Algebraic Aspects of the Dirichlet Problem

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Outline

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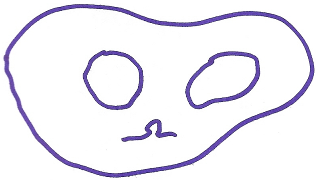
$$\begin{cases} \Delta u = 0, & \text{in } \Omega \\ u = v & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where the data $v \in C(\partial\Omega)$.



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QUESTION. When is the solution u a rational function of x and y as well?

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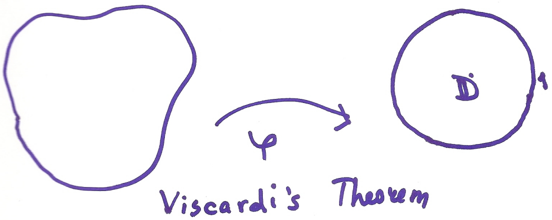

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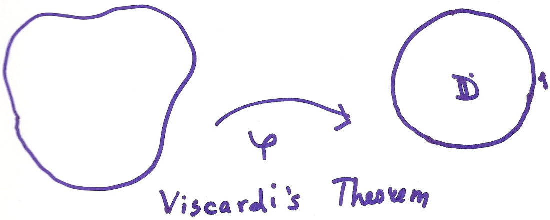

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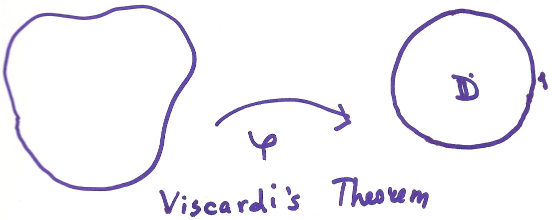

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- Accordingly, the Schwarz function $S(z)$ of $\partial\Omega$ ($\bar{z} = S(z)$on $\partial\Omega$) is a rational function. P. Davis' theorem $\Rightarrow$ that Ω is a disk.

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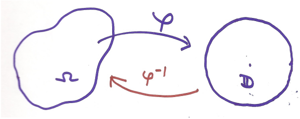

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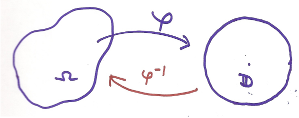

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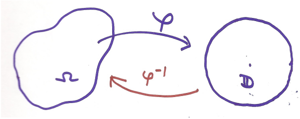

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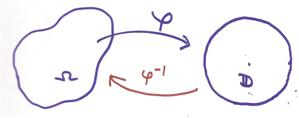

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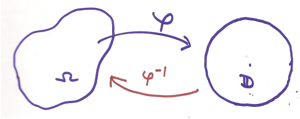

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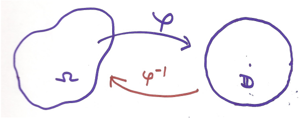

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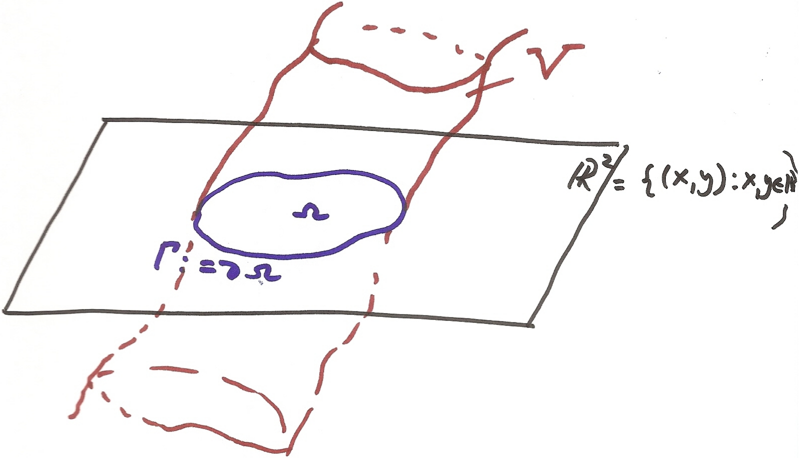
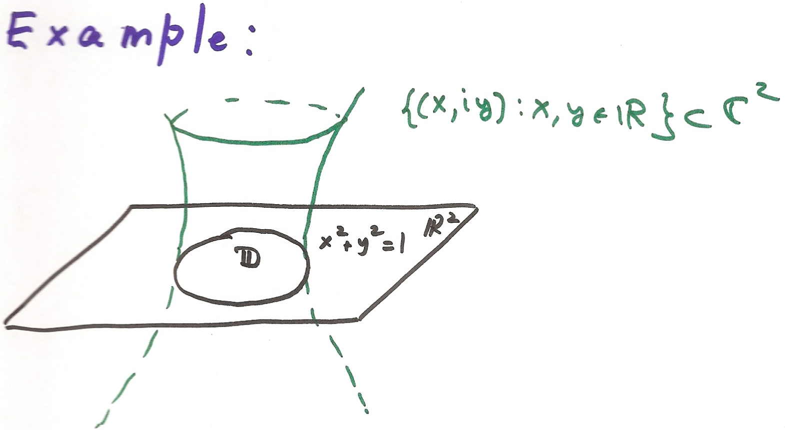
$\mathbb{C}^2$ views

$$\mathbb{C}^2 = \{(x,y) : x, y \in \mathbb{C}\}$$



example



Example:
$\{(x,iy): x,y \in \mathbb{R}\} \subset \mathbb{C}^2$
$x^2+y^2=1$
$\mathbb{R}^2$
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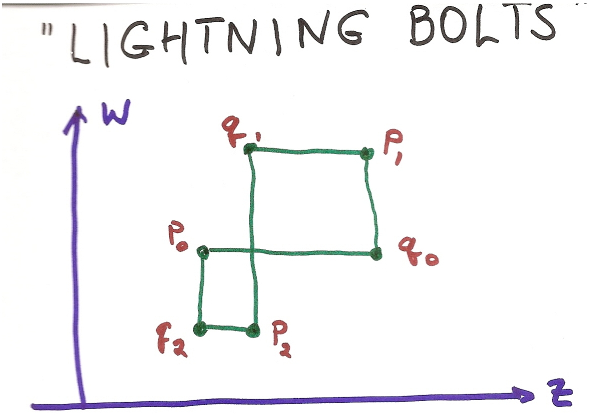
- An obstruction to matching on V any polynomial $R(z,w)$ with a sum of meromorphic functions $R(z,w) = F(z) + G(w)$, F, G meromorphic near V, would be, for example, a continuum family of finite atomic measures μ_t on V which annihilate ALL sums $f(z) + g(w)$, with f, g holomorphic near *supp* μ_t.

Lightning Bolts: an example



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- A complex "lightning bolt" (LB) is a finite set of points (vertices) $p_0, q_0, p_1, \ldots, p_n, q_n$ in $\mathbb{C}^2$ such that each complex line connecting p_j to q_j or q_j to p_{j+1} is either "horizontal" or "vertical".

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$$\mu := \sum_{j=0}^{n} \delta_{p_j} - \sum_{j=0}^{n} \delta_{q_j},$$

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- The measure μ is an annihilating measure for all holomorphic functions in $\mathbb{C}^2$ representable in the form $f(z) + g(w)$.

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- Independently, L. Hansen and H. S. Shapiro were the first to consider in 1994 the simplest closed complex LB with 4 vertices ("rectangles") in connection with the functional equations arising from continuation of solutions to the DP.

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Let Ω be a bounded simply connected domain in **C** *with smooth boundary. Assume that a Riemann mapping $\phi \colon \Omega \to \mathbb{D}$ is algebraic and that the inverse $\phi^{-1} \colon \mathbb{D} \to \Omega$ is NOT rational. Then, the complexified boundary $V_{\partial\Omega}$ contains a continuum family of closed LB.*

The crux in the proof is an explicit construction.

Further Discussion

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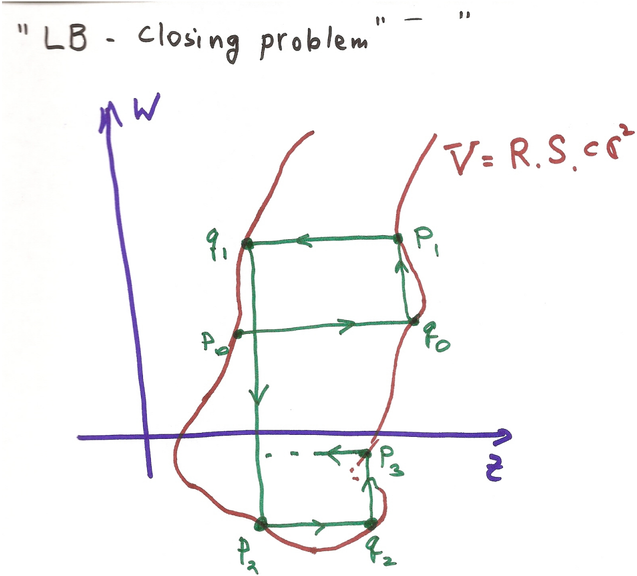
For this we have to resort to the specific construction of a rather special family of grids of points obtained as orbits of a special finite subgroup of the monodromy group with two generators.

Closing a LB



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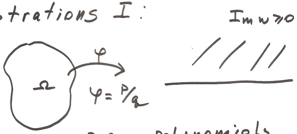
An illustration I

Illustrations I:



P, q - polynomials

$$\partial\Omega \subseteq \left\{ z : P(z)/q(z) = \overline{P(z)}/\overline{q(z)} \right\}$$

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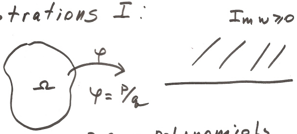

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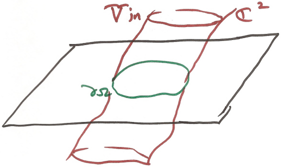
Illustrations II $\mathbb{C}^2$:

$$\partial\Omega \subseteq V := \left\{(z,w) : \frac{p(z)}{q(z)} = \frac{p^*(w)}{q^*(w)}\right\}$$

$$= \left\{(z,w) : \varphi(z) = \varphi^*(w)\right\}$$



Sketch of Proof of Thm IV (ii)

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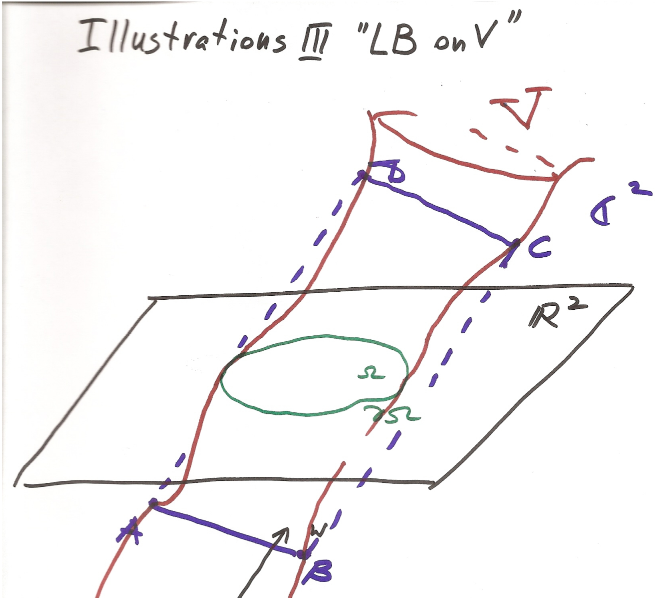
  $u(A) + u(C) = u(B) + u(D$ holds for all such u.
- Taking the data to be a polynomial v in z and w such that $v(A) = v(B) = v(C) = 0$ while $v(D) = 1$ we arrive at a contradiction. Hence the degree of ϕ must be 1 and Ω is a disk.

An illustration II: the LB we want



Final Punch in the Proof: Construction of LB

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E. Lundberg 2009, LB for the TV-screen

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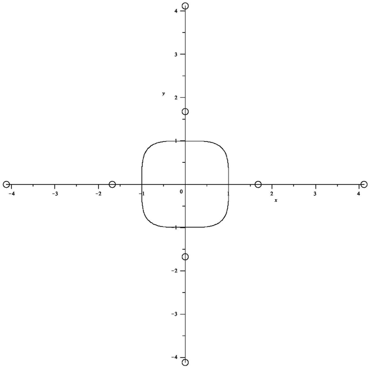

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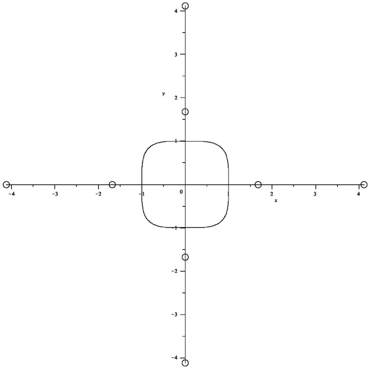

(The fact that the solution of the DP with the data $x^2 + y^2$ in the TV screen $x^4 + y^4 \leq 1$ develops countably many singularities outside is due to P. Ebenfelt, 1992.)

More examples of LB for cubics, E. Lundberg, 2009

The cubic $8x(x^2 - y^2) + 57x^2 + 77y^2 - 49 = 0$

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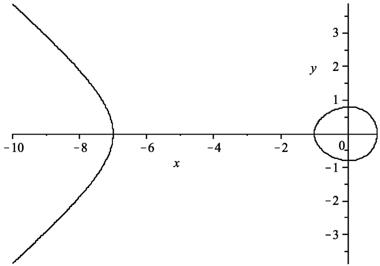

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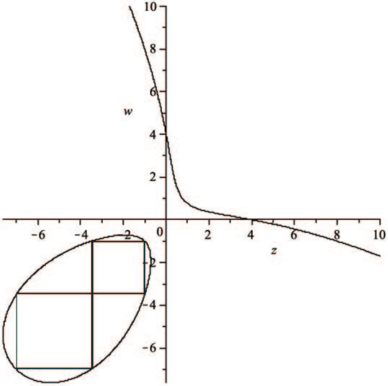

THANK YOU!