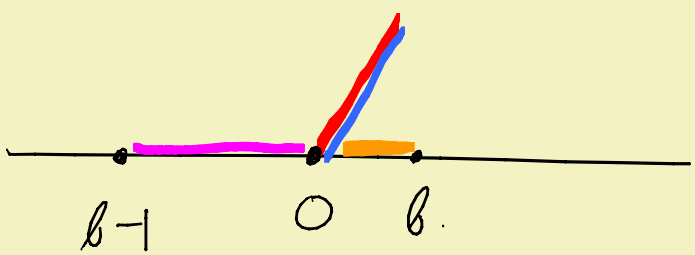
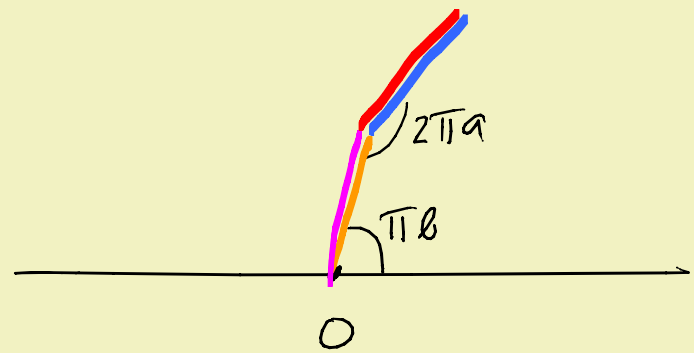


USE COMPOSITION TO BUILD A CURVE:





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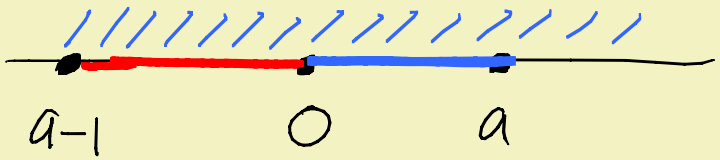

$a, a-1 \to 0$

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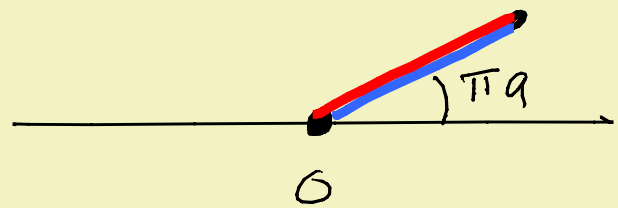

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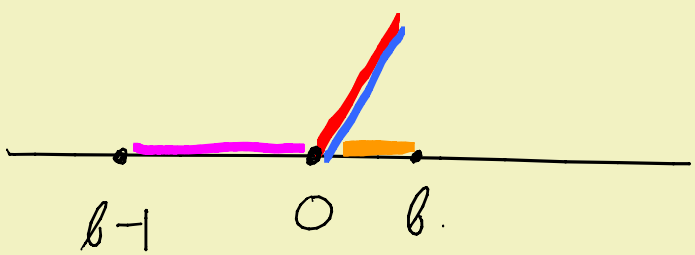
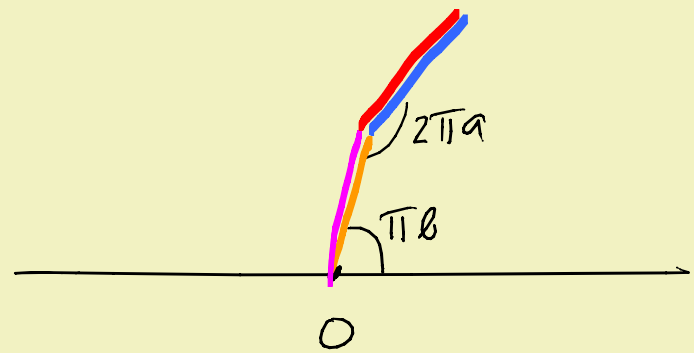
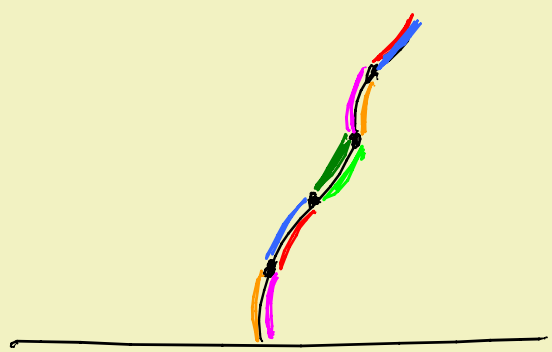
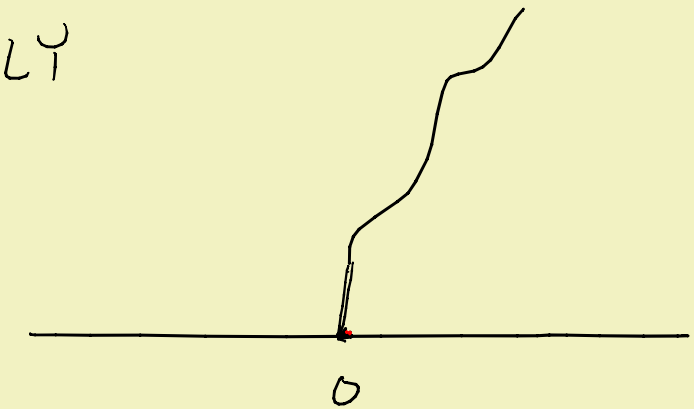




REPEAT



CONVERSELY



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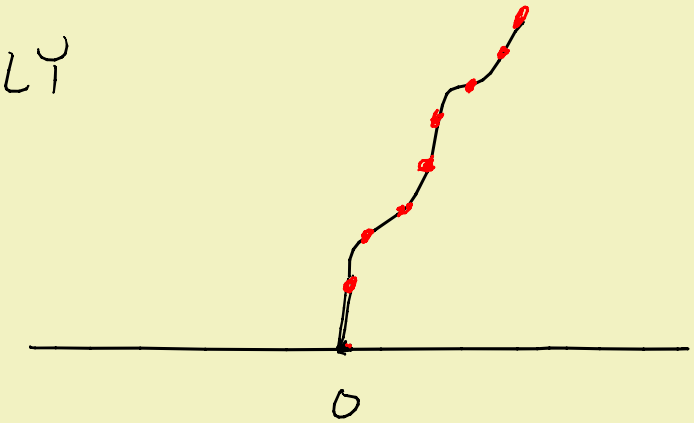

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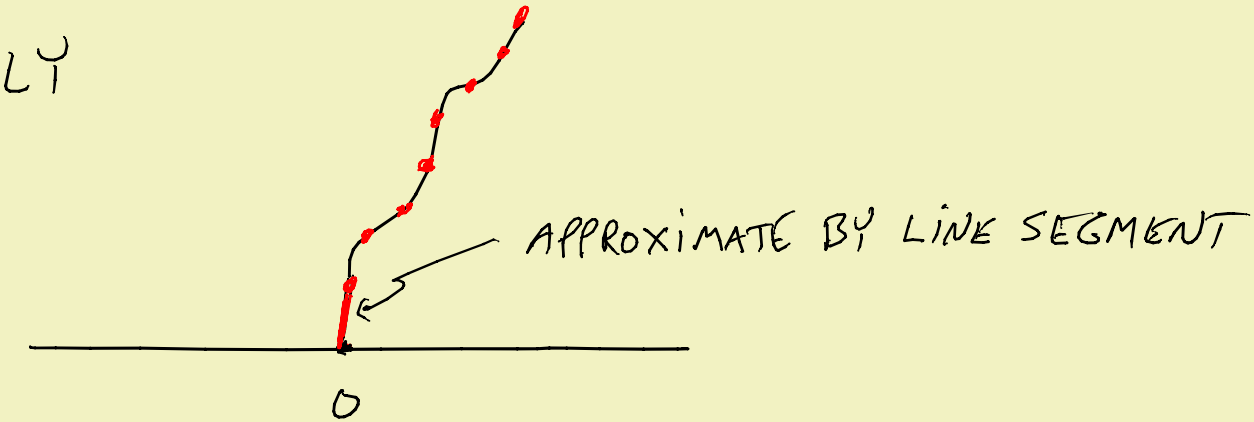

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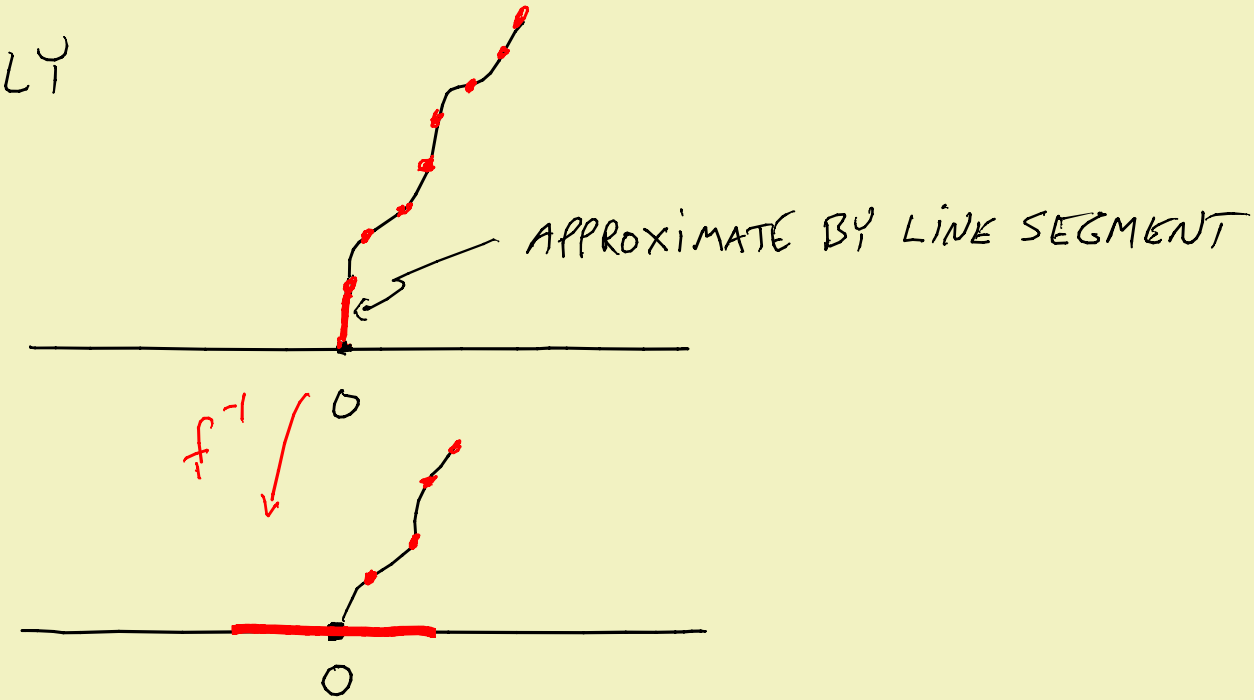

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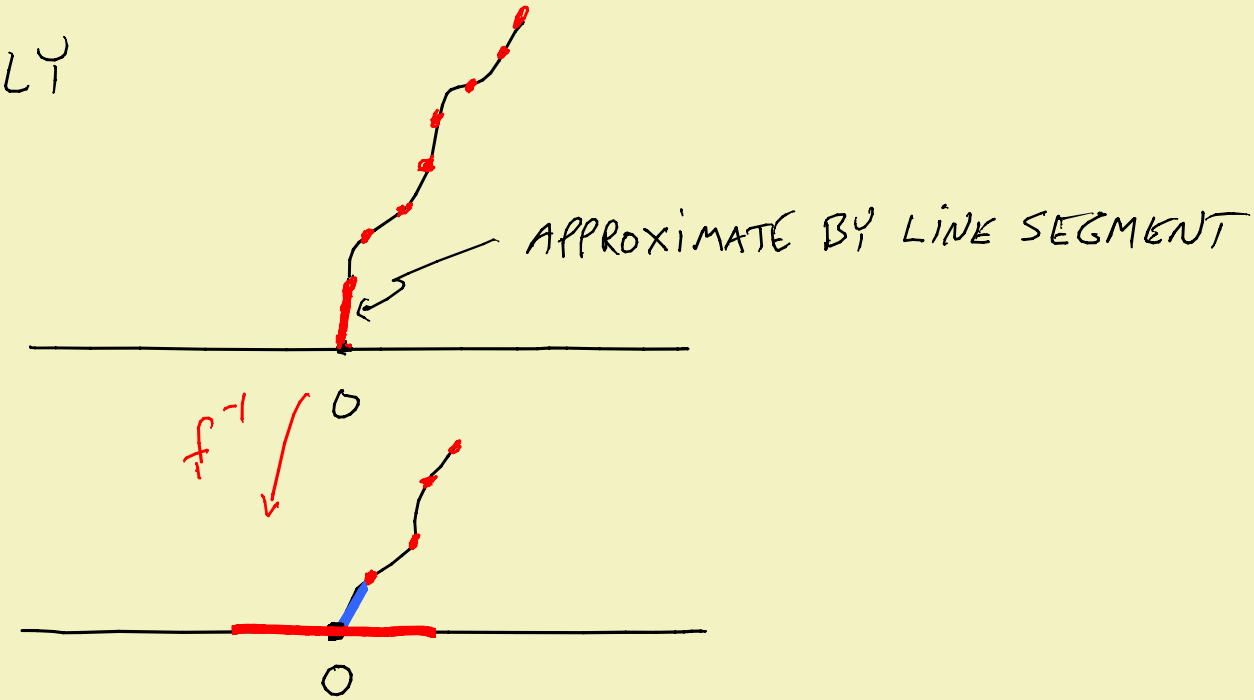

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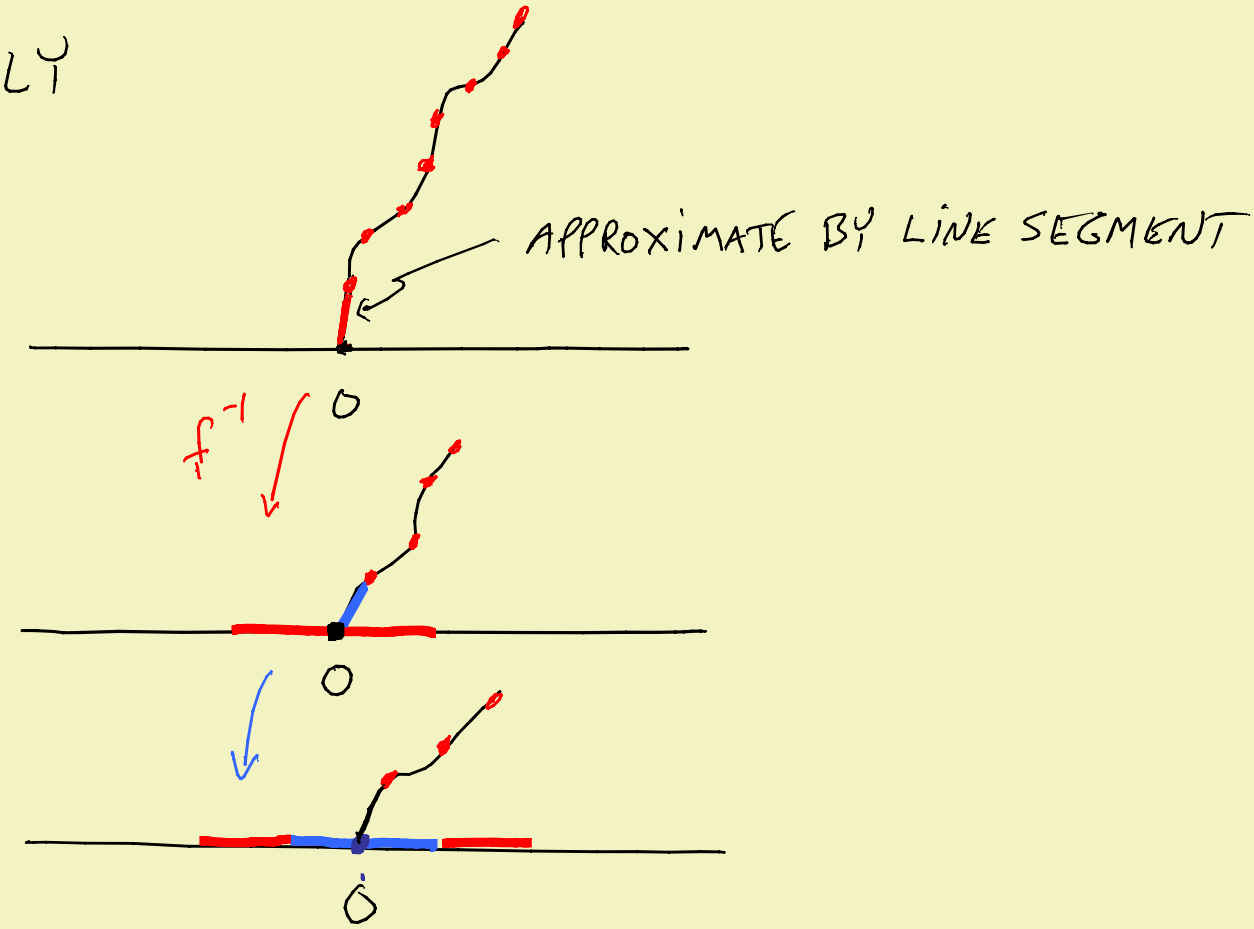

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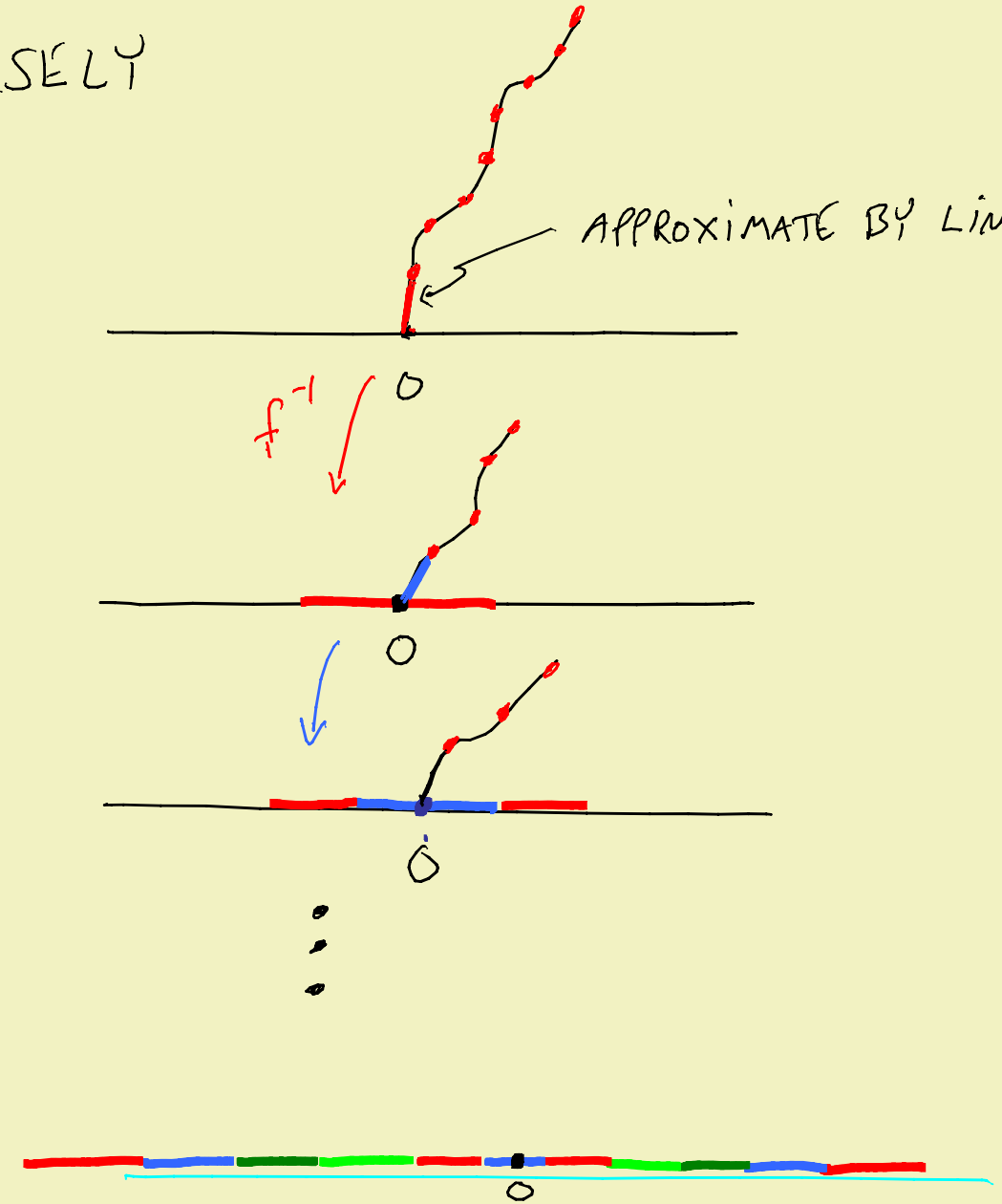
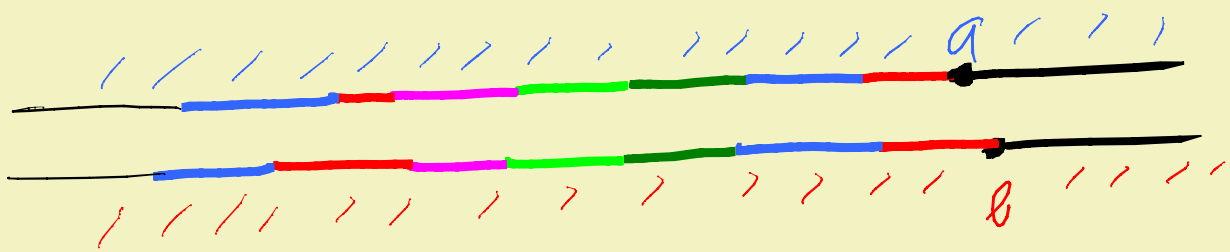


SIMPLE, FAST
ACCURATE
EASILY 10^4 INT.

TO WELD A CLOSED JORDAN CURVE:



$C\sqrt{z-a}$

$D\sqrt{z-b}$

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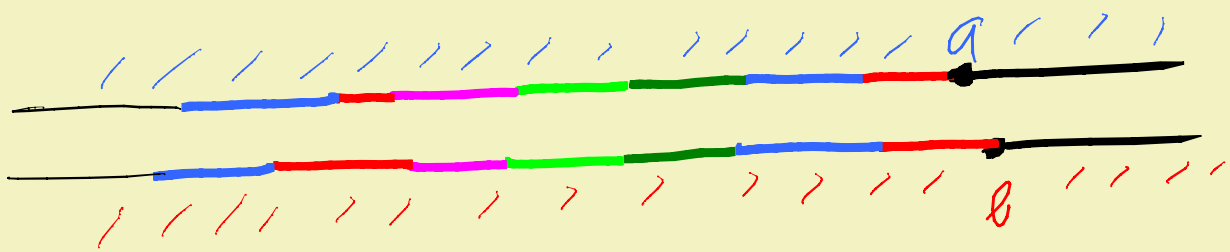

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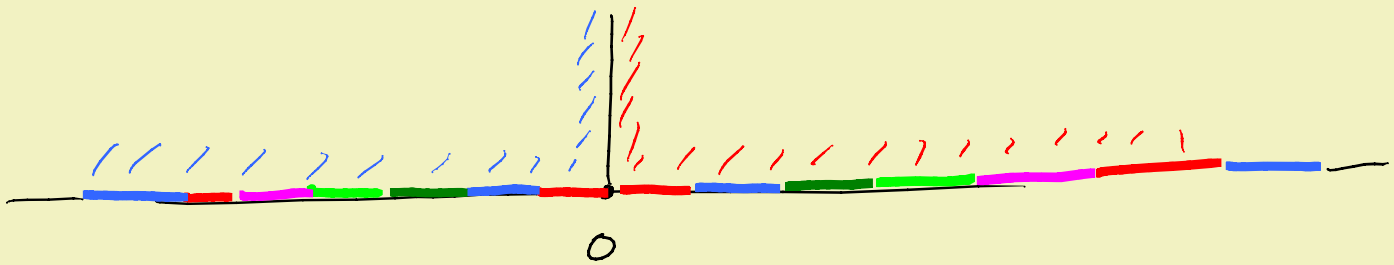

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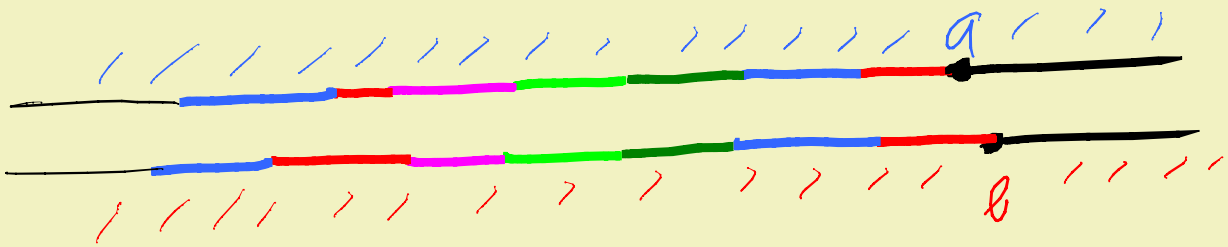

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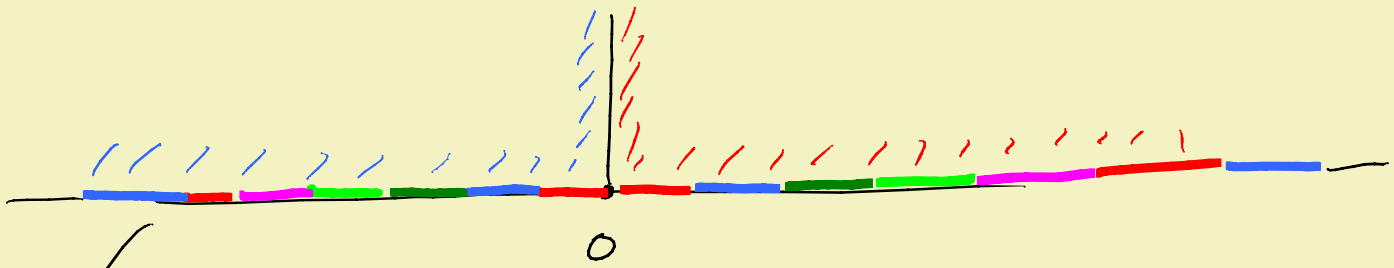
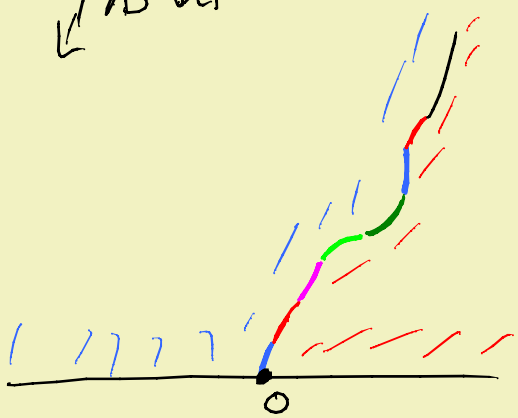


AS BEFORE



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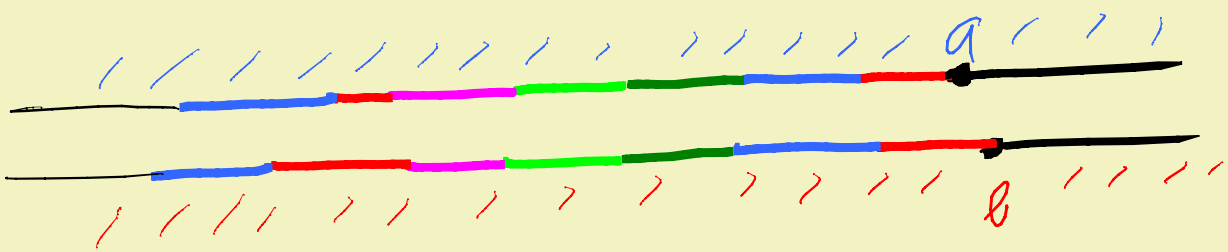

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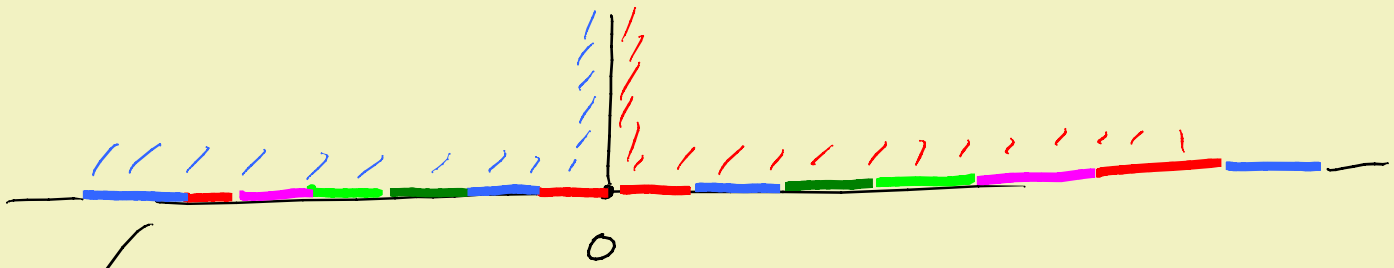
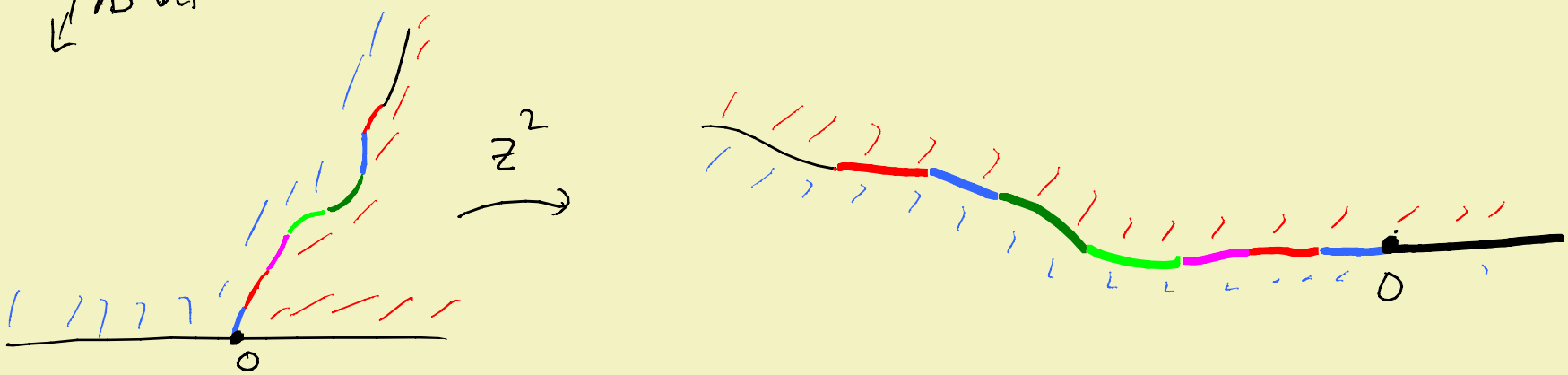
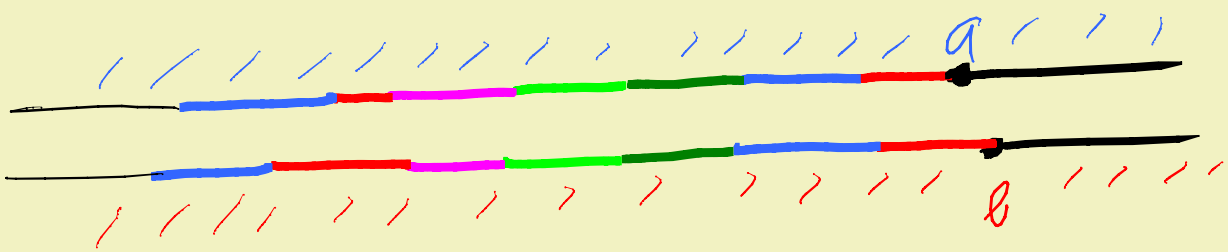


AS BEFORE

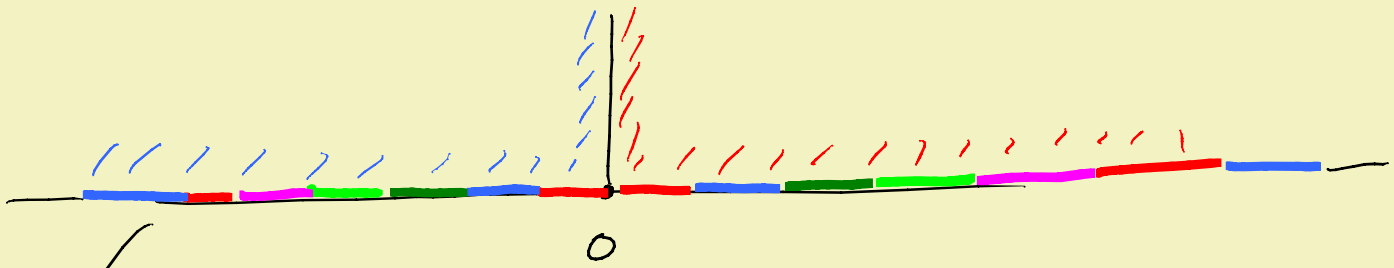


TO WELD A CLOSED JORDAN CURVE:



$C\sqrt{z-a}$

$D\sqrt{z-b}$



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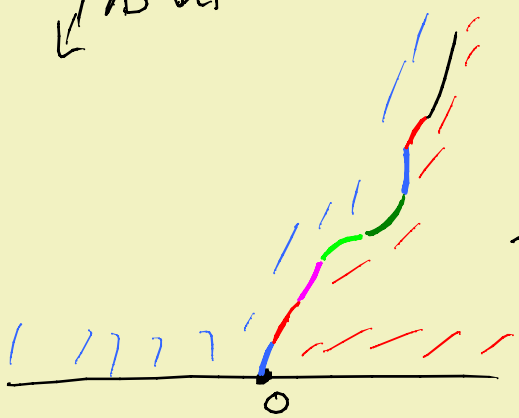
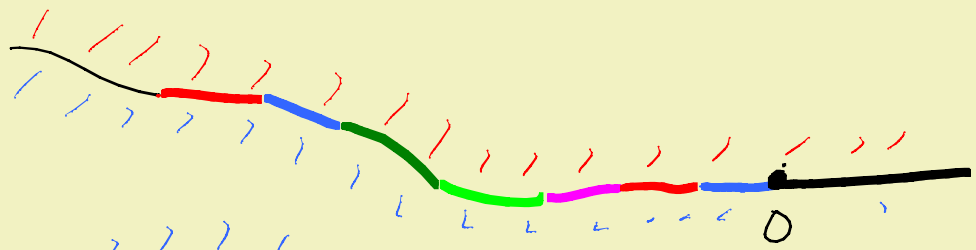
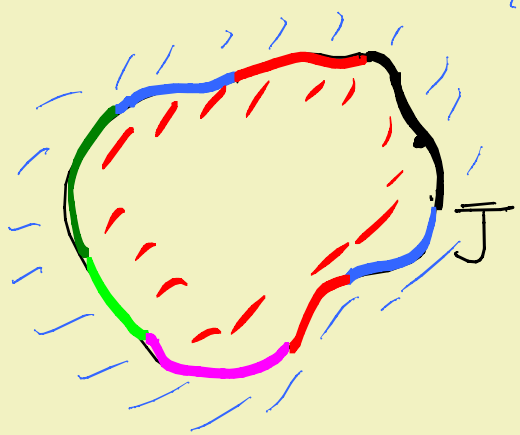


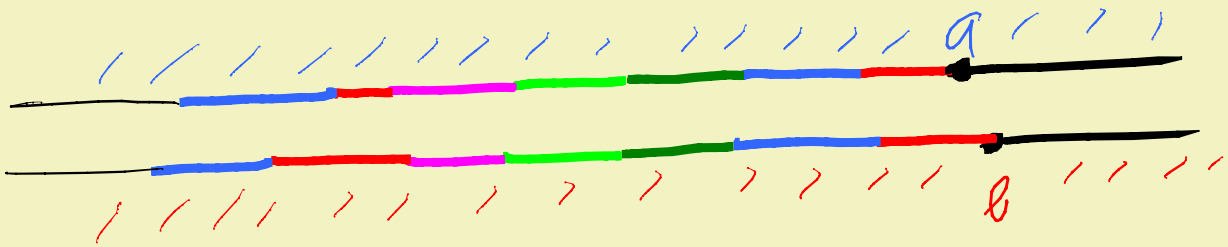
z^2





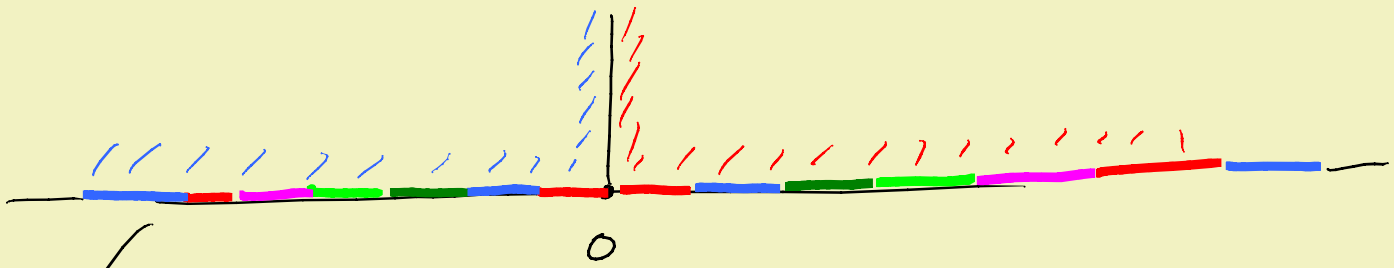
$\lambda\left(\frac{z-a}{z-b}\right)$

TO WELD A CLOSED JORDAN CURVE:



$C\sqrt{z-a}$

$D\sqrt{z-b}$



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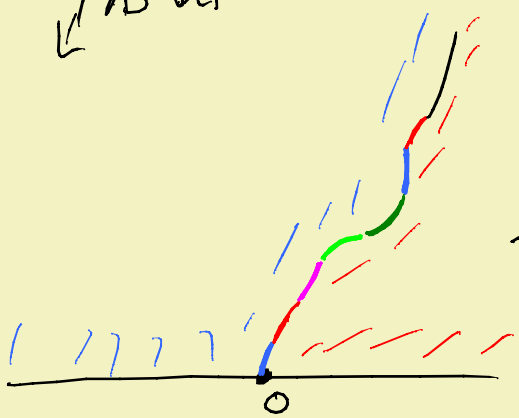
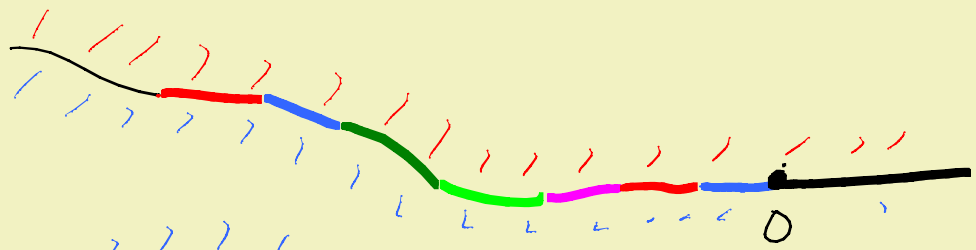
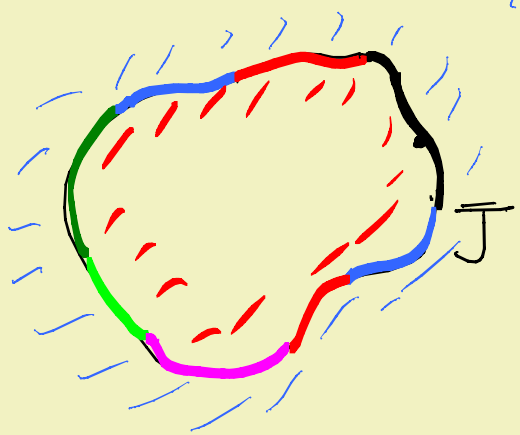


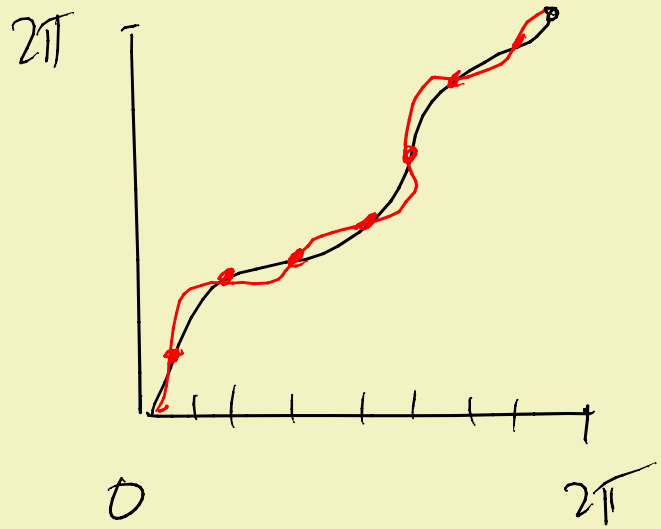
z^2





$\lambda\left(\frac{z-a}{z-b}\right)$

CONVERSELY, GIVEN J
REVERSE STEPS



— GIVEN H

— APPROX H_c

AGREE AT DATA POINTS

UNIFORMLY CLOSE

SINCE BOTH INCREASING.

RECENT APPLICATIONS:

STOCHASTIC LÖWNER EVOLUTION (SLE)

JORDAN BOUNDARY

- VISION/SHAPE RECOGNITION
- LEMNISCATES

TREE BOUNDARIES & LAMINATIONS

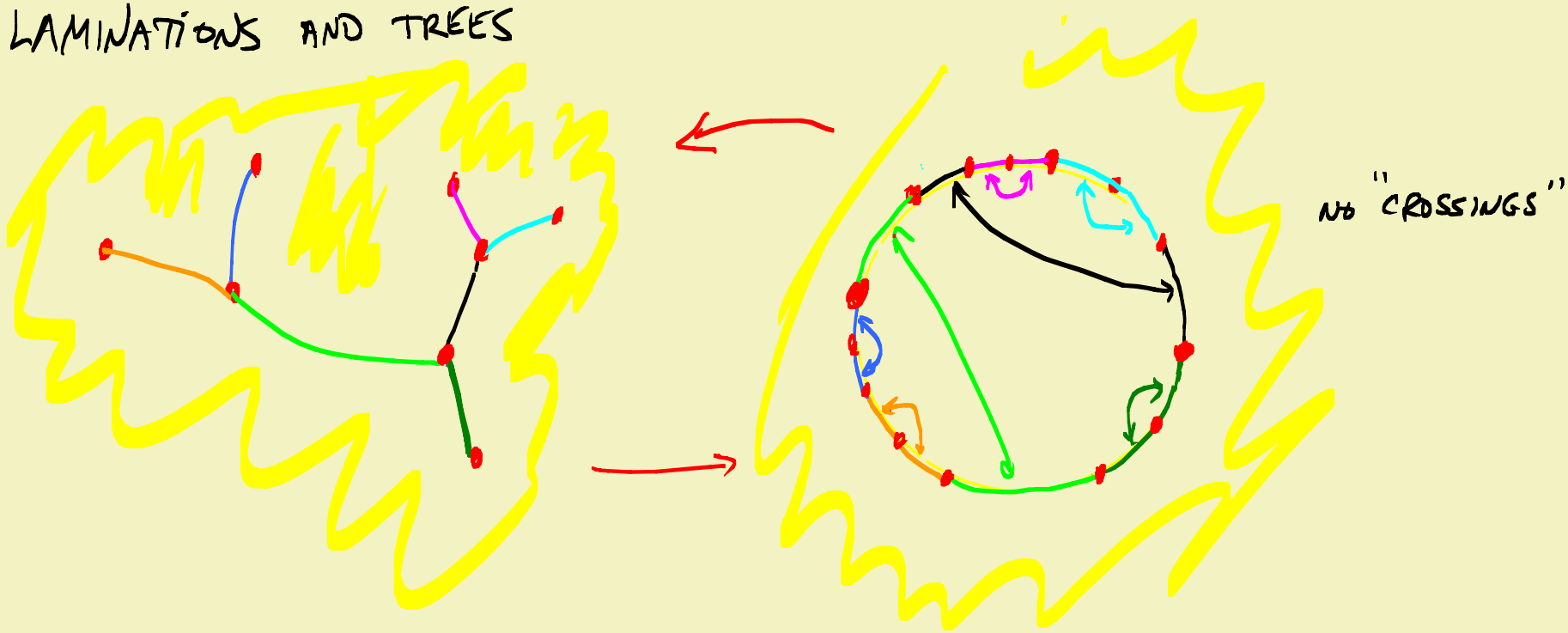
- DENDRITES – JULIA SETS
- RANDOM TREES
- TRUE TREES & SHABAT POLYNOMIALS
  CLASSIFICATION/CATALOG
- ITERATED FUNCTION SYSTEMS
- TRAJECTORIES OF QUADRATIC/CUBIC DIFFERENTIALS

UNIFORMIZATION

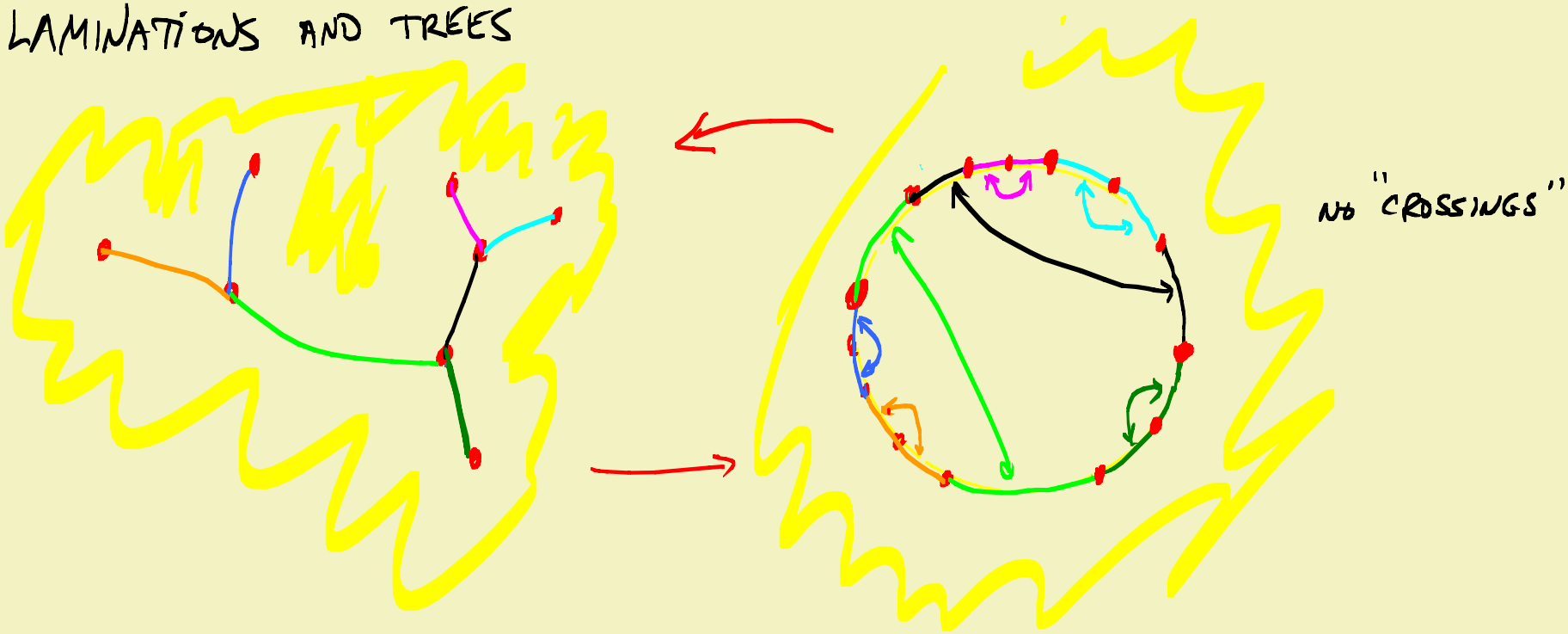
- TRIANGULATIONS
- EMBEDDED GRAPHS (MAPS) & BELYĬ FUNCTIONS
  GROTHENDIECK'S DESSIN'S D'ENFANTS
- PLANE DOMAINS

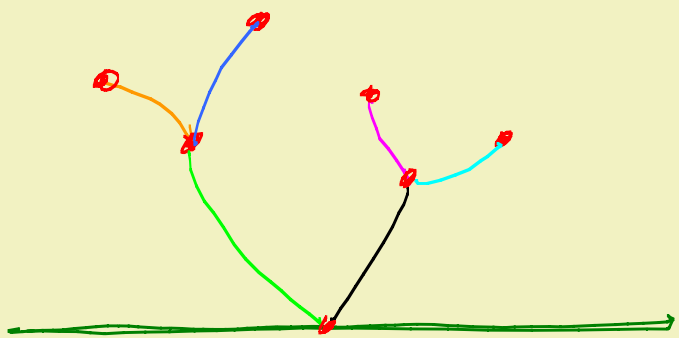
— NUMERICAL EXPERIMENTATIONS TO GIVE INSIGHT & IDEAS FOR PROVING THEOREMS.

LAMINATIONS AND TREES

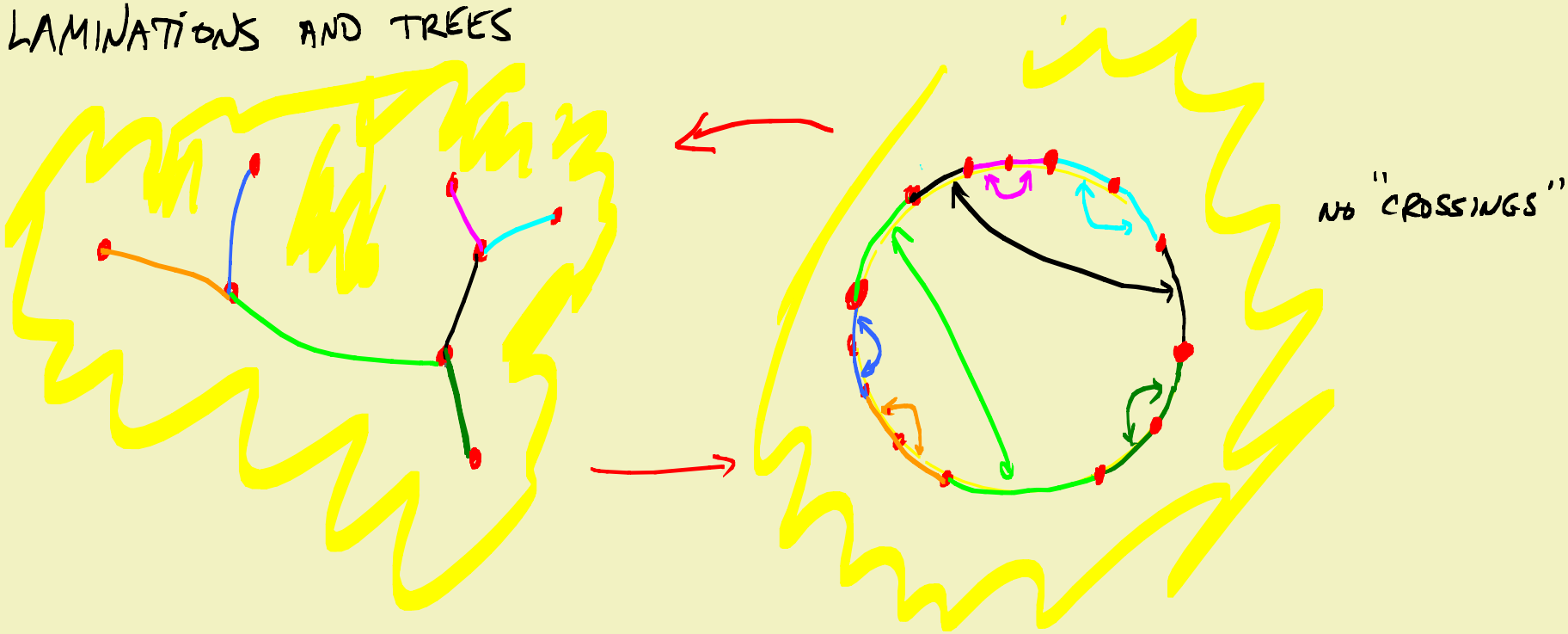


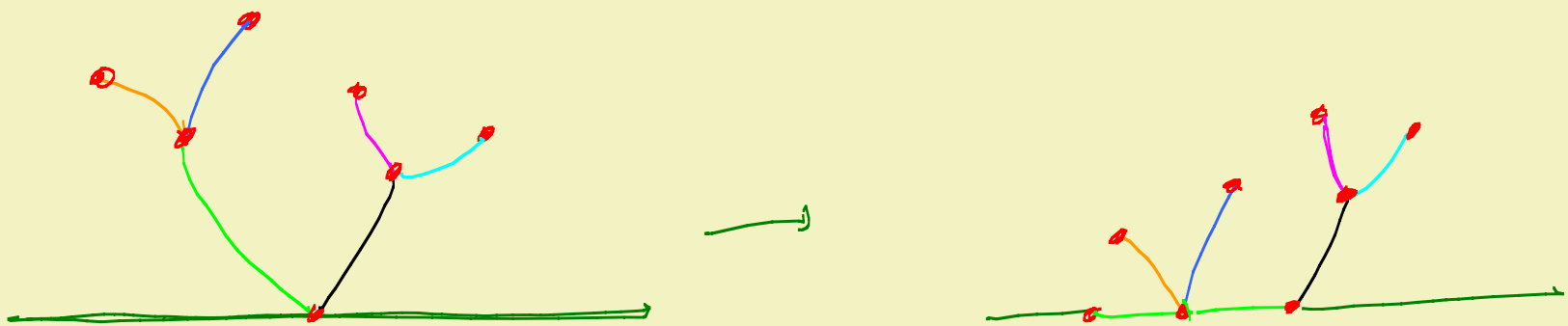
LAMINATIONS AND TREES



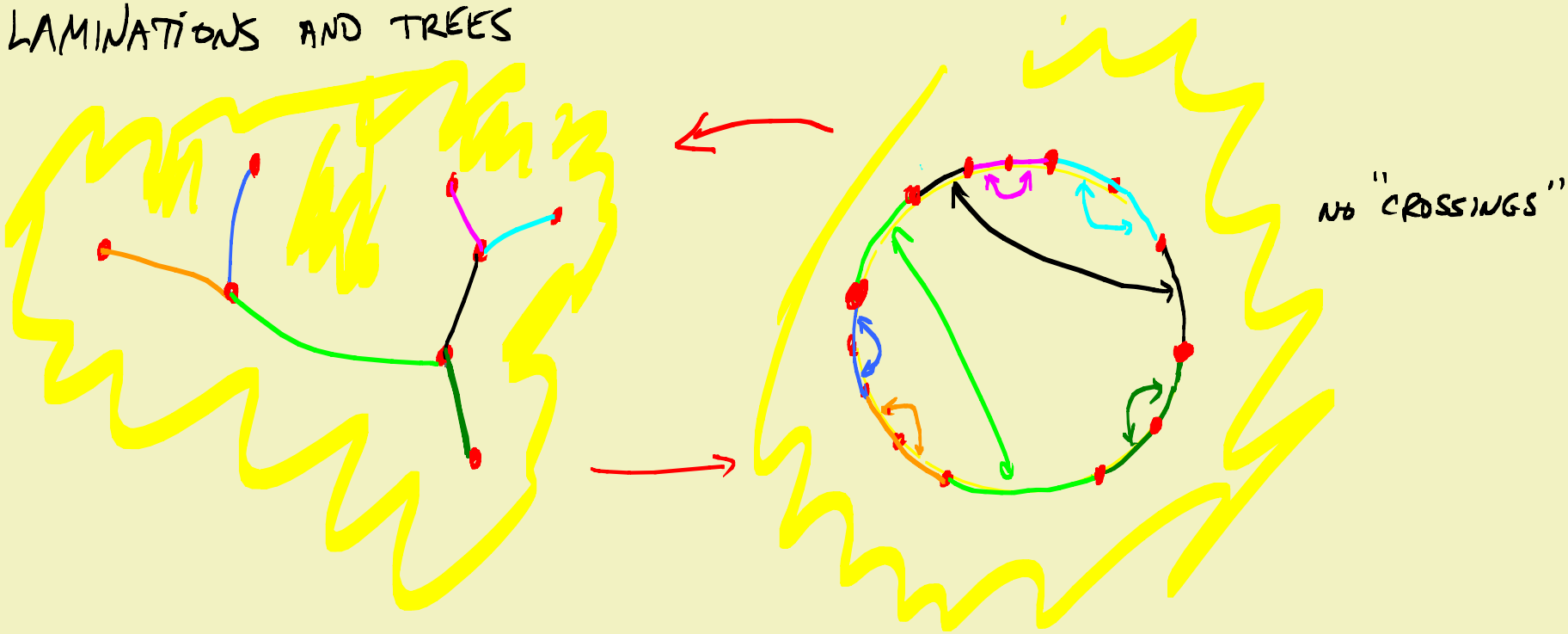


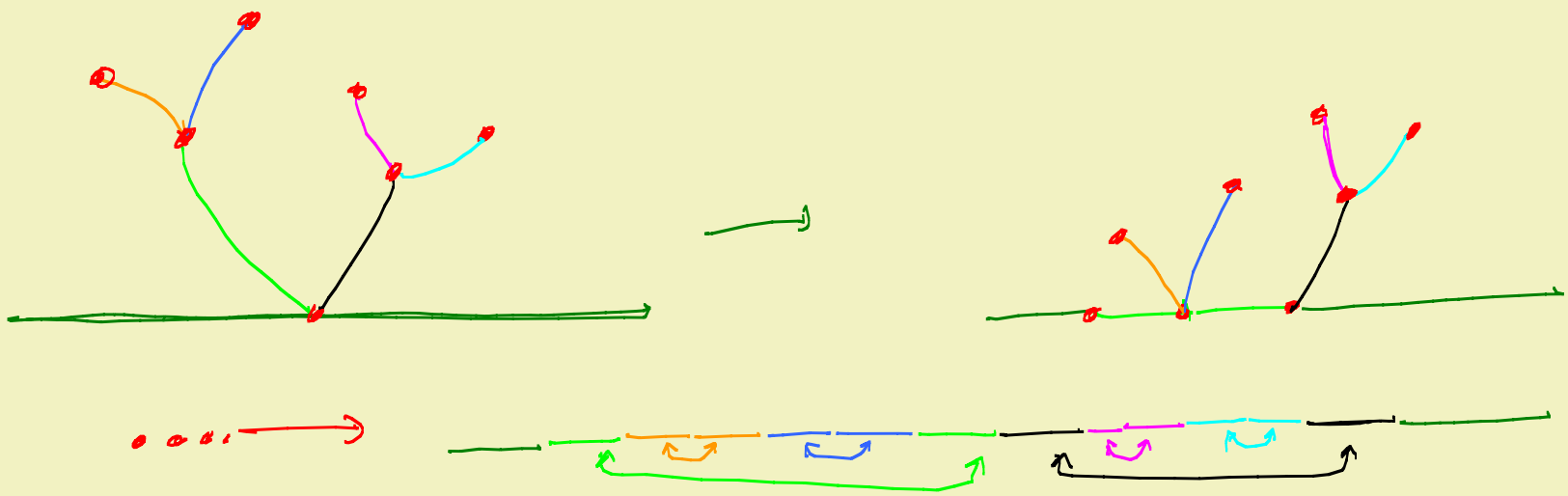
LAMINATIONS AND TREES





LAMINATIONS AND TREES

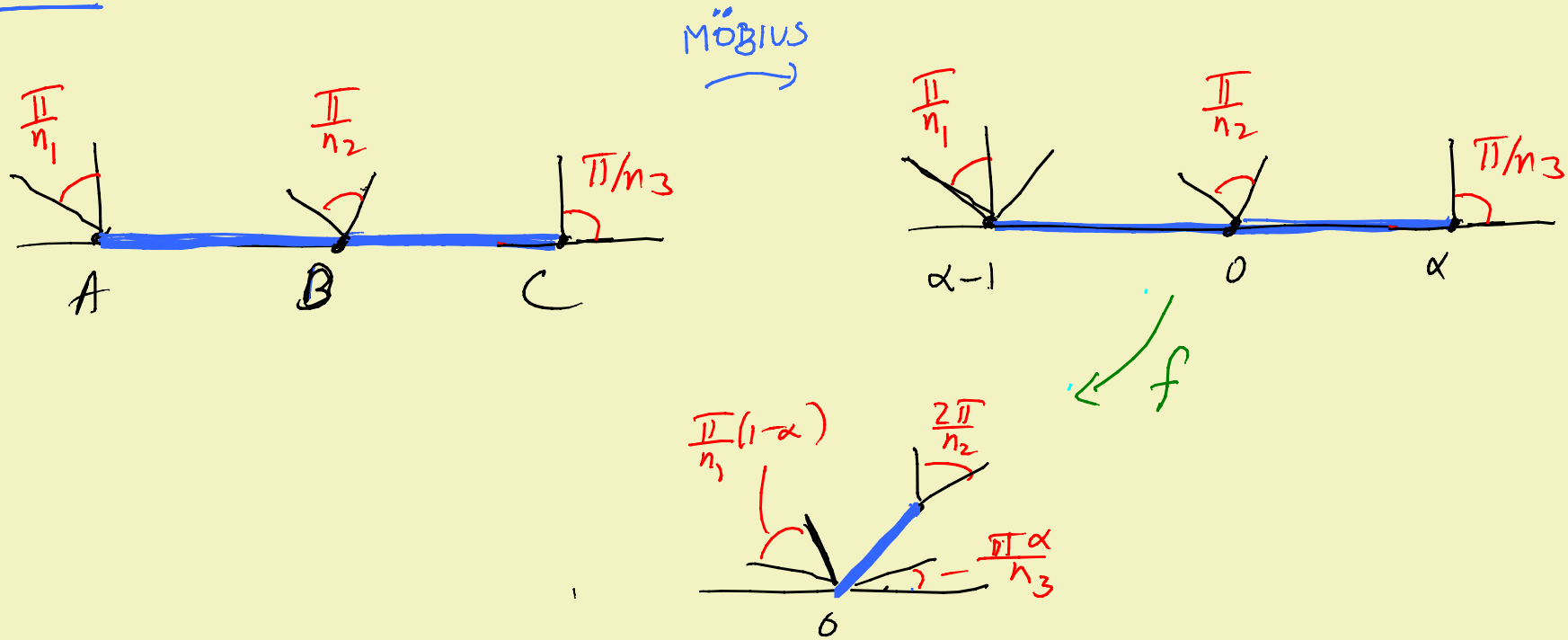




& CONVERSELY

MODIFICATION OF ALGORITHM:

BETTER: KEEP ANGLES AT EACH VERTEX EQUAL USING:

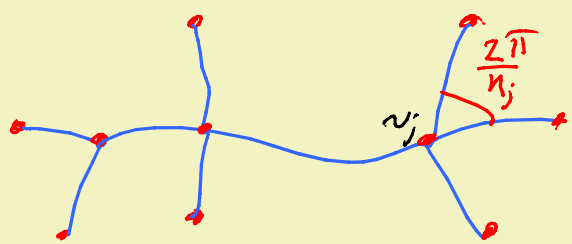


CHOOSE α $\ni$: $\frac{\pi}{n_1}(1-\alpha) = \frac{\pi\alpha}{n_3}$

THEOREM (BARNES, M., ROHDE)

GIVEN A FINITE LAMINATION, THE ALGORITHM PRODUCES

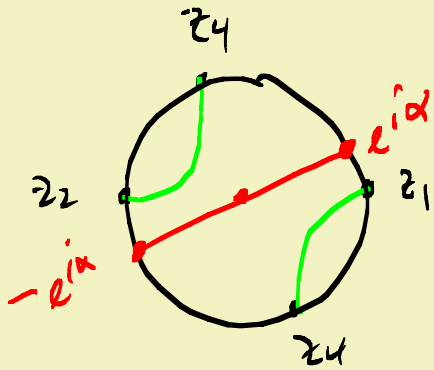
- A FINITE TREE T - WITH ANALYTIC BRANCHES,
  - EQUAL ANGLES AT VERTICES
- CONFORMAL MAPS φ, φ^{-1} BETWEEN $\mathbb{D}^c$ & $\mathbb{C} \setminus T$
- $\varphi(z) \sim v_j + c_j (z - \zeta_j)^{2/n_j}$ near ζ_j



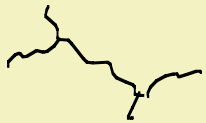
DENDRITES

$z^2 + c$

$$f(z^2) = [f(z)]^2 + c$$



$e^{i\alpha}, -e^{i\alpha}$ — 4 SQUARE ROOTS



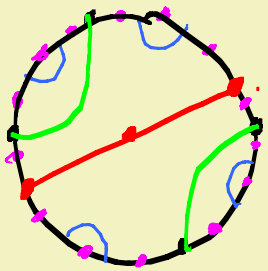
PAIR: (z_1, z_4) (z_2, z_3)

$\exists$: $G_{z_i, z_j} \cap G_{e^{i\alpha}, -e^{i\alpha}} = \phi$ (1)

REPEAT FOR EACH PAIR (z_k, z_ℓ):

4 SQUARE ROOTS GIVE 2 NEW PAIRS

$\exists$: (1) HOLDS



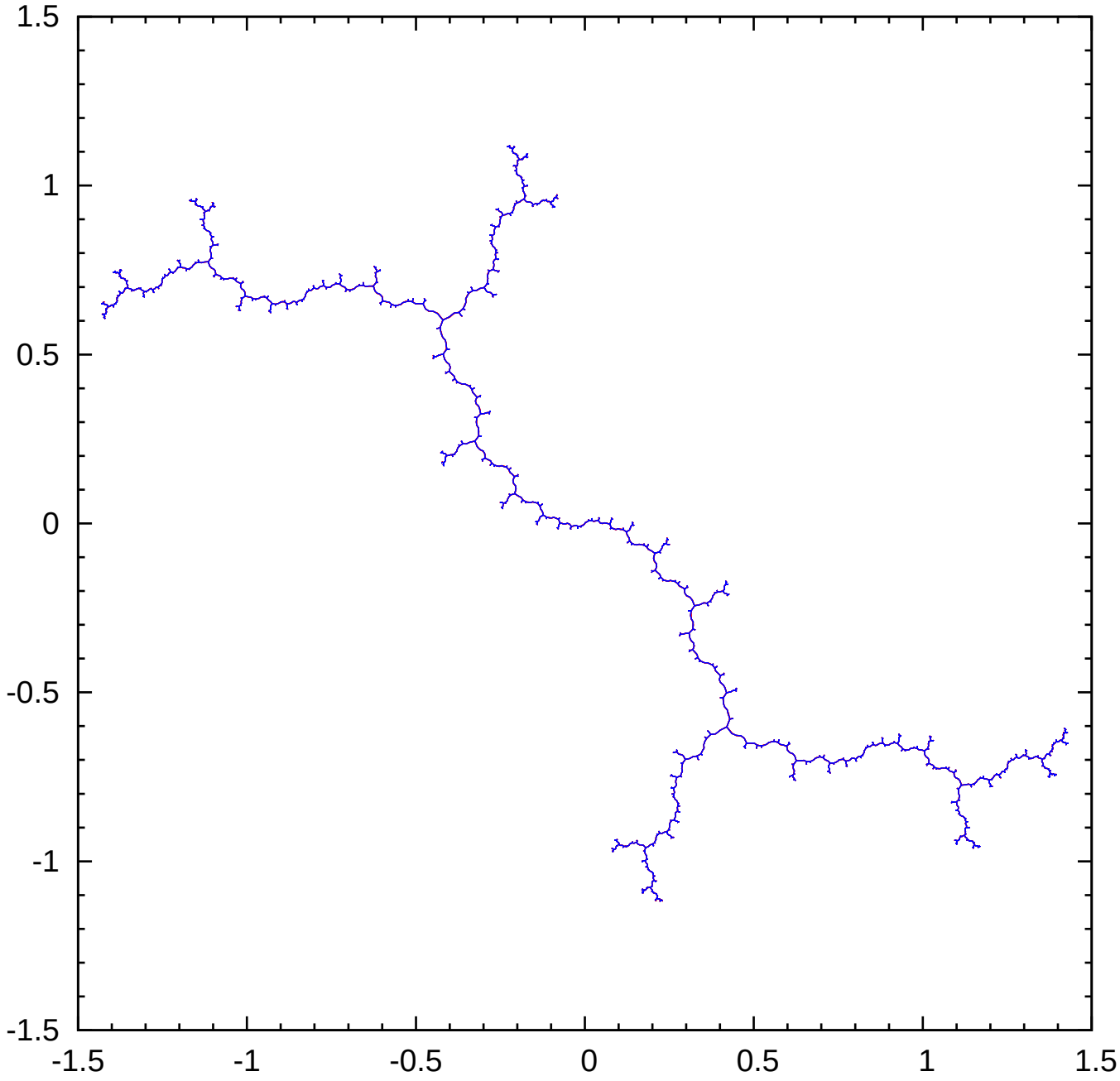
VERTICES = MIDPOINTS

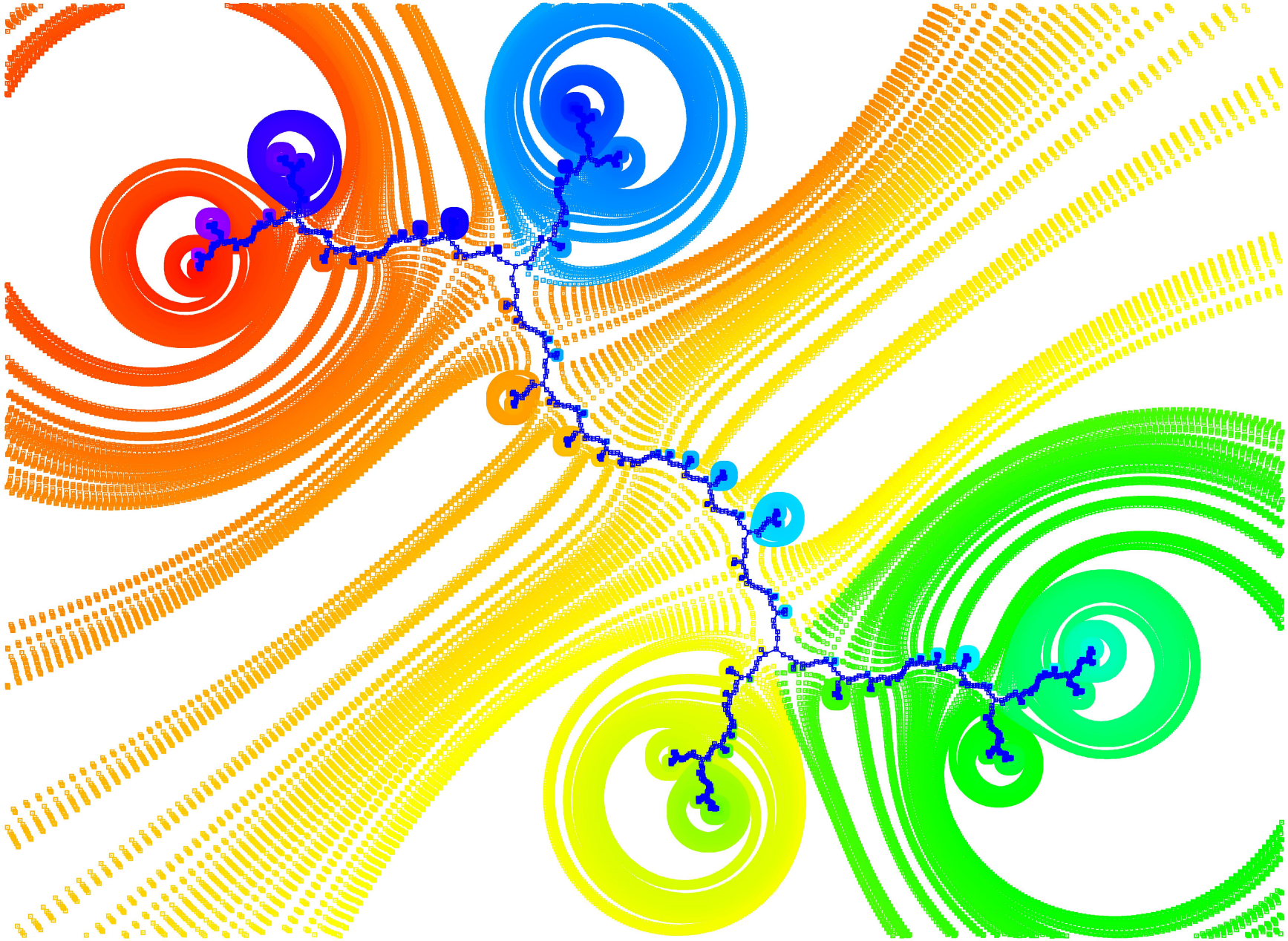
LAMINATION SHOWS WHICH INTERVALS TO IDENTIFY

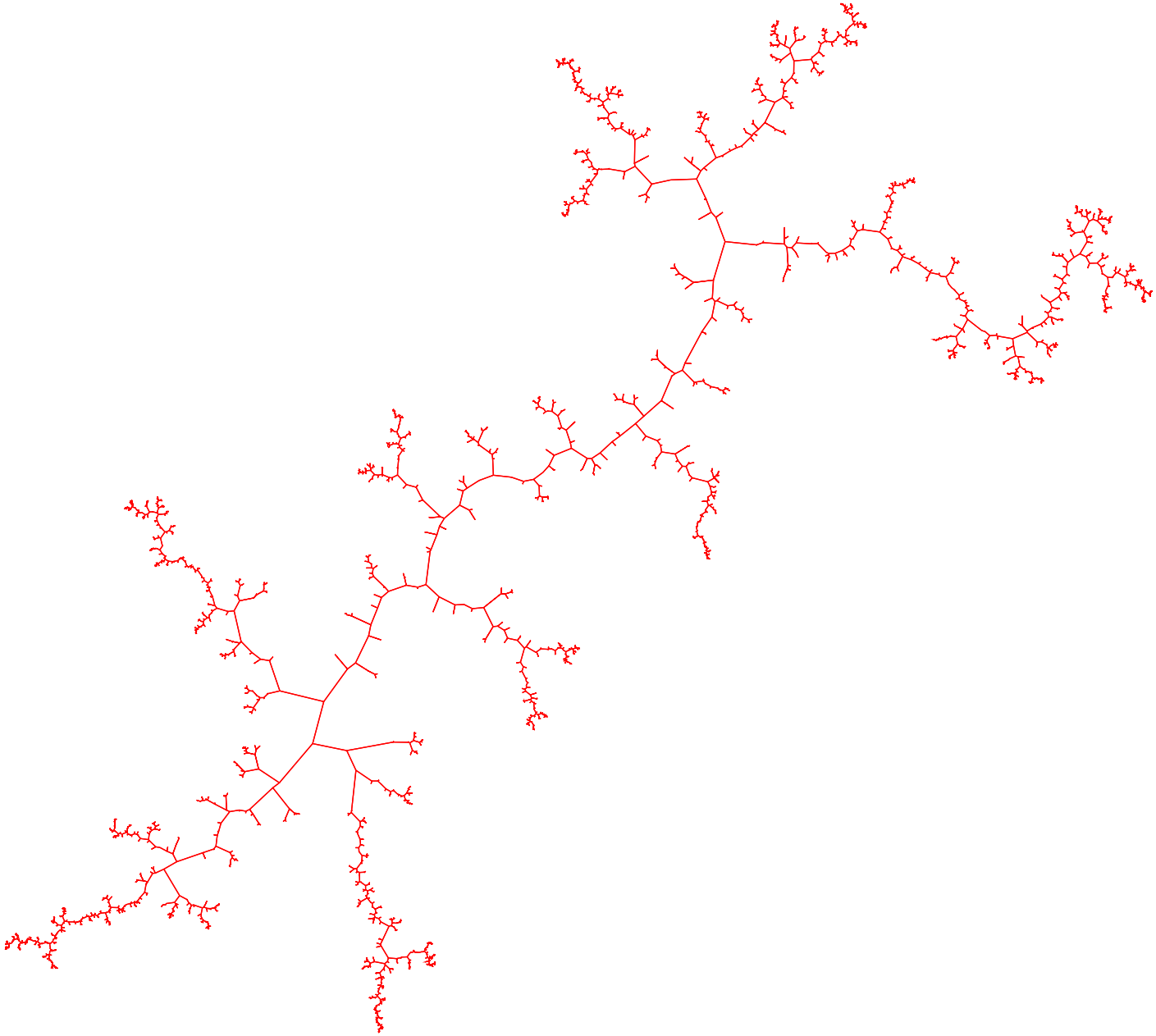
TREE $\longleftrightarrow$ JULIA SET FOR $z^2 + c$

GET c FROM COEFF. AT ∞

ZIPPER GIVES A TREE (DENDRITE) FOR THIS LAMINATION.


1.5
1
0.5
0
-0.5
-1
-1.5
-1.5
-1
-0.5
0
0.5
1
1.5





CLASSIFICATION OF PLANAR TREES (UP TO HOMEOMORPHISMS OF $\mathbb{C}$)

SHABAT / BELYI

TREES $\longleftrightarrow$ SHABAT POLYNOMIALS p

$p'(z) = 0 \Rightarrow p(z) = 0$ or $p(z) = 1$

$$T = p^{-1}[0,1]$$

SHABAT POLYNOMIALS AND THEIR RATIONAL COUNTERPARTS (WHICH WE'LL SEE SOON) CALLED BELYI FUNCTIONS CONNECT TO MANY AREAS:

GROUP THEORY, RIEMANN SURFACES, ALGEBRAIC GEOMETRY

GALOIS THEORY, MODULI SPACES, QUANTUM FIELDS, KNOTS

NICE REFERENCE BOOK: "GRAPHS ON SURFACES AND THEIR APPLICATIONS" BY S. LANDO & K. ZVONKIN, SPRINGER (2004)

NOT EASY TO FIND SUCH P.

METHODS:

- GRÖBNER BASIS ALGORITHM — BIRCH, SHABAT, SCHNERS
- PUISEUX SERIES — OESTERLÉ
- ↳ + APPROXIMATION, ITERATION & INTUITION — COUVEIGNES & GRANBOULAN
- DEFORM z^n : ADD EXTRA CRITICAL VALUE C & MOVE C FROM 0 to 1 — MATIYASEVICH, Y.V

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METHOD OF COUVEIGNES & GRANBOULAN — 3 STAGES

1. COMPUTE APPROXIMATIONS OF THE VERTICES WITH AD HOC METHODS
2. USE AN ITERATIVE ALGORITHM TO IMPROVE ACCURACY TO 1000–2000 DIGITS
   WORK WITH P & $P-1$ IN SQUARE FREE FORM $\prod p_j^{i}$
   OBTAIN EQUATIONS INVOLVING POWER SUMS OF THE ROOTS
3. USE LATTICE REDUCTION ALGORITHM (LLL) TO FIND ALGEBRAIC EQUATIONS SATISFIED BY THE COEFFICIENTS OF THE POLYNOMIAL (MORE EFFICIENT THAN GRÖBNER BASIS)

"THE FORMULATION AND TESTING OF CONJECTURES ARE HAMPERED BY THE FACT THAT EFFICIENT COMPUTATIONS OF THE DRAWINGS, CONJUGATION CLASSES AND ASSOCIATED GALOIS GROUPS IS NOT EASY, EVEN FOR SMALL TREES"

— BETREMA & ZVONKIN.

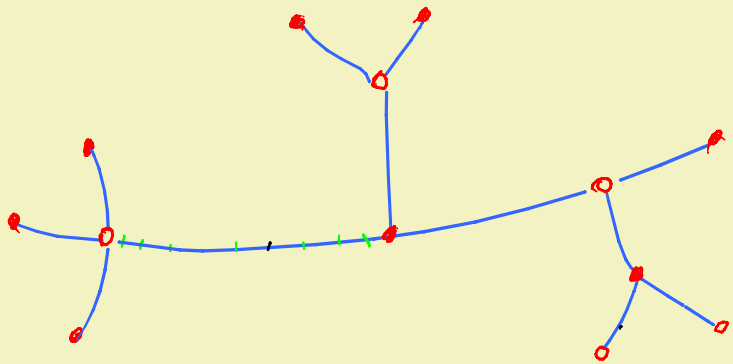
1992 BETREMA & ZVONKIN — CATALOG OF TREES WITH ≤ 8 EDGES

2009 KOCHETKOV — CATALOG OF TREES WITH 9 EDGES

"THE COMPLETE STUDY OF TREES WITH 10 EDGES IS A DIFFICULT WORK, AND PROBABLY NOBODY WILL DO IT IN THE FORESEEABLE FUTURE."

— KOCHETKOV

2 TREES ARE HOMEOMORPHIC
IF THE HAVE THE SAME PATTERN OF INTERVAL
MATCHING — LET'S BUILD A CANONICAL ONE
BY TAKING INTERVALS OF EQUAL LENGTH ON $\partial \mathbb{D}$



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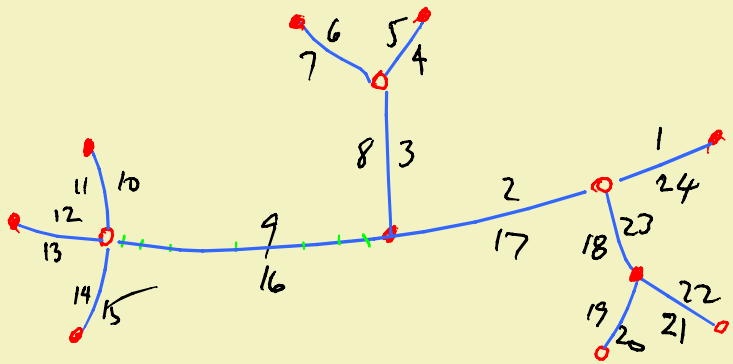

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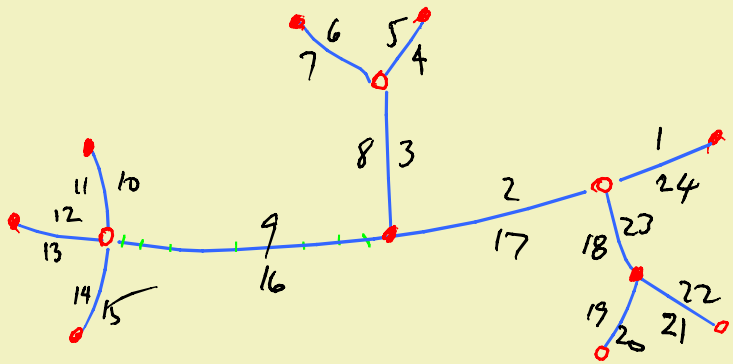
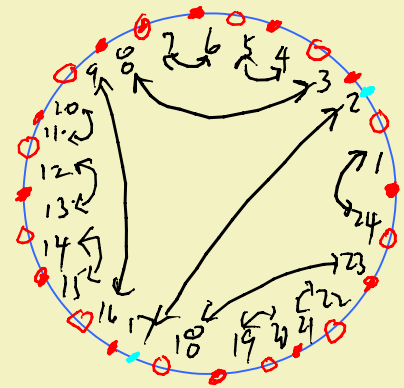


φ

2n intervals



CHOOSE

$h(e^{i\theta}) = e^{i(\theta_j - \theta_j')}$

on I_j

(ROTATE & REFLECT)

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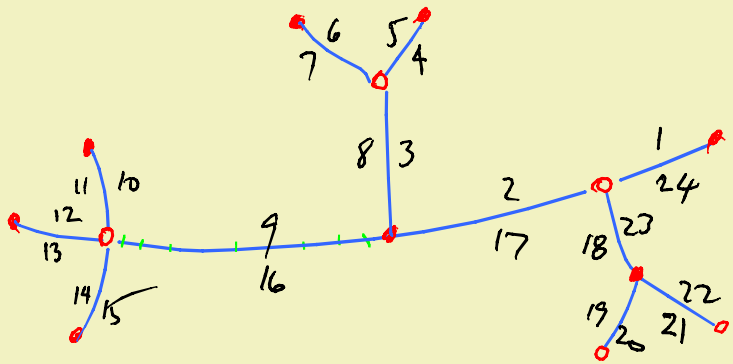

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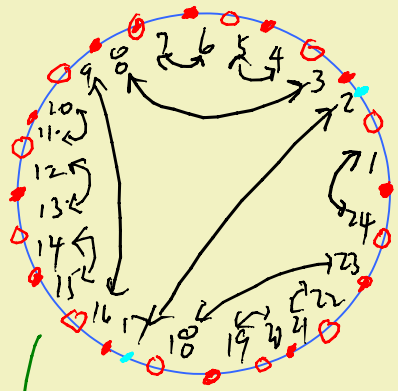
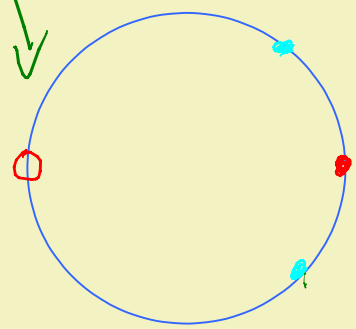
2n intervals



z^n



CHOOSE

$h(e^{i\theta}) = e^{i(\theta_i - \theta)}$

on I_i

(ROTATE & REFLECT)

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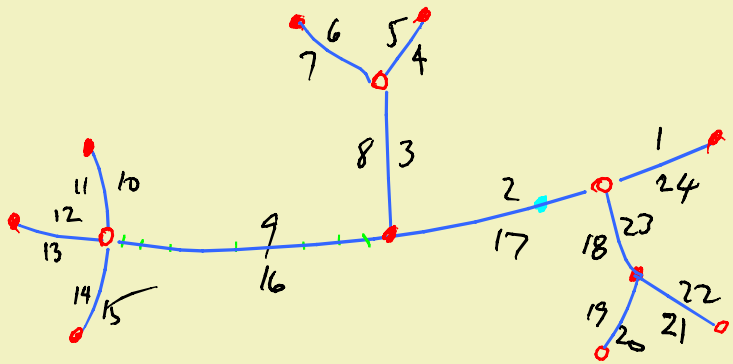
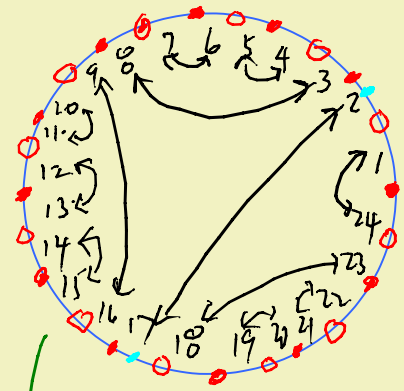


φ

2n intervals



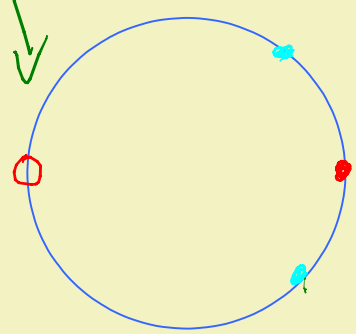
CHOOSE

$h(e^{i\theta}) = e^{i(\theta_j - \theta_j')}$

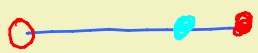
on I_j

(ROTATE & REFLECT)

z^n



$\frac{1}{4}\left(z + \frac{1}{z}\right) + \frac{1}{2}$



$$p(z) = \frac{1}{4}\left[(\varphi^{-1})^n + \frac{1}{(\varphi^{-1})^n}\right] + \frac{1}{2}$$

<u>COROLLARY</u> (BARNES, M., ROHDE)

THERE ARE QC MAPS α & β s.t.

$$\varphi = \alpha \circ \varphi_c \circ \beta \quad \text{is CONFORMAL ON } \mathbb{D}^c \text{ \&}$$

HAS WELDING MAP h, WHERE φ_c IS THE MAP CONSTRUCTED BY THE ZIPPER ALGORITHM, WELDING h_c

$\beta = h_c \circ h$ on EVEN INTERVALS; IDENTITY ON ODD – EXTEND TO $\mathbb{C}$

α COMES FROM SOLUTION OF BELTRAMI EQ. TO MAKE φ ANALYTIC OFF $\partial\mathbb{D}$.

COROLLARY (BARNES, M., ROHDE)

THERE ARE QC MAPS α & β ∋:

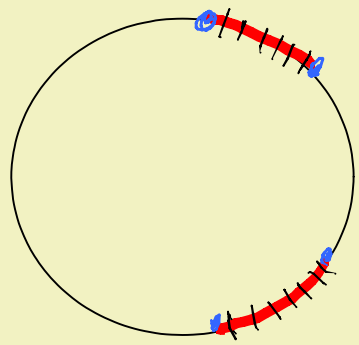
$$\varphi = \alpha \circ \varphi_c \circ \beta$$ is CONFORMAL ON $\mathbb{D}^c$ &

HAS WELDING MAP h, WHERE φ_c IS THE MAP CONSTRUCTED BY THE ZIPPER ALGORITHM, WELDING h_c

$\beta = h_c \circ h$ on EVEN INTERVALS; IDENTITY ON ODD - EXTEND TO $\mathbb{C}$

α COMES FROM SOLUTION OF BELTRAMI EQ. TO MAKE φ ANALYTIC OFF $\partial\mathbb{D}$.

BETTER:



JUST "REFINE" THE LAMINATION

THE VERTICES OF THE TREE ARE THE IMAGES: $\varphi\left(e^{2\pi i k/2^n}\right)$

PICTURES NOT GREAT SINCE THE IMAGES WON'T BE EQUALLY SPACED SINCE $\varphi \sim$ POWER NEAR VERTICES.

COROLLARY (BARNES, M., ROHDE)

THERE ARE QC MAPS α & β s.t.

$$\varphi = \alpha \circ \varphi_c \circ \beta$$ is CONFORMAL ON $\mathbb{D}^c$ &

HAS WELDING MAP h, WHERE φ_c IS THE MAP CONSTRUCTED BY THE ZIPPER ALGORITHM, WELDING h_c

$\beta = h_c \circ h$ on EVEN INTERVALS; IDENTITY ON ODD – EXTEND TO $\mathbb{C}$

α COMES FROM SOLUTION OF BELTRAMI EQ. TO MAKE φ ANALYTIC OFF $\partial\mathbb{D}$.

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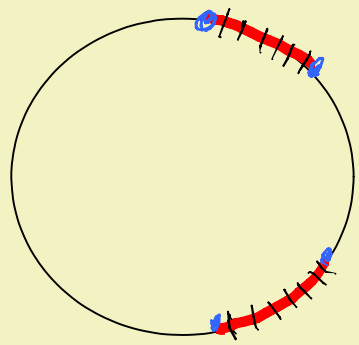

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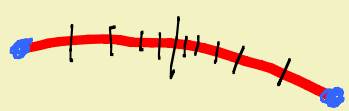
PICTURES NOT GREAT SINCE THE IMAGES WON'T BE EQUALLY SPACES SINCE $\varphi \sim$ POWER NEAR VERTICES.

BEST: REFINE LAMINATION ACCORDING TO DEGREE OF VERTEX (KNOWN)



STILL "EUCLIDEAN" IDENTIFICATION SO LONG AS SUBDIVISION IS THE SAME FOR A GIVEN POWER.

WE GIVE SOME IMPROVEMENTS OF C & G METHOD

1. USE CONFORMAL WELDING - COMPLETELY HANDS FREE

2. A DIFFERENT ALGORITHM IN STAGE 2

3. PSLQ ALGORITHM INSTEAD OF LLL

FOR STAGE 2, WE SIMPLY WRITE $p'(z) = c\prod_{j=1}^{k}(z - v_j)^{n_j}$

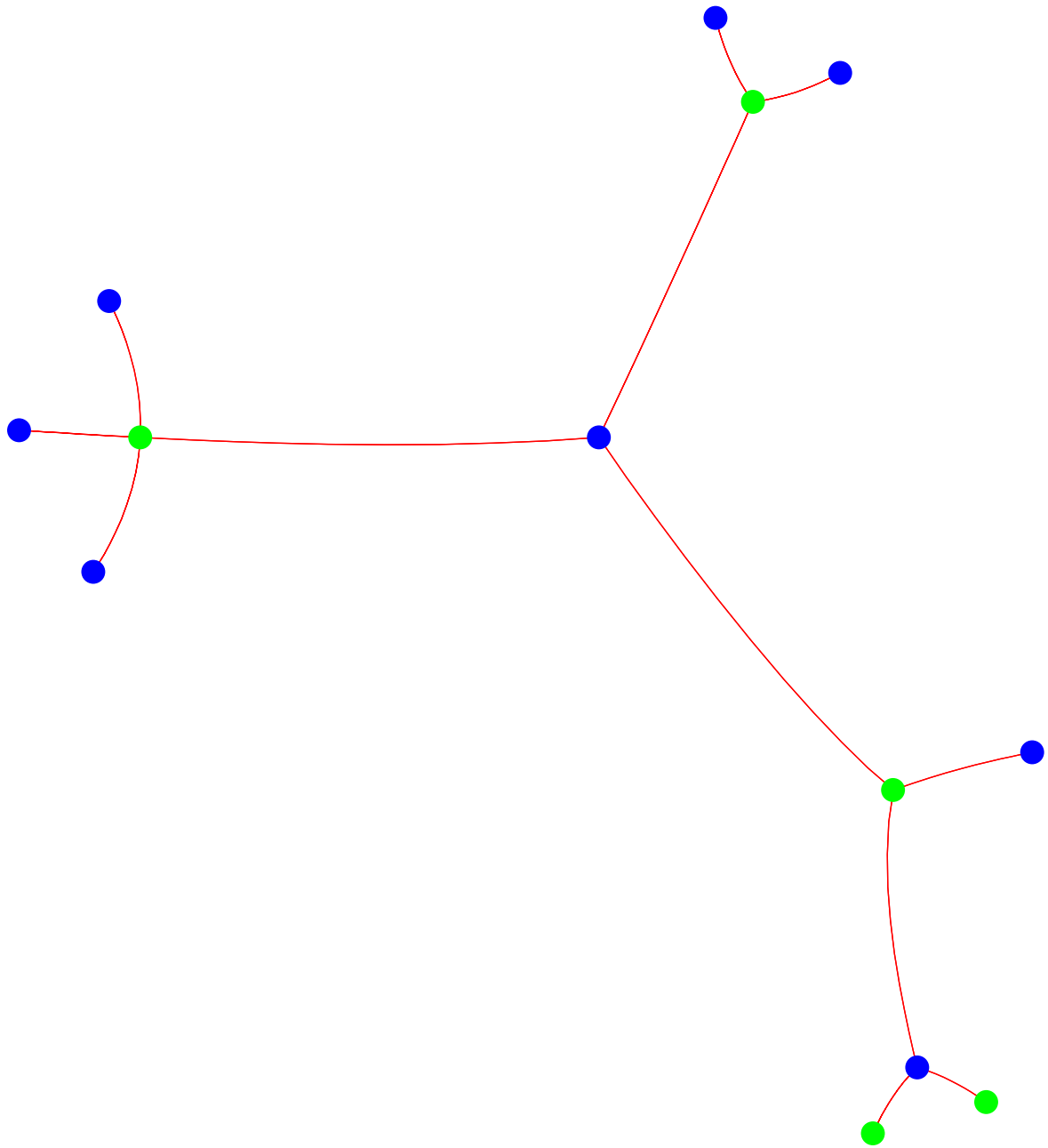
FIND p FROM COEFFICIENTS $v_1, \ldots, v_k$ unknowns

Solve $p(v_j) = 0$ or 1 WLOG $v_1 = 0$ $v_2 = 1$

$k-1$ unknowns $v_3, \ldots, v_k, c$

$k-1$ equations.

JUST USE NEWTON'S METHOD. – ONCE v_j ARE FOUND, USE NEWTON AGAIN TO FIND OTHER ROOTS TO $p=0$, $p=1$ (ALL OF THESE ROOTS ARE SIMPLE).



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10 ^       0 x  0.,
        1
10 ^       0 x  0.,
10 ^       0 x  0.,
        2
10 ^       0 x  0.,
10 ^       0 x  0.,
        3
10 ^       0 x  0.,
10 ^       0 x  0.,
        4
10 ^       0 x  3.856983139614529149333146433902169255569686449117718222435 1
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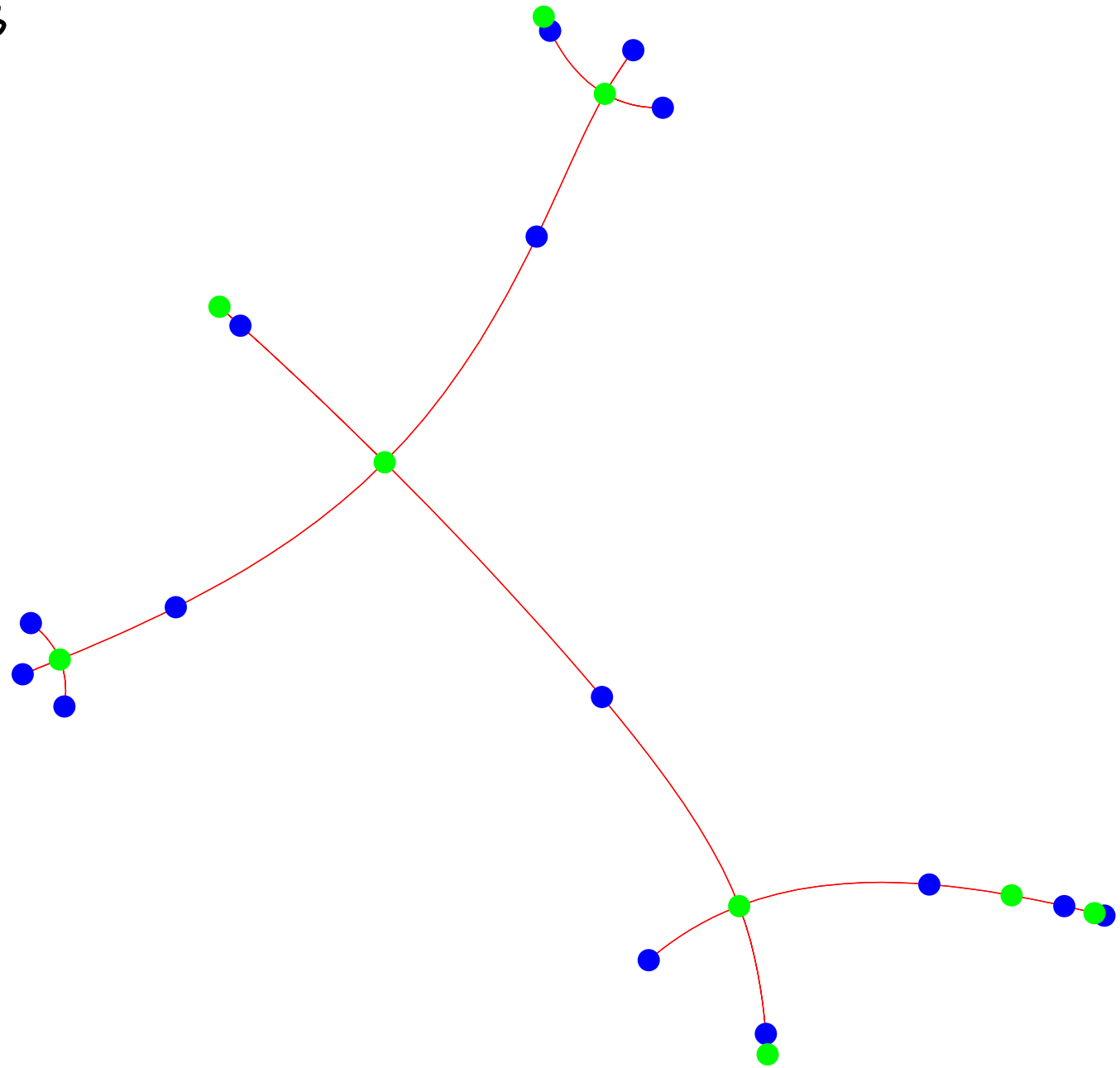
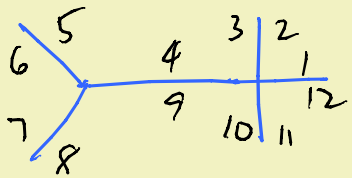
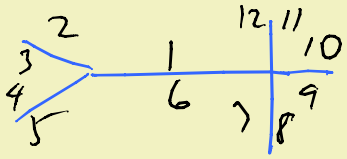
M_{23}



TO GENERATE A CATALOG AUTOMATICALLY WE NEED TO BE ABLE TO LIST ALL "NON-CROSSING" MATCHINGS OF $1, \ldots, 2N$.

TWO MATCHINGS GENERATE THE SAME TREE IFF ONE IS A "ROTATION" OF THE OTHER





SAME.

L_{N-1} = MATCHINGS FOR $N-1$ EDGES (ELIM. ROTATIONS)

$$1 \quad 2 \cdots\cdots 2N-2$$
$$m_1 \quad m_2 \cdots m_{2N-2}$$

CONSTRUCT AN ELEMENT OF L_N

$$1 \quad 2 \cdots i \quad i+1 \cdots 2N-2$$
$$m_1 \quad m_2 \quad m_i \quad m_{i+1} \cdots m_{2N-2}$$

INSERT $2N-1 \quad 2N$
$2N \quad 2N-1$

RELABEL $i+j \to i+j+2$
$2N-1 \to i+1$
$2N \to i$

LOOK AT ALL "ROTATIONS" IF NONE ARE IN N-LIST SO FAR, ADD IT

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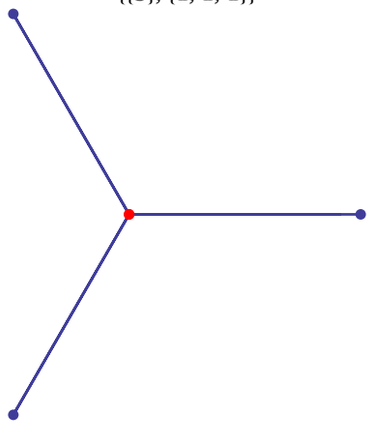
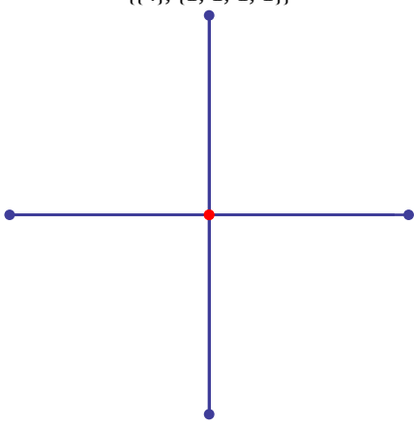
{{3}, {1, 1, 1}}

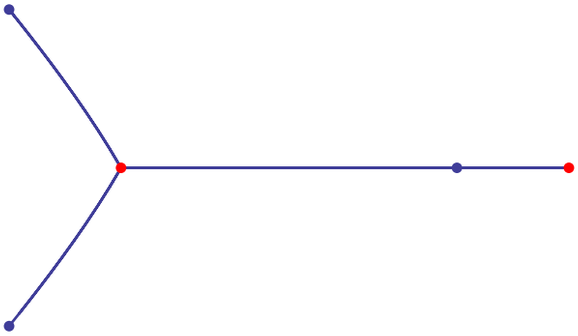


{{2, 1}, {2, 1}}

{{4}, {1, 1, 1, 1}}

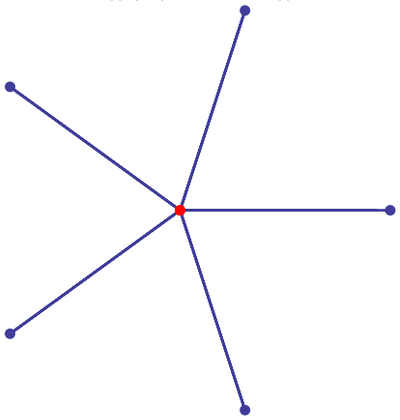


{{3, 1}, {2, 1, 1}}

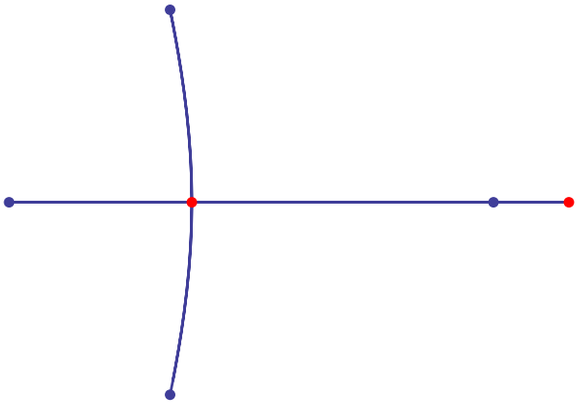


{{2, 1, 1}, {2, 2}}

{{5}, {1, 1, 1, 1, 1}}



{{4, 1}, {2, 1, 1, 1}}



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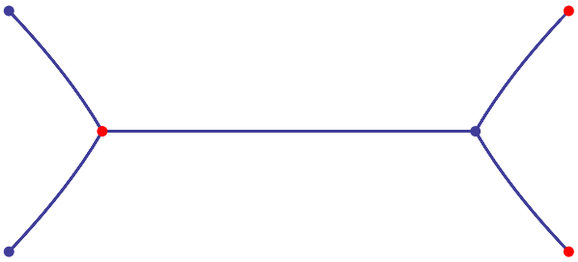

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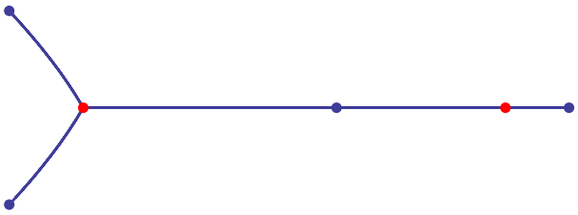

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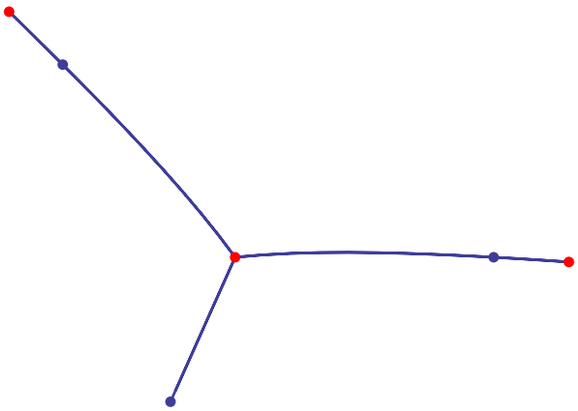

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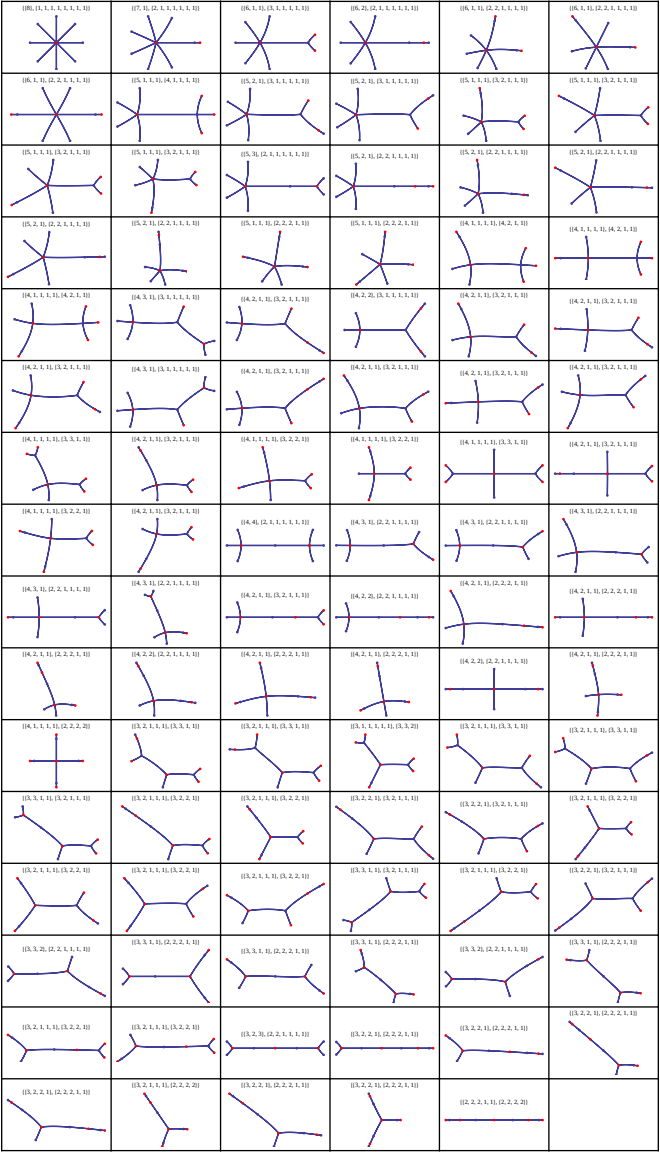
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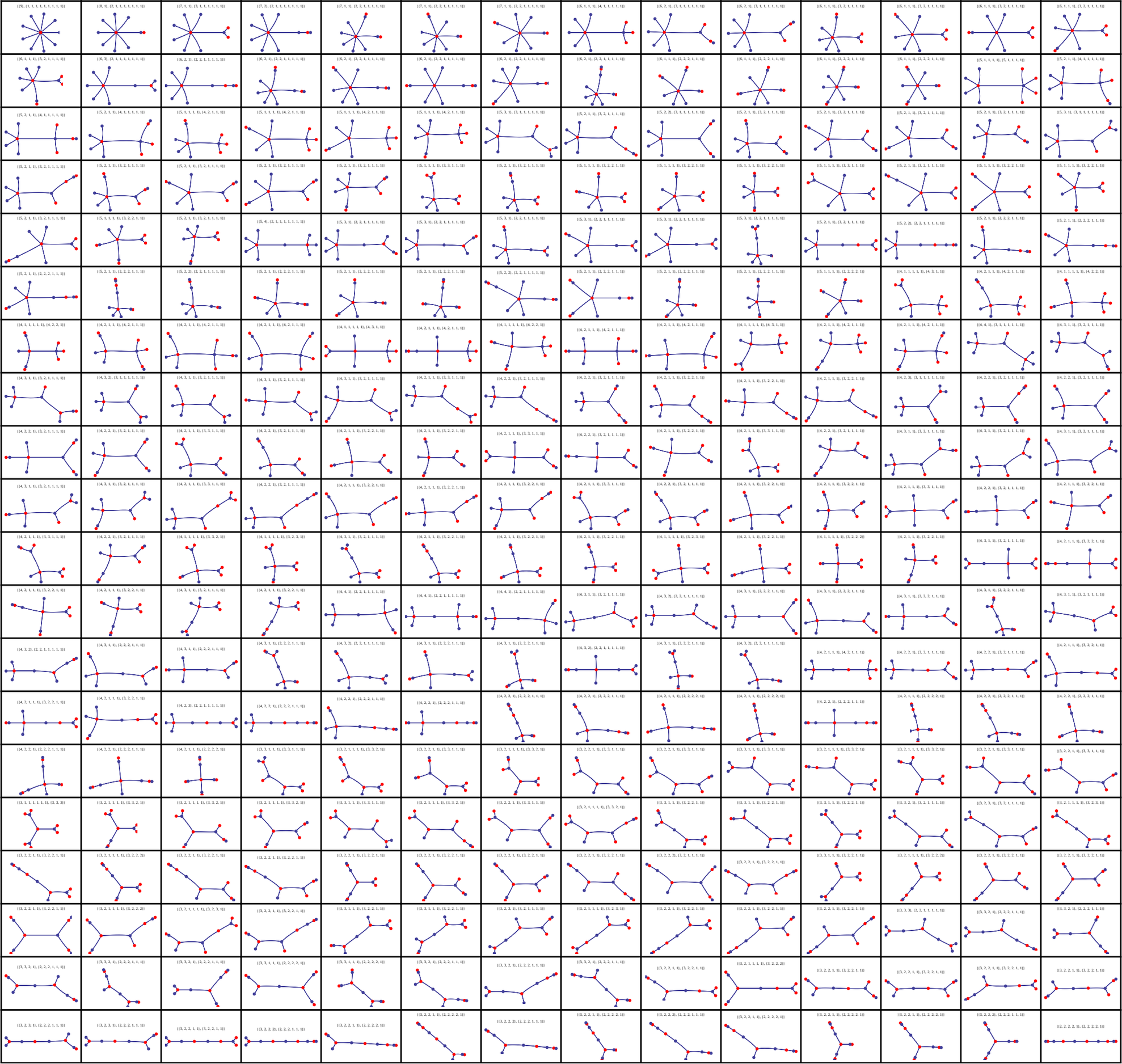
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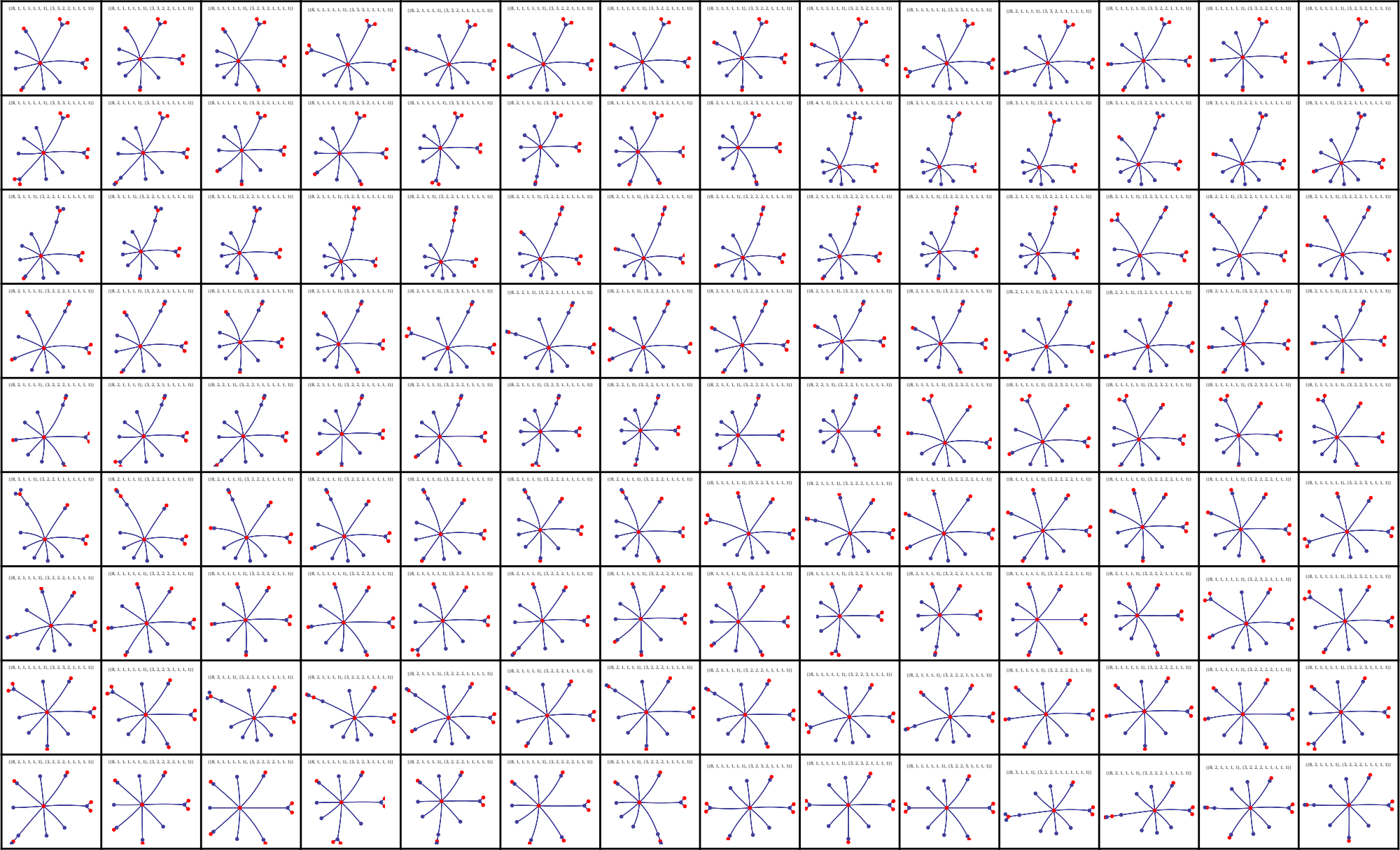
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EDGES	TREES
2	1
3	2
4	3
5	6
6	14
7	34
8	95
9	280
10	854
11	2694
12	8,714
13	28,640
14	95,640

B & Z K.

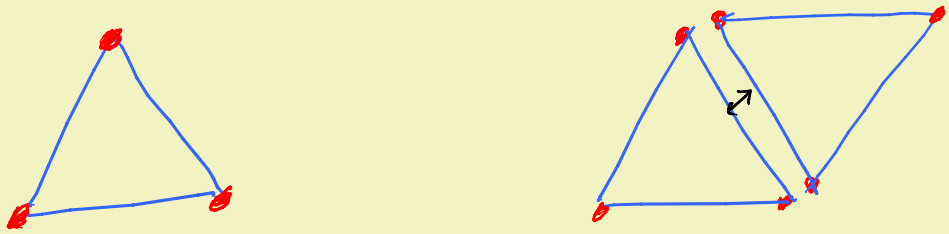
GROWTH RATE RELATED TO CATALAN NUMBERS.

- USE ~10 ADDITIONAL POINTS PER EDGE
- WELDING GIVES ABOUT 3 DIGITS OF ACCURACY TO VERTICES
- ZIPPER IS VERY FAST & SIMPLE
- NEWTON ITERATION IS WITHIN REGION OF QUADRATIC CONVERGENCE. 9-10 ITERATIONS TO GET ~1000 DIGITS.
- NEED LOTS OF DIGITS OF ACCURACY LOSE ABOUT 8-9 DIGITS DUE TO OSCIL. OF POLY. NEEDED QUAD PRECISION ~34 DIGITS TO DRAW ACCURATE TREES.

(WALK UP)

UNIFORMIZATION OF "TRIANGULATIONS" OF $\mathbb{C}^*$

CUT OUT A BUNCH OF EQUILATERAL TRIANGLES



PASTE TOGETHER ALONG EDGES.

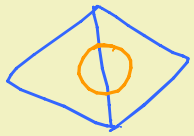
IF 6 MEET AT A VERTEX — FLAT

BUT FEWER THAN 6, FORM "CONE"

MORE THAN 6 "CRINKLY"

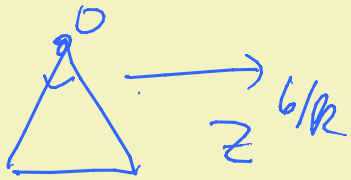
DEFINE A RIEMANN SURFACE. W

CHARTS: EUCLIDEAN DISKS ON INTERIOR OF TRIANGLES



TWO HALF DISKS ON ADJACENT TRIANGLES AT THE EDGES.

IF K MEET AT A VERTEX



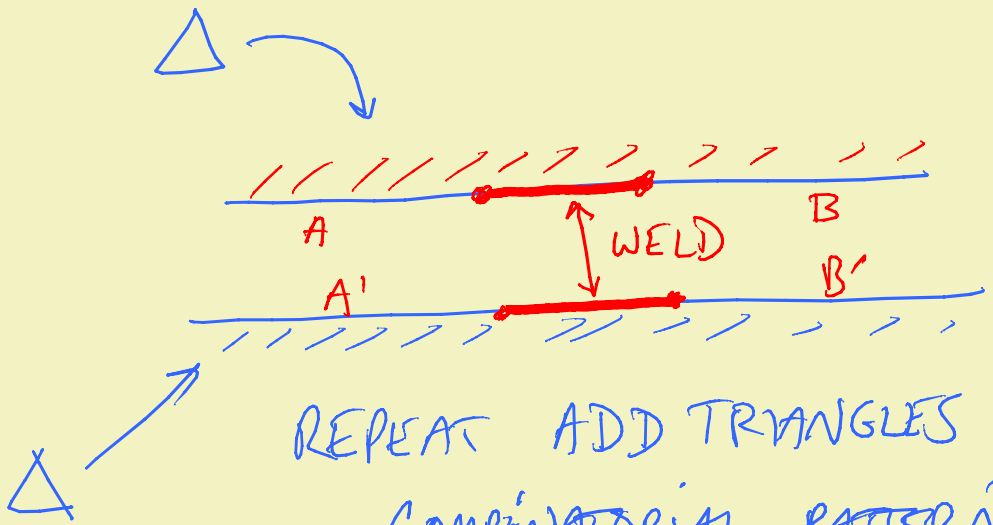
$\frac{6}{k} \cdot \pi/3 = \frac{2\pi}{k}$.

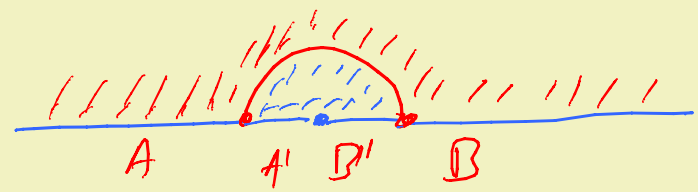
UNION OF SECTORS IS CHART.

IF W IS SIMPLY CONNECTED & COMPACT, IT IS CONFORMALLY $\approx \mathbb{S}^2$

THE IMAGES OF THE EDGES ON $\mathbb{S}^2$ ARE ANALYTIC ARCS.

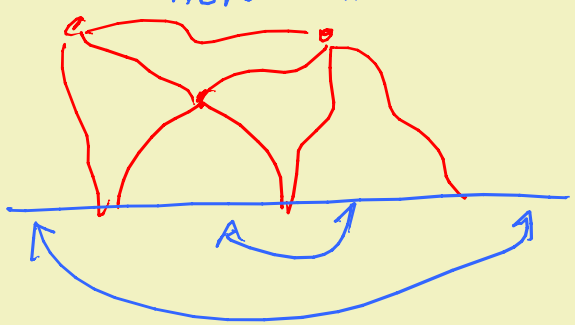
THIS CAN BE ACCOMPLISHED BY THE ZIPPER WELDING ALGORITHM





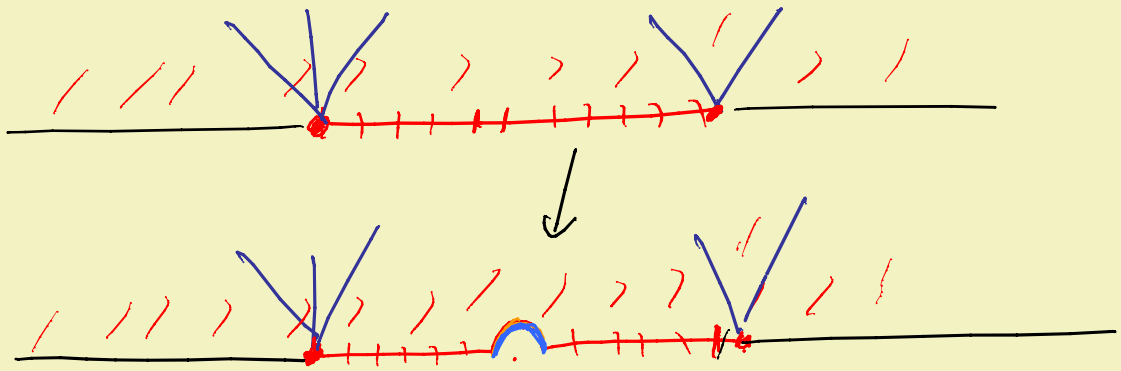
REPEAT ADD TRIANGLES ACCORDING TO THE GIVEN COMBINATORIAL PATTERN

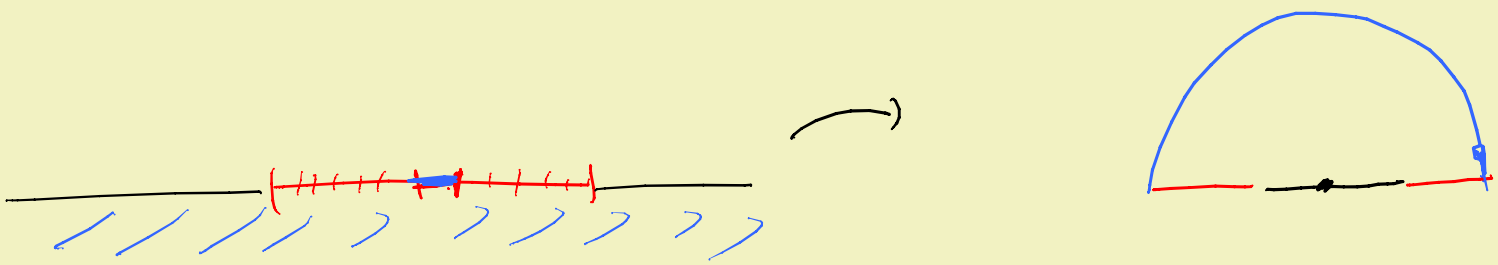
WHEN THERE ARE NO MORE TRIANGLES TO ADD (FINITE) THE REMAINING INTERVALS ON $\mathbb{R}$ ARE IDENTIFIED

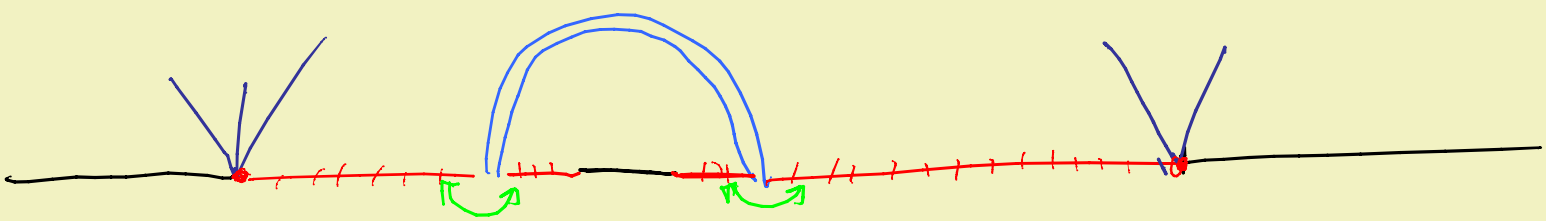


WANT ANGLES EQUAL AT VERTICES AS BEFORE

KEY: START MID-INTERVAL (ELSE SPIRALS)



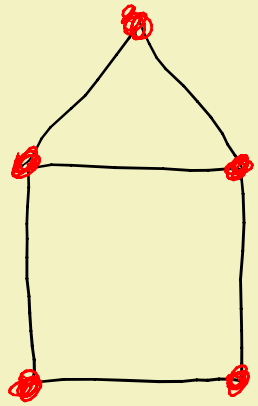




THEN MAKE ANGLES EQUAL AS BEFORE

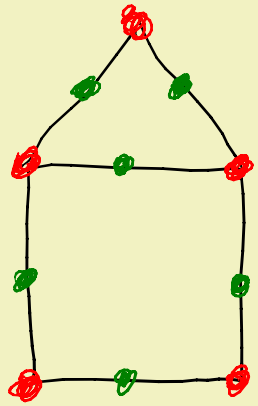
APPLICATION

EMBEDDED GRAPHS
(MAPS)



APPLICATION

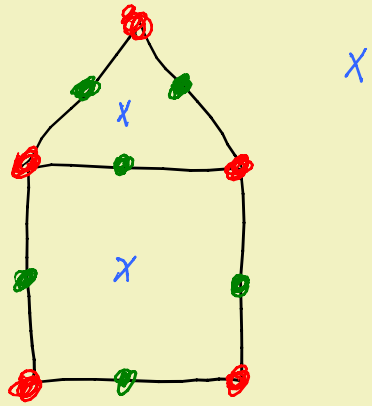
EMBEDDED GRAPHS
(MAPS)



ADD ONE VERTEX PER EDGE

APPLICATION

EMBEDDED GRAPHS
(MAPS)

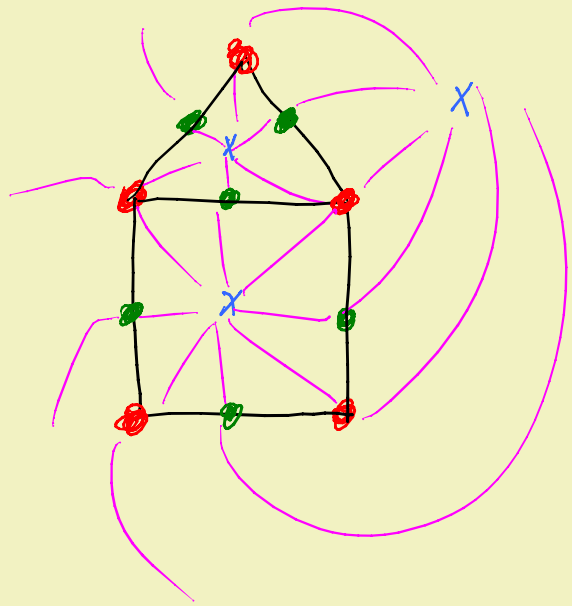


ADD ONE VERTEX PER EDGE

AND ONE VERTEX PER FACE

APPLICATION

EMBEDDED GRAPHS
(MAPS)



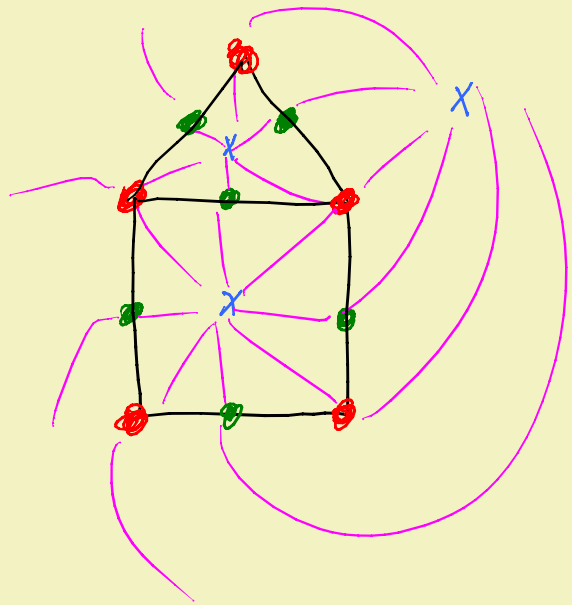
ADD ONE VERTEX PER EDGE

AND ONE VERTEX PER FACE

& CONNECT IT TO NEIGHBORS

APPLICATION

EMBEDDED GRAPHS
(MAPS)



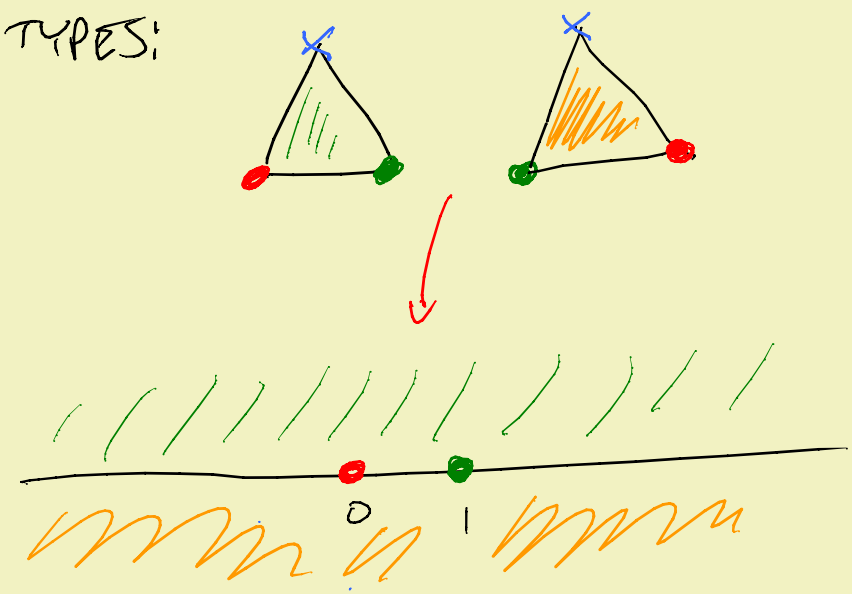
ADD ONE VERTEX PER EDGE

AND ONE VERTEX PER FACE

& CONNECT IT TO NEIGHBORS

OBTAIN A "TRIANGULATION OF $\mathbb{C}^*$

TWO TYPES:



AGAIN GET A RIEMANN SURFACE

BUT ALSO A MAP OF $S^2 \setminus$ ANALYTIC ARCS

ONTO $\mathbb{C} \setminus \mathbb{R}$ WHICH IS CONTINUOUS UP TO THE ARCS

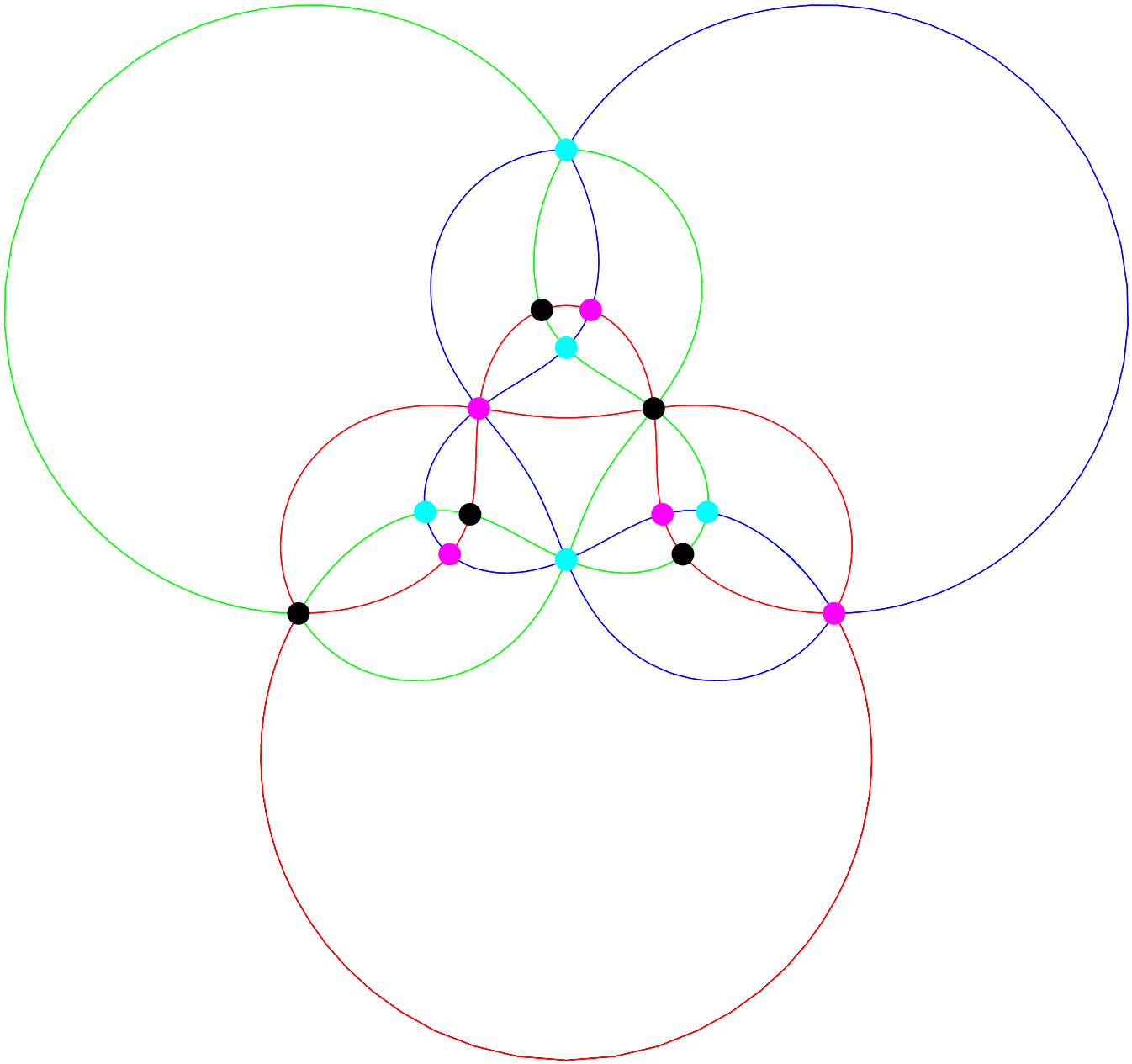
AND HENCE A RATIONAL FUNCTION R. (BELYĬ)

- CRITICAL POINTS $\subset \{R = 0, 1, \infty\}$.

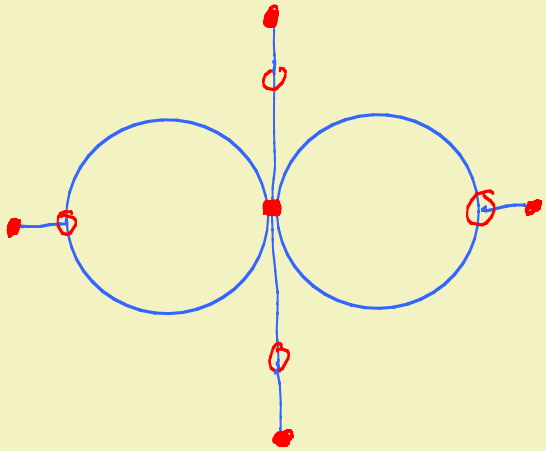
GROTHENDIECK WAS SO STRUCK BY BELYI'S THEOREM HE CONSIDERED IT "LIFE-CHANGING" & WORKED ON THEM UNTIL HE QUIT MATH. — HE CALLED THEM "DESSINS D'ENFANTS"

$$\text{DESSIN} = R^{-1}[0,1]$$

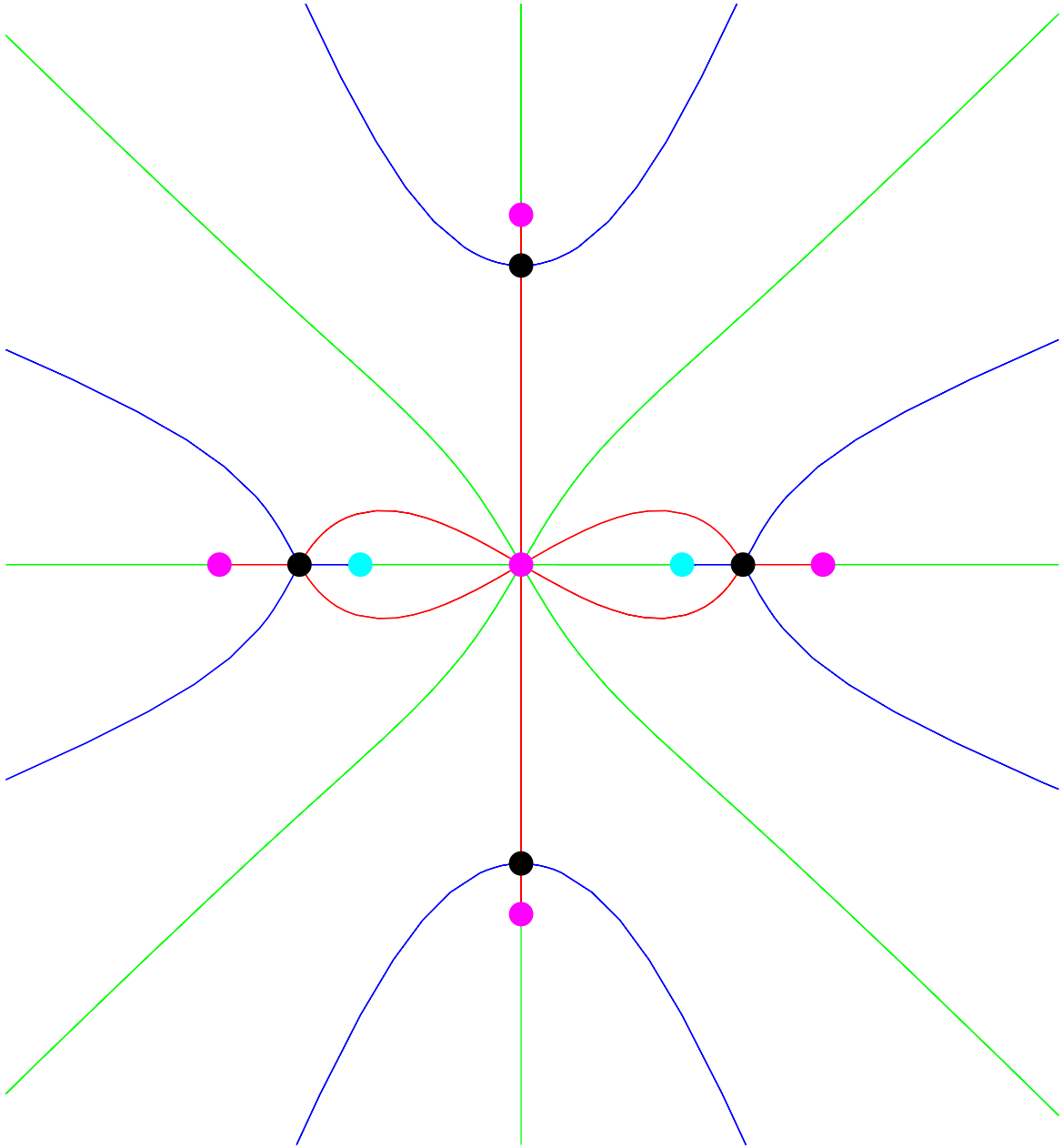
G. Cui

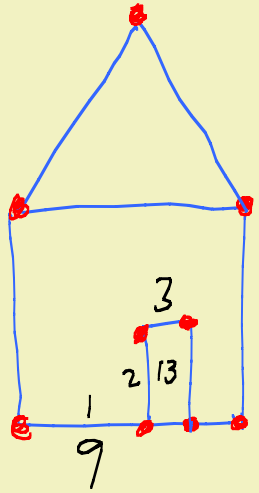


CP AT VERTICES OF EQUILATERAL TRIANGLE



SOMETIMES NEED ONLY ADD FACE VERTICES AND EDGES



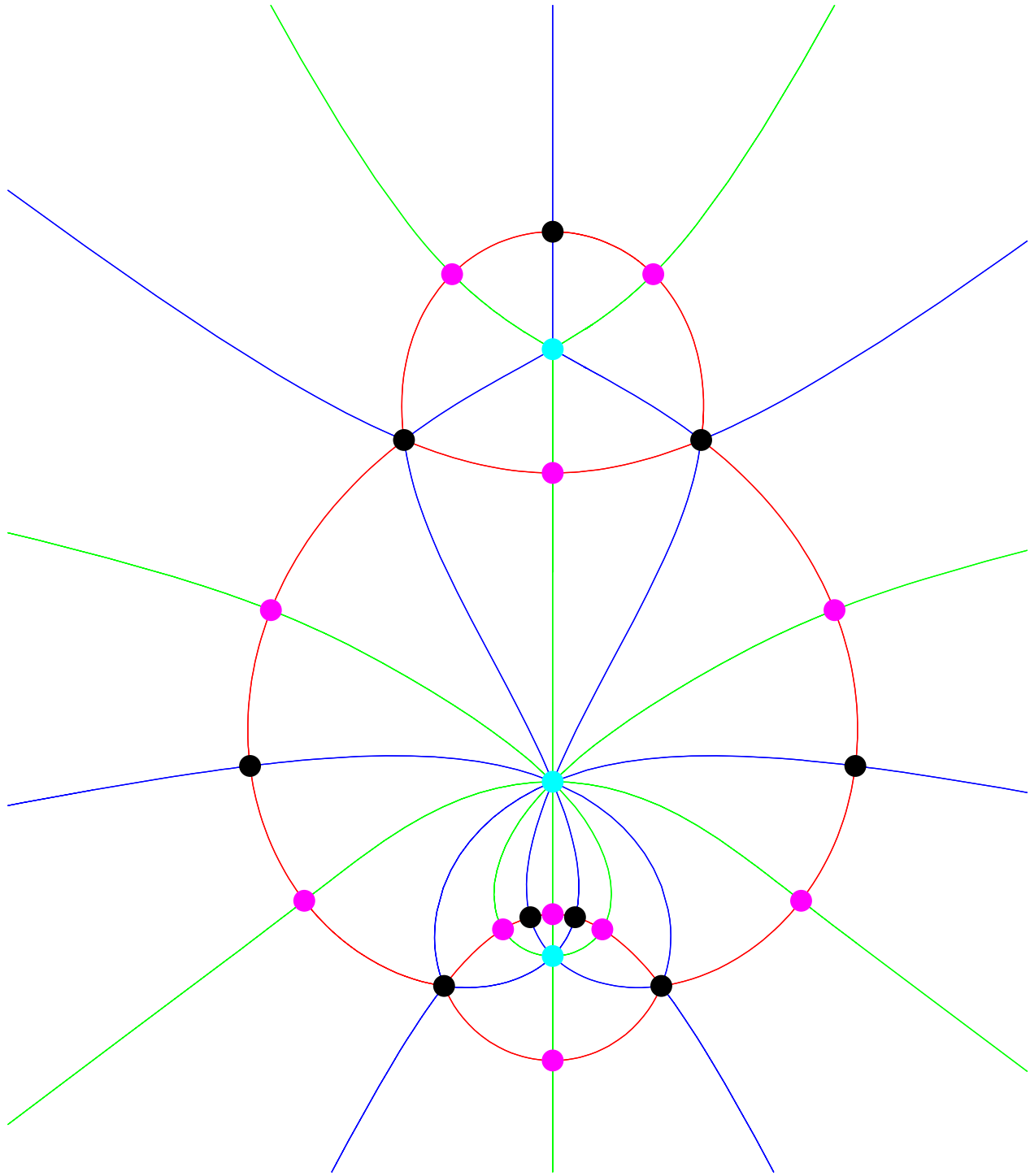


INPUT

EDGE,	MATCH,	NEXT
1	9	2
2	13	3

ETC.

66 x 2 EDGES 5 SEC ERROR < 10^{-24}



CURRENT PROJECT:

CONSTRUCT CATALOG OF (0-GENUS) BELYI FUNCTIONS)

— GRAPHS EMBEDDED IN $\mathbb{C}$. (MAPS)

WORK WITH QUADRANGULATIONS

TO GENERATE "MATCHINGS" —

(others don't occur in finite quadrangulations)

- METHOD FOR REDUCING N-QUAD. TO (N-1) QUAD. THAT DOES NOT DEPEND ON LABELLING.

THANK YOU

GO RAIBH MAITH AGAIBH