The Peak Point Conjecture and Function Algebras Invariant under Group Actions

“If I omit some of the hypotheses from the statement of a theorem, then one of the exercises is to determine the hypotheses under which the theorem holds.”

Definition: A **uniform algebra** (or **function algebra**) on compact space X is an algebra of continuous complex-valued functions on X that is uniformly closed, contains the constants, and separates points.

Examples:

1. $C(X)$= all continuous complex-valued functions on X

$X \subset \mathbb{C}^n$ compact

2. $P(X)$= the uniform closure of the polynomials in $z_1, \ldots, z_n$ on X
3. $R(X)$= the uniform closure of the rational functions holomorphic on X
4. $A(X)$= the continuous functions on X holomorphic on the interior of X

$\mathcal{M}_A =$ the maximal ideal space of A

"$X \subset \mathcal{M}_A$" $\qquad (x \sim \phi_x : f \mapsto f(x))$

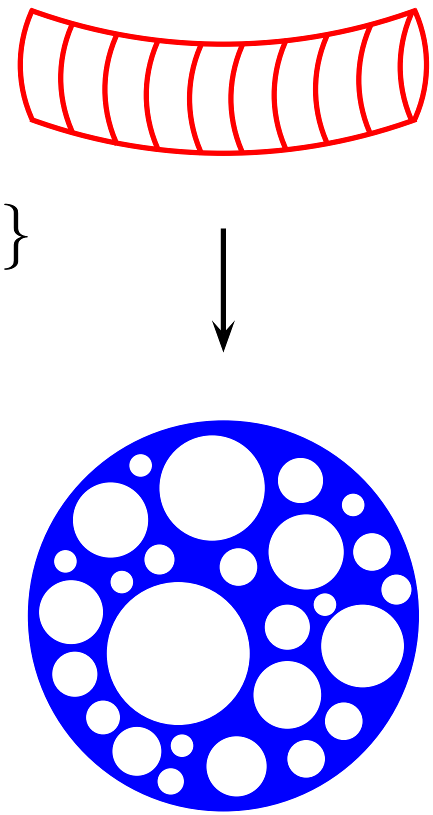
Definition: A point $x \in X$ is a **peak point** if there is a function $f \in A$ such that $f(x) = 1$ while $|f(y)| < 1$ for $y \neq x$.

Peak point conjecture: If A is a uniform algebra on a compact space X, $\mathcal{M}_A = X$, and every point of X is a peak point for A, then $A = C(X)$.

Bishop's peak point criterion (1959): Let $K \subset \mathbb{C}$ be compact. If every point of K is a peak point for $R(K)$, then $R(K) = C(K)$.

The peak point conjecture is False. (Brian Cole 1968)

Let $E \subset \{z \in \mathbb{C} : |z| < 1\}$ be compact and such that $R(E) \neq C(E)$ but the only Jensen measures for $R(E)$ are the point masses.



Basener's Example (1973):

$$X = \{(z, w) \in \mathbb{C}^2 : z \in E, |z|^2 + |w|^2 = 1\}$$

Basener showed $R(X)$ is a counterexample to the "peak point conjecture".

Corollary: There exists a counterexample to the peak point conjecture on $S^3 \subset \mathbb{C}^2$.

Question: If $f_1, \ldots, f_n \in A(\overline{B}_n)$ and $(\partial f_j / \partial z_k)$ is invertible at every point of B_n, does $A(\overline{B}_n)[\,\overline{f}_1, \ldots, \overline{f}_n\,] = C(\overline{B}_n)$?

Question: Is $P(X) = C(X)$ for every common level set $X \subset S = \{z \in \mathbb{C}^n : |z| = 1\}$ of $f_1, \ldots, f_n$?

Definition: The **polynomially convex hull** of $X \subset \mathbb{C}^n$ is

$$\hat{X} = \{z \in \mathbb{C}^n : |p(z)| \leq \sup_{x \in X} |p(x)| \quad \text{for every polynomial } p\}.$$

X is said to be **polynomially convex** if $\hat{X} = X$.

The maximal ideal space of $P(X)$ is $\hat{X}$.

The maximal ideal space of $C(X)$ is X.

$\therefore P(X) = C(X) \Rightarrow X$ is polynomially convex.

Question: If $X \subset S = \{z \in \mathbb{C}^n : |z| = 1\}$ and X is polynomially convex, is $P(X) = C(X)$?

Answer: No.

Theorem (I. 1996): There exists a polynomially convex subset X of the unit sphere in $\mathbb{C}^3$ such that $P(X) \neq C(X)$.

The proof is a modification of Basener's counterexample to the peak point conjecture.

The question is still open in $\mathbb{C}^2$.

Peak Point Theorem for 2-Manifolds (Anderson–I., 2001): Let M be a compact C^1 2-dimensional manifold-with-boundary, and let A be a uniform algebra on M generated by C^1 functions. If $\mathcal{M}_A = M$ and every point of M is a peak point for A, then $A = C(M)$.

Basener's counterexample to the peak point conjecture on S^3 shows that this result fails for 3-dimensional manifolds.

Theorem (Anderson-I.-Wermer, 2001/2009): Let M be a real-analytic 3-dimensional manifold-with-boundary, and let A be a uniform algebra on M generated by real-analytic functions. If $\mathcal{M}_A = M$ and every point of M is a peak point for A, then $A = C(M)$.

Peak point theorems for algebras generated by real-analytic functions on manifolds of arbitrary dimension.

More general forms of these theorems for uniform algebras on compact subsets of manifolds.

Theorem (Stout, 2006): If X is a compact, polynomially convex, real-analytic subvariety of $\mathbb{C}^n$, then $P(X) = C(X)$.

Definition: The **essential set** E for a uniform algebra A on X is the smallest closed set $E \subset X$ such that

$$f \in C(X), f \equiv 0 \text{ on } E \Rightarrow f \in A.$$

Note: $f \in C(X)$ and $f|E \in A|E \Rightarrow f \in A$.

Theorem (Anderson-I., 2009): Let A be a uniform algebra generated by C^1 functions on a compact C^1 manifold-with-boundary M. Assume $\mathcal{M}_A = M$ and every point of M is a peak point for A. Then the essential set for A has empty interior in M.

Localization theorem for 2-manifolds (I., 2010): Let M be a compact C^1 2-dimensional manifold-with-boundary, and let A be a uniform algebra on M generated by C^1 functions. If $\mathcal{M}_A = M$ and for each point $x \in M$ and each function $f \in C(M)$ there is a neighborhood of x on which f is uniformly approximable by A, then $A = C(M)$.

Question: Does this hold for arbitrary uniform algebras?

Question: Is every uniform algebra whose maximal ideal space is a 2-dimensional manifold local?

Example: On every compact C^∞ manifold of dimension ≥ 4, there exists a nonlocal algebra generated by C^∞ functions.

Question: Does there exist a nonlocal counterexample to the peak point conjecture?

Definition: $A(S) = A(\overline{B}_n)|S \qquad (S = \{z \in \mathbb{C}^n : |z| = 1\})$

We will refer to $A(S)$ as the **ball algebra**.

Question (Douglas): If A is a uniform algebra on the sphere $S \subset \mathbb{C}^n$ that contains the ball algebra $A(S)$ and whose maximal ideal space is S, and if A is invariant under the action of the n-torus on S, must $A = C(S)$?

Motivation: A conjecture of Arveson in operator theory.

$n = 1$: Yes. (Wermer's maximality theorem)

$n = 2$: Yes (provided A is generated by C^1 functions).

Theorem (I., 2009): Let A be a uniform algebra on the sphere $S \subset \mathbb{C}^2$ that contains $A(S)$. Assume the maximal ideal space of A is S, the algebra A is invariant under the torus action on S, and the C^1 smooth functions in A are dense in A. Then $A = C(S)$.

The proof is an application of the peak point theorem of Anderson-I. concerning the essential set of smooth counter-examples to the peak point conjecture on manifolds.

$n \geq 3$: No.

Theorem (I., 2009): There exists a uniform algebra on $S \subset \mathbb{C}^3$ such that $A(S) \subsetneqq A \subsetneqq C(S)$, the maximal ideal space of A is S, and A is invariant under the action of the 3-torus on S. (A can be taken to be generated by C^∞ functions.)

The construction is a modification of Basener's counter-example to the peak point conjecture and the example of a compact polynomially convex subset of the sphere in $\mathbb{C}^3$ on which the polynomials fail to be dense in the continuous functions.

What about transitive actions?

Theorem (I., 2009): Let A be a uniform algebra on a compact, smooth manifold M. Assume the maximal ideal space of A is M, and A is invariant under the action of a Lie group that acts transitively on M. Then $A = C(M)$.

Smoothness, manifolds, Lie groups are irrelevant?

Just need transitive topological group action?

Conjecture: Let A be a uniform algebra on a compact space X. Assume the maximal ideal space of A is X, and A is invariant under the action of a topological group G that acts transitively on X. Then $A = C(X)$.

This conjecture can be regarded as a replacement for the disproved peak point conjecture.

Definition: A topological group G is said to be **approximated by Lie groups** if every neighborhood of the identity in G contains a normal subgroup N such that G/N is a Lie group.

Theorem (I., 2010): The conjecture holds whenever the topological group G is locally compact and can be approximated by Lie groups.

Corollary (I., 2010): The conjecture holds for compact groups, for locally compact groups whose quotient modulo the identity component is compact, and for abelian groups.

Can the conjecture be generalized so as to apply to algebras on spaces without a transitive group of homeomorphisms?

What if the action is not transitive, but we assume that the orbit space of X under G coincides with the orbit space under the action of $\mathrm{Homeo}(X)$? Must then $A = C(X)$?

Answer: No.

Counterexample:
$A = A(\overline{D})$, $G = \{\text{the conformal self-maps of the disc}\}$,
Orbit space $= \{D, \partial D\}$

What if A is invariant under all self-homeomorphisms of X? Must then $A = C(X)$?

Answer: No again.

Theorem: There exists a compact space X with no nontrivial self-homeomorphisms on which there is a uniform algebra $A \neq C(X)$.

However, the answer is yes for "nice" spaces.

The proofs of all the positive results discussed above use some form (or generalization) of the following theorem.

Theorem (Wermer, etc.): If X is a compact polynomially convex subset of a smooth submanifold of $\mathbb{C}^n$ with no complex tangents, then $P(X) = C(X)$.