Vector valued multivariate spectral multipliers, Littlewood-Paley functions, and Sobolev spaces in the Hermite setting

Alejandro Sanabria
Department de Mathematical Analysis
University of La Laguna

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$\diamondsuit$ jointly with Jorge J. Betancor and Juan C. Fariña (University of La Laguna, Tenerife).
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The Hermite operator (also called harmonic oscillator) $\tilde{H}$ on $\mathbb{R}^n$ is defined by

$$\tilde{H} = -\Delta + |x|^2,$$

where Δ denotes the usual Laplacian operator.

For every $m \in \mathbb{N}$ we denote by h_m the m-th Hermite function given by

$$h_m(u) = (2^m m! \sqrt{\pi})^{-\frac{1}{2}} P_m(u) e^{-\frac{u^2}{2}}, \;\; u \in \mathbb{R},$$

where P_m represents the m-th Hermite polynomial.

If $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$, the k-th Hermite function h_k is defined by

$$h_k(x) = \prod_{j=1}^{n} h_{k_j}(x_j), \;\; x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

We have that, for every $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$,

$$\tilde{H} h_k = (2|k| + n) h_k,$$

where $|k| = k_1 + \cdots + k_n$.

- The sequence $\{h_k\}_{k \in \mathbb{N}^n}$ is an orthonormal basis in $L^2(\mathbb{R}^n)$.
- The linear space span$\{h_k\}_{k \in \mathbb{N}^n}$ generated by $\{h_k\}_{k \in \mathbb{N}^n}$ is dense in $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$.

The Hermite operator H is defined by

$$Hf = \sum_{k\in\mathbb{N}^n} (2|k|+n)c_k(f)h_k, \ \ f \in D(H),$$

where

$$D(H) = \{f \in L^2(\mathbb{R}^n) \colon \sum_{k\in\mathbb{N}^n} (2|k|+n)^2 |c_k(f)|^2 < \infty\}$$

and, for every $k \in \mathbb{N}^n$

$$c_k(f) = \int_{\mathbb{R}^n} h_k(x)f(x)dx, \ \ f \in L^2(\mathbb{R}^n).$$

- The space $C_c^\infty(\mathbb{R}^n)$ of smooth compactly supported functions in $\mathbb{R}^n$ is contained in the domain $D(H)$ of H and $Hf = \tilde{H}f$, $f \in C_c^\infty(\mathbb{R}^n)$

Hermite polynomial setting:

- Muckenhoupt (1969): one dimensional setting.
- Sjögren (1982), Urbina (1990), Fabes, Gutiérrez and Scotto (1994): harmonic analysis operators associated with the Ornstein-Uhlenbech operator in $\mathbb{R}^n$.
- In the last decade this topic has been studied by a host of authors: García-Cuerva, Mauceri, Meda, Sjögren, and Torrea (2000), Pérez, and Soria (2000), Harboure, Torrea, and Viviani (2003).

Hermite function setting:

- Thangavelu (1993) , K. Stempak and J.L. Torrea (2003): harmonic analysis operators associated with the Hermite operator (Hermite function expansions setting).
- Lust Piquard (2006) , Huang (2012) among others.

The heat semigroup $\{W_t^H\}_{t>0}$ generated by $-H$ in $L^2(\mathbb{R}^n)$ is defined by

$$W_t^H(f) = \sum_{k\in\mathbb{N}^n} e^{-(2|k|+n)t} c_k(f) h_k, \;\; f \in L^2(\mathbb{R}^n) \text{ and } t > 0.$$

According to Mehler's formula we can write, for every $t > 0$ and $f \in L^2(\mathbb{R}^n)$,

$$W_t^H(f)(x) = \int_{\mathbb{R}^n} W_t^H(x,y) f(y) dy,$$

where

$$W_t^H(x,y) = \frac{1}{\pi^{\frac{n}{2}}} \left(\frac{e^{-2t}}{1-e^{-4t}} \right)^{\frac{n}{2}} e^{-\frac{1}{4}\left(|x-y|^2 \frac{1+e^{-2t}}{1-e^{-2t}} + |x+y|^2 \frac{1-e^{-2t}}{1+e^{-2t}} \right)},$$

where $x, y \in \mathbb{R}^n$ and $t > 0$.

- $\{W_t^H\}_{t>0}$ is the semigroup of contractions in $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, generated by $-H$.

- The Poisson semigroup $\{P_t^H\}_{t>0}$ associated with the Hermite operator (generated by $-\sqrt{H}$) in $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, is defined by using the subordination formula as follows:

  For every $t > 0$,

$$P_t^H(f)(x) = \frac{t}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-\frac{t^2}{4u}}}{u^{\frac{3}{2}}} W_u^H(f)(x)du, \quad f \in L^p(\mathbb{R}^n).$$

- $\{P_t^H\}_{t>0}$ is a semigroup of contractions in $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$.

- The Hermite semigroups $\{W_t^H\}_{t>0}$ and $\{P_t^H\}_{t>0}$ are not conservative.

- $\{W_t^H\}_{t>0}$ and $\{P_t^H\}_{t>0}$ are not diffusion semigroups (in the sense of Stein).

- Hytönen's results (Revista Iberoaméricana de Matemáticas, ...) about vector valued Littlewood-Paley functions do not apply in the Hermite setting.

- The square function (also called Littlewood-Paley function) $g^H_{W,k}$ associated with the Hermite semigroup $\{W^H_t\}_{t>0}$ is defined by

$$g^H_{W,k}(f)(x) = \left(\int_0^\infty |t^k \partial_t^k W^H_t(f)(x)|^2 \frac{dt}{t}\right)^{\frac{1}{2}}, \; x \in \mathbb{R}^n,$$

for every $f \in L^p(\mathbb{R}^n)$ and $k \in \mathbb{N} \setminus \{0\}$.

- L^p-boundedness properties of the operator $g^H_{W,k}$, $k \in \mathbb{N} \setminus \{0\}$, were established by Thangavelu (Lectures on Hermite and Laguerre expansions) and Stempak and Torrea (Acta Math. Hungar.).
- For every $1 < p < \infty$ and $k \in \mathbb{N} \setminus \{0\}$, there exists $C > 0$ such that

$$\frac{1}{C}\|f\|_{L^p(\mathbb{R}^n)} \leq \|g^H_{W,k}(f)\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}, \; f \in L^p(\mathbb{R}^n).$$

- Equivalence above allows to obtain L^p-boundedness for spectral Hermite multipliers (Thangavelu - Lectures on Hermite and Laguerre expansions).

- The Littlewood-Paley functions associated with the Hermite-Poisson semigroup $\{P_t^H\}_{t>0}$ are defined by

$$g_{P,k}^H(f)(x) = \left(\int_0^\infty |t^k \partial_t^k P_t^H(f)(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}}, \ x \in \mathbb{R}^n, \ k \in \mathbb{N} \setminus \{0\}.$$

- For every $1 < p < \infty$ and $k \in \mathbb{N} \setminus \{0\}$, there exists $C > 0$ such that

$$\frac{1}{C} \|f\|_{L^p(\mathbb{R}^n)} \leq \|g_{P,k}^H(f)\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}, \ f \in L^p(\mathbb{R}^n).$$

- C. Segovia and R. Wheeden (J. Math. Mech.; 19 (1969-70); 247-262) introduced the notion of fractional derivative:

$$\partial_t^\sigma f(t,x) = \frac{e^{-\pi(m-\sigma)i}}{\Gamma(m-\sigma)} \int_0^\infty \partial_t^m f(t+s,x) s^{m-\sigma-1}\,ds, \;\; x \in \mathbb{R}^n \text{ and } t \in (0,\infty).$$

  where $\sigma > 0$ and $m \in \mathbb{N} \setminus \{0\}$ such that $m-1 \leq \sigma < m$.

- The generalized Littlewood-Paley function associated with the Poisson semigroup for the Hermite operator is defined by

$$g^H_{P,\sigma}(f)(x) = \left(\int_0^\infty |t^\sigma \partial_t^\sigma P_t^H(f)(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}}, \;\; \sigma > 0,$$

- For every $1 < p < \infty$ there exists $C > 0$ such that

$$\frac{1}{C}\|f\|_{L^p(\mathbb{R}^n)} \leq \|g^H_{P,\sigma}(f)\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}, \;\; f \in L^p(\mathbb{R}^n), \; \sigma > 0.$$

- J.L. Torrea and C. Zhang, Fractional vector-valued Littlewood-Paley-Stein theory for semigroups (arXiv: 1105.6022.v3) considered generalized Littlewood-Paley g-functions associated with diffusion semigroups.

- Suppose $\mathbb{B}$ is a Banach space and $\sigma > 0$. A first and natural definition of the Littlewood-Paley function on $L^p(\mathbb{R}^n,\mathbb{B})$, $1 < p < \infty$, is the following one. If $L^p(\mathbb{R}^n,\mathbb{B})$, $1 < p < \infty$, we define

$$G^H_{P,\sigma,\mathbb{B}}(f)(x) = \left(\int_0^\infty \|t^\sigma \partial_t^\sigma P_t^H(f)(x)\|^2_{\mathbb{B}} \frac{dt}{t}\right)^{\frac{1}{2}}, \ \ \sigma > 0,$$

- M. Martínez, J.L. Torrea, Q. Xu, C. Zhang, amongst others.

Theorem

Let $\mathbb{B}$ is a Banach space. The following assertions are equivalent:

i) $\mathbb{B}$ is isomorphic to a Hilbert space

ii) For some (equivalently, for any) $1 < p < \infty$ and $\sigma > 0$, we have that

$$\frac{1}{C}\|f\|_{L^p(\mathbb{R}^n,\mathbb{B})} \le \|G^H_{P,\sigma,\mathbb{B}}(f)\|_{L^p(\mathbb{R}^n,\mathbb{B})} \le C\|f\|_{L^p(\mathbb{R}^n,\mathbb{B})}, \ \ f \in L^p(\mathbb{R}^n,\mathbb{B}).$$

- Motivated by the ideas in:
  - C. Kaiser, Wavelet transforms for functions with values in Lebesgue spaces, Proceedings of SPIE Optics and Photonics, Conf. on Mathematical Methods. Wavelets XI 5914, 2005.
  - C. Kaiser and L. Weis, Wavelet transform for functions with values in UMD spaces, Studia Math., 186 (2), 101-126, 2008.

  Take a different point of view of vector valued Littlewood-Paley functions looking for the validity of the equivalence

$$\frac{1}{C}\|f\|_{L^p(\mathbb{R}^n,\mathbb{B})} \leq \|G^H_{P,\sigma,\mathbb{B}}(f)\|_{L^p(\mathbb{R}^n,\mathbb{B})} \leq C\|f\|_{L^p(\mathbb{R}^n,\mathbb{B})}, \;\; f \in L^p(\mathbb{R}^n,\mathbb{B}),$$

  for other Banach spaces that are not Hilbert spaces.
- UMD Banach spaces

- The Hilbert transform of $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$ is defined by

$$\mathrm{H}(f)(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|x-y|>\varepsilon} \frac{f(y)}{y-x} dy, \text{ a.e. } x \in \mathbb{R}.$$

Theorem

The operator H is a bounded operator from $L^p(\mathbb{R})$ into itself, for every $1 < p < \infty$, and from $L^1(\mathbb{R})$ into $L^{1,\infty}(\mathbb{R})$

- If $1 < p < \infty$ and $\mathbb{B}$ is a Banach space, the Hilbert transform is defined on $L^p(\mathbb{R}) \otimes \mathbb{B}$ in a natural way as the operator $\mathrm{H} \otimes \mathrm{Id}_{\mathbb{B}}$.

Definition

A Banach space $\mathbb{B}$ is said to be a UMD space when the Hilbert transform H can be extended to the Bochner-Lebesgue space $L^p(\mathbb{R}, \mathbb{B})$ as a bounded operator from $L^p(\mathbb{R}, \mathbb{B})$ into itself, for some $1 < p < \infty$.

- The UMD property does not depend on p.
- $\mathbb{B}$ is a UMD Banach space if and only if the Hilbert transform can be extended to $L^1(\mathbb{R},\mathbb{B})$ as a bounded operator from $L^1(\mathbb{R},\mathbb{B})$ into $L^{1,\infty}(\mathbb{R},\mathbb{B})$.
- There exist a lot of other characterizations for the UMD Banach spaces (Burkholder, Bourgain, Hytönen, Torrea, Xu, Harboure, Viviani, Guerre-Delabrière,...).

Examples

- Hilbert spaces,
- $L^p(\Omega;\mu;\mathbb{B})$, $1 < p < \infty$ and $\mathbb{B}$ is UMD,
- $L^{p_1}(L^{p_2}(...(L^{p_n})...))$ (mixed norm), $1 < p_j < \infty$, $j = 1, \dots, n$,
- $L^{p,q}(\Omega)$ (Lorentz spaces), $1 < p, q < \infty$,
- C_p (Schatten class), $1 < p < \infty$.

- Hytönen (Revista Iberoaméricana...) used stochastic integration to define Littlewood-Paley functions associated with diffusion semigroups.
- Kaiser and Weis studied vector valued square functions defined by convolutions by using γ-radonifying operators.

- Let $\{g_n\}_{n=1}^{\infty}$ be a sequence of independent standard Gaussian variables on a probability space $(\Omega, \mathbb{P})$.

Theorem

Let $\mathcal{H}$ be a separable real Hilbert space with orthonormal basis $\{h_n\}_{n=1}^{\infty}$ and let $\mathbb{B}$ be a real Banach space. For a bounded linear operator $T \in \mathcal{L}(\mathcal{H}, \mathbb{B})$ the following assertions are equivalent:

- The operator TT^* is the covariance of a Gaussian measure m on $\mathbb{B}$;
- Almost surely the series $\sum_{n=1}^{\infty} g_n T h_n$ converges in $\mathbb{B}$;
- For some $1 \leq p < \infty$ the series $\sum_{n=1}^{\infty} g_n T h_n$ converges in $L^p(\Omega, \mathbb{B})$;
- For every $1 \leq p < \infty$ the series $\sum_{n=1}^{\infty} g_n T h_n$ converges in $L^p(\Omega, \mathbb{B})$.

In this situation, for every $1 \leq p < \infty$, and $x = \sum_{n=1}^{\infty} g_n T h_n$

$$\int_{\mathbb{B}} \|x\|^p d\mu(x) = \mathbb{E} \left\| \sum_{n=1}^{\infty} g_n T h_n \right\|^p = sup_{N \geq 1} \mathbb{E} \left\| \sum_{n=1}^{N} g_n T h_n \right\|^p$$

Definition

A bounded operator $T \in \mathcal{L}(\mathcal{H}, \mathbb{B})$ is called γ-radonifying if it satisfies the equivalent conditions of the theorem. For such an operator we define

$$\|T\|_\gamma = \left(\int_{\mathbb{B}} \|x\|^2 d\mu(x) \right)^{\frac{1}{2}} = \left(\mathbb{E} \left\| \sum_{n=1}^{\infty} g_n T h_n \right\|^2 \right)^{\frac{1}{2}}$$

Theorem

- $(\gamma(\mathcal{H}, \mathbb{B}), \|\cdot\|_\gamma)$ is a real Banach space, which is separable provided that $\mathbb{B}$ is separable.
- If $T \in \mathcal{L}(\mathcal{H}, \mathbb{B})$ is γ-radonifying, then T is compact.
- If $\mathbb{B}$ is a real Hilbert space, then $T \in \mathcal{L}(\mathcal{H}, \mathbb{B})$ is γ-radonifying if and only if T is Hilbert-Schmidt.
- $\gamma(\mathcal{H}, \mathbb{C}) = \mathcal{H}$.

- Suppose that
  - $\mathcal{H} = L^2(W,\mu)$ where $(W,\mathfrak{B},\mu)$ is a σ-finite measure space with a countably generated σ-algebra $\mathfrak{B}$ and that
  - $f\colon W \to \mathbb{B}$ is a strongly measurable function such that f is weakly $\mathcal{H}$, that is, for every $S \in \mathbb{B}^*$, the dual space of $\mathbb{B}$, the function $S \circ f \in \mathcal{H}$.
- Then there exists $T_f \in \mathcal{L}(\mathcal{H},\mathbb{B})$ satisfying that

$$\langle S, T_f(\varphi)\rangle_{\mathbb{B}^*,\mathbb{B}} = \int_W \langle S, f(w)\rangle_{\mathbb{B}^*,\mathbb{B}} \varphi(w) d\mu(w), \;\; \varphi \in \mathcal{H}.$$

where $\langle \cdot,\cdot\rangle_{\mathbb{B}^*,\mathbb{B}}$ denotes the $(\mathbb{B}^*,\mathbb{B})$ duality.

- We say that $f \in \gamma(W,\mu;\mathbb{B})$ when $T_f \in \gamma(\mathcal{H};\mathbb{B})$ and then we define

$$\|f\|_{\gamma(W,\mu;\mathbb{B})} = \|T_f\|_{\gamma(\mathcal{H};\mathbb{B})}$$

It is usual to identify f with T_f .

- C. Kaiser and L. Weis, Suppose that $\psi \in L^2(\mathbb{R}^n)$. We consider $\psi_t(x) = \frac{1}{t^n}\psi\left(\frac{x}{t}\right)$, $x \in \mathbb{R}^n$ and $t > 0$. The wavelet transform W_ψ associated with ψ is defined by

$$W_\psi(f)(x,t) = (f * \psi_t)(x), \;\; x \in \mathbb{R}^n \text{ and } t > 0,$$

where $f \in \mathcal{S}(\mathbb{R}^n, \mathbb{B})$, the $\mathbb{B}$-valued Schwartz space.

- Kaiser and Weis gave sufficient conditions for ψ in order to

$$\|W_\psi f\|_{E(\mathbb{R}^n, \gamma(\mathcal{H},\mathbb{B}))} \sim \|f\|_{E(\mathbb{R}^n;\mathbb{B})},$$

for every $f \in E(\mathbb{R}^n;\mathbb{B})$, where $\mathbb{B}$ is a UMD Banach space and E represents L^p, $1 < p < \infty$, H^1 or BMO.

- Taking $\psi(x) = \partial_t P_t(x)_{|t=1}$, $x \in \mathbb{R}^n$, we have that

$$W_\psi(f)(x,t) = t\partial_t P_t(f)(x), \;\; x \in \mathbb{R}^n \text{ and } t > 0.$$

- J. Betancor, A. Castro, J. Curbelo, J. Fariña, and L. Rodríguez-Mesa. J. Funct. Anal., 263 (12): 3804-3856, 2012.
- The univariate fractional g-function operator associated with the Hermite-Poisson semigroup is defined by

$$\mathcal{G}^H_{P,\sigma;\mathbb{B}}(f)(t,x) = \int_{\mathbb{R}} t^\sigma \partial_t^\sigma P_t^H(x,y) f(y)dy, \quad \sigma > 0, \ x \in \mathbb{R} \text{ and } t > 0,$$

for every $f \in L^p(\mathbb{R}^n, \mathbb{B})$.

Theorem

Let $\mathcal{H} = L^2\left((0,\infty), \frac{dt}{t}\right)$, $\mathbb{B}$ be a UMD Banach space and $\sigma > 0$. Then, the operator $\mathcal{G}^H_{P,\sigma,\mathbb{B}}$ is bounded

- from $L^p(\mathbb{R}^n, \mathbb{B})$ into $L^p(\mathbb{R}^n, \gamma(\mathcal{H}, \mathbb{B}))$, for every $1 < p < \infty$,
- from $L^1(\mathbb{R}^n, \mathbb{B})$ into $L^{1,\infty}(\mathbb{R}^n, \gamma(\mathcal{H}, \mathbb{B}))$, and
- from $H^1(\mathbb{R}^n, \mathbb{B})$ into $L^1(\mathbb{R}^n, \gamma(\mathcal{H}, \mathbb{B}))$.

Moreover, for every $1 < p < \infty$, $\|f\|_{L^p(\mathbb{R}^n,\mathbb{B})} \sim \|\mathcal{G}^H_{P,\sigma,\mathbb{B}}(f)\|_{L^p(\mathbb{R}^n,\gamma(\mathcal{H},\mathbb{B}))}, \ f \in L^p(\mathbb{R}^n, \mathbb{B})$.

- Note that, since $\gamma(\mathcal{H}, \mathbb{C}) = \mathcal{H}$, if $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, then

$$\|\mathcal{G}^H_{P,\sigma,\mathbb{C}}(f)\|_{\gamma(\mathcal{H},\mathbb{C})} = g^H_{P,\sigma}(f).$$

- Let $k = (k_1, \ldots, k_n) \in (\mathbb{N} \setminus \{0\})^n$. We define the multivariate square function $g^H_{P,k}$ as follows

$$g^H_{P,k}(f)(x) = \left(\int_{(0,\infty)^n} \left| \int_{\mathbb{R}^n} \prod_{j=1}^n t_j^{k_j} \partial_{t_j}^{k_j} P^H_{t_j}(x_j, y_j) f(y_1, \ldots, y_n) dy \right|^2 \frac{dt_1 \cdots dt_n}{t_1 \cdots t_n} \right)^{\frac{1}{2}},$$

  where $x \in \mathbb{R}^n$, for every $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$.
- By proceeding as in the proof of Theorem 2.4 by Wróbel. (Monatsh. Math., 168 (1): 125-149, 2012) we can show the following property.

Theorem

Let $1 < p < \infty$ and $k = (k_1, \ldots, k_n) \in (\mathbb{N} \setminus \{0\})^n$. Then, there exists $C > 0$ such that

$$\frac{1}{C} ||f||_{L^p(\mathbb{R}^n)} \leq ||g^H_{P,k}(f)||_{L^p(\mathbb{R}^n)} \leq C||f||_{L^p(\mathbb{R}^n)}, \;\; f \in L^p(\mathbb{R}^n).$$

- If $\{\varepsilon_j\}_{j=1}^{\infty}$ is a sequence of independent symmetric ± 1-valued random variables (usually called Rademacher variables) on some probability space, we denote by $\mathbb{E}_\varepsilon$ the corresponding expectation operator.
- Suppose that $\{\varepsilon_j\}_{j=1}^{\infty}$ and $\{\eta_j\}_{j=1}^{\infty}$ are two independent sequences of Rademacher variables. We say that a Banach space $\mathbb{B}$ has (Pisier's) property (α) when there exists $C > 0$ such that

$$\mathbb{E}_\varepsilon \mathbb{E}_\eta \Big\| \sum_{i,j=1}^{N} \alpha_{i,j} \varepsilon_i \eta_j x_{i,j} \Big\|_{\mathbb{B}} \le \mathbb{E}_\varepsilon \mathbb{E}_\eta \Big\| \sum_{i,j=1}^{N} \varepsilon_i \eta_j x_{i,j} \Big\|_{\mathbb{B}},$$

  for every $\alpha_{i,j} \in \{+1, -1\}$, $x_{i,j} \in \mathbb{B}$, $i, j = 1, \dots, N$, and $N \in \mathbb{N} \setminus \{0\}$.
- UMD and (α) properties of Banach spaces are crucial in order to prove Banach valued Fourier multipliers of Mikhlin type.

Examples

- Hilbert spaces.
- $L^p(\Omega, \mu, \mathbb{B})$ if $\mathbb{B}$ has property-(α).

- Let $\mathcal{H}^n = L^2\left((0,\infty)^n, \frac{dt_1\cdots dt_n}{t_1\cdots t_n}\right)$. If $k = (k_1,\dots,k_n) \in (\mathbb{N}\setminus\{0\})^n$, we consider the g-function associated with the Hermite operator defined by

$$G^H_{P,k;\mathbb{B}}(f)(t,x) = \int_{\mathbb{R}^n} \prod_{j=1}^{n} t_j^{k_j} \partial_{t_j}^{k_j} P^H_{t_j}(x_j,y_j) f(y_1,\dots,y_n)\, dy_1 \dots dy_n,$$

for every $f \in L^p(\mathbb{R}^n,\mathbb{B})$, $1 < p < \infty$, where $t = (t_1,\dots,t_n) \in (0,\infty)^n$ and $x = (x_1,\dots,x_n) \in \mathbb{R}^n$.

Theorem

Let $\mathbb{B}$ be a UMD Banach space with the property (α), $k \in (\mathbb{N}\setminus\{0\})^n$ and $1 < p < \infty$. Then, there exists $C > 0$ such that

$$\frac{1}{C}||f||_{L^p(\mathbb{R}^n,\mathbb{B})} \le ||G^H_{P,k;\mathbb{B}}(f)||_{L^p(\mathbb{R}^n,\gamma(\mathcal{H}^n,\mathbb{B}))} \le C||f||_{L^p(\mathbb{R}^n,\mathbb{B})}, \;\; f \in L^p(\mathbb{R}^n,\mathbb{B}).$$

- Note that $\gamma(\mathcal{H}^n,\mathbb{C}) = \mathcal{H}^n$.

Proof.

- We proceed by induction on n. (We have seen that is true when $n = 1$)
- Assume that $f \in L^p(\mathbb{R}) \otimes L^p(\mathbb{R}^n) \otimes \mathbb{B}$, that is $f = \sum_{i=1}^k g_i v_i b_i$, where $g_i \in L^p(\mathbb{R})$, $v_i \in L^p(\mathbb{R}^n)$ and $b_i \in \mathbb{B}$, $i = 1, \dots, k \in \mathbb{N} \setminus \{0\}$.
- $L^p(\mathbb{R}) \otimes L^p(\mathbb{R}^n) \otimes \mathbb{B}$ is dense in $L^p(\mathbb{R}^{n+1}, \mathbb{B})$.
- $$G^H_{P,\beta;\mathbb{B}}(f)(t,x) = \sum_{i=1}^k G^H_{P,\beta_1;\mathbb{C}}(g_i)(t_1, x_1) G^H_{P,\tilde{\beta};\mathbb{C}}(v_i)(\tilde{t}, \tilde{x}) b_i,$$
  where $t = (t_1, \tilde{t}) = (t_1, t_2, \dots, t_{n+1}) \in (0, \infty)^{n+1}$, $x = (x_1, \tilde{x}) = (x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1}$, and $\tilde{\beta} = (\beta_2, \dots, \beta_{n+1}) \in (\mathbb{N} \setminus \{0\})^n$.
- $G^H_{P,\beta;\mathbb{B}}(f)(t, \cdot)$ is strongly measurable

- Since $\mathbb{B}$ has the property (α) we have that $\gamma(\mathcal{H}^{n+1},\mathbb{B}) \simeq \gamma(\mathcal{H}^1,\gamma(\mathcal{H}^n,\mathbb{B}))$ (γ-Fubini property - Van Neerven, Weis). By using the induction hypothesis and taking into account that $\gamma(\mathcal{H}^l,\mathbb{B})$, $l \in \mathbb{N}\setminus\{0\}$, is UMD with the property (α) we obtain

$$\begin{aligned}
&\|G^H_{P,\beta;\mathbb{B}}(f)\|_{L^p(\mathbb{R}^{n+1},\gamma(\mathcal{H}^{n+1},\mathbb{B}))} \\
&= \left(\int_{\mathbb{R}^{n+1}} \|G^H_{P,\beta;\mathbb{B}}(f)(\cdot,x)\|^p_{\gamma(\mathcal{H}^{n+1},\mathbb{B})}dx\right)^{\frac{1}{p}} \\
&= \left(\int_{\mathbb{R}^{n+1}} \left\|\int_{\mathbb{R}} t_1^{\beta_1}\partial_{t_1}^{\beta_1}P^H_{t_1}(x_1,y_1)G^H_{P,\tilde{\beta};\mathbb{B}}(f)(\tilde{t},\tilde{x})dy_1\right\|^p_{\gamma(\mathcal{H}^1,\gamma(\mathcal{H}^n,\mathbb{B}))}dx\right)^{\frac{1}{p}} \\
&\leq C\left(\int_{\mathbb{R}}\int_{\mathbb{R}^n} \|G^H_{P,\tilde{\beta};\mathbb{B}}(f(x_1,\tilde{y}))(\tilde{t},\tilde{x})\|^p_{\gamma(\mathcal{H}^n,\mathbb{B})}d\tilde{x}dx_1\right)^{\frac{1}{p}} \\
&\leq C\|f\|_{L^p(\mathbb{R}^{n+1})}
\end{aligned}$$

- The operator $G^H_{P,\beta;\mathbb{B}}$ can be extended to $L^p(\mathbb{R}^{n+1},\mathbb{B})$ as a bounded operator from $L^p(\mathbb{R}^{n+1},\mathbb{B})$ into $L^p(\mathbb{R}^{n+1},\gamma(\mathcal{H}^{n+1},\mathbb{B}))$. This extension operator is denoted by $\tilde{G}^H_{P,\beta;\mathbb{B}}$.
- We show that, for every $f \in L^p(\mathbb{R}^{n+1},\mathbb{B})$,

$$\tilde{G}^H_{P,\beta;\mathbb{B}}(f)(x) = G^H_{P,\beta;\mathbb{B}}(f)(\cdot,x), \text{ a.e. } x \in \mathbb{R}^{n+1}.$$

- The induction is completed and we prove that the operator $G^H_{P,\beta;\mathbb{B}}$ is bounded from $L^p(\mathbb{R}^{n+1},\mathbb{B})$ into $L^p(\mathbb{R}^{n+1},\gamma(\mathcal{H}^{n+1},\mathbb{B}))$, for every $\beta \in (\mathbb{N}\setminus\{0\})^{n+1}$ and $n \in \mathbb{N}$.
- Our next objective is to see that there exists $C > 0$ such that

$$\|f\|_{L^p(\mathbb{R}^{n+1},\mathbb{B})} \leq C\|G^H_{P,\alpha;\mathbb{B}}\|_{L^p(\mathbb{R}^{n+1},\gamma(\mathcal{H}^{n+1},\mathbb{B}))}, \quad f \in L^p(\mathbb{R}^{n+1},\mathbb{B}).$$

- By using standard spectral arguments we can show that, for every $f \in L^p(\mathbb{R}^{n+1}) \otimes \mathbb{B}$ and $g \in L^{p'}(\mathbb{R}^{n+1}) \otimes \mathbb{B}^*$, where p' is the conjugated of p, that is $p' = \frac{p}{p-1}$,

$$\int_{\mathbb{R}^{n+1}} \int_{(0,\infty)^{n+1}} \langle G^H_{P,\alpha;\mathbb{B}^*}(g)(t,x), G^H_{P,\alpha;\mathbb{B}}(f)(t,x)\rangle_{\mathbb{B}^*,\mathbb{B}} \frac{dt_1\cdots dt_{n+1}}{t_1\cdots t_{n+1}}dx$$
$$= \prod_{j=1}^{n+1} \frac{\Gamma(2\alpha_j)}{2^{2\alpha_j}} \int_{\mathbb{R}^{n+1}} \langle g(x), f(x)\rangle_{\mathbb{B}^*,\mathbb{B}} dx$$

- Let $f \in L^p(\mathbb{R}^{n+1}) \otimes B$. We have that

$$||f||_{L^p(\mathbb{R}^{n+1},\mathbb{B})} = \sup_{\substack{g \in L^{p'}(\mathbb{R}^{n+1}) \otimes \mathbb{B}^* \\ ||g||_{L^{p'}(\mathbb{R}^{n+1},\mathbb{B}^*)} \leq 1}} \left| \int_{\mathbb{R}^{n+1}} \langle g(x), f(y) \rangle_{\mathbb{B}^*,\mathbb{B}} dy \right|$$

$$\leq \quad C \sup_{\substack{g \in L^{p'}(\mathbb{R}^{n+1}) \otimes \mathbb{B}^* \\ ||g||_{L^{p'}(\mathbb{R}^{n+1},\mathbb{B}^*)} \leq 1}} \left| \int_{\mathbb{R}^{n+1}} \int_{(0,\infty)^{n+1}} \langle G^H_{P,\alpha;\mathbb{B}^*}(g)(t,x), G^H_{P,\alpha;\mathbb{B}}(f)(t,x) \rangle_{\mathbb{B}^*,\mathbb{B}} \frac{dt_1 \cdots dt_{n+1}}{t_1 \cdots t_{n+1}} dx \right|$$

$$\leq \quad C \sup_{\substack{g \in L^{p'}(\mathbb{R}^{n+1}) \otimes \mathbb{B}^* \\ ||g||_{L^{p'}(\mathbb{R}^{n+1},\mathbb{B}^*)} \leq 1}} \int_{\mathbb{R}^{n+1}} \|G^H_{P,\alpha;\mathbb{B}^*}(g)(\cdot,x)\|_{\gamma(\mathcal{H}^{n+1},\mathbb{B}^*)} \|G^H_{P,\alpha;\mathbb{B}}(f)(\cdot,x)\|_{\gamma(\mathcal{H}^{n+1},\mathbb{B})} dx.$$

- Since $\mathbb{B}^*$ is UMD and it has the property (α), $G^H_{P,\alpha;\mathbb{B}^*}$ is bounded from $L^{p'}(\mathbb{R}^{n+1},\mathbb{B}^*)$ into $L^{p'}(\mathbb{R}^{n+1},\gamma(\mathcal{H}^{n+1},\mathbb{B}^*))$, and by using Hölder's inequality we get

$$\|f\|_{L^p(\mathbb{R}^{n+1},\mathbb{B})} \leq C \|G^H_{P,\alpha;\mathbb{B}}(f)\|_{L^p(\mathbb{R}^{n+1},\gamma(\mathcal{H}^{n+1},\mathbb{B}))}$$

- Since $L^p(\mathbb{R}^{n+1}) \otimes \mathbb{B}$ is a dense subspace of $L^p(\mathbb{R}^{n+1},\mathbb{B})$ and $G^H_{P,\alpha;\mathbb{B}}$ is a bounded operator from $L^p(\mathbb{R}^{n+1},\mathbb{B})$ into $L^p(\mathbb{R}^{n+1},\gamma(\mathcal{H}^{n+1},\mathbb{B}))$ we conclude that

$$\|f\|_{L^p(\mathbb{R}^{n+1},\mathbb{B})} \leq C \|G^H_{P,\alpha;\mathbb{B}}(f)\|_{L^p(\mathbb{R}^{n+1},\gamma(\mathcal{H}^{n+1},\mathbb{B}))}, \quad f \in L^p(\mathbb{R}^{n+1},\mathbb{B}).$$

- Suppose now that m is a bounded Borel measurable function from $(0,\infty)^n$ into $\mathbb{C}$. The Hermite multivariate multiplier T_{m} associated with m is defined by

$$T_{\mathrm{m}} f = \sum_{k=(k_1,\dots,k_n)\in\mathbb{N}^n} \mathrm{m}(\lambda_{k_1},\cdots,\lambda_{k_n}) c_k(f) h_k, \;\; f \in L^2(\mathbb{R}^n),$$

- T_{m} is a bounded operator from $L^2(\mathbb{R}^n)$ into itself.

- Thangavelu (Theorem 4.2.1, Lectures on Hermite and Laguerre operators, Princeton Univ. Press, Princeton, N.J., 1993) established Mikhlin-Hörmander type condition on m under that T_{m} is bounded from $L^p(\mathbb{R}^n)$ into itself, for every $1 < p < \infty$.

- We define the operator $T_{\mathrm{m}} \otimes \mathrm{Id}_{\mathbb{B}}$ on $L^2(\mathbb{R}^n) \otimes \mathbb{B}$ in the usual way.

- We are motivated by the results of Meda (Proc. Amer. Math. Soc., 110 (3), (1990), 639-647) and Wróbel (Monatsh. Math., 168 (1) (2012), 125-149).

Notation:

- For every $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, we consider

$$m_\alpha(t,\lambda) = \prod_{j=1}^{n} (t_j\lambda_j)^{\alpha_j} e^{-\frac{t_j\lambda_j}{2}} M(\lambda),$$

  where $t = (t_1, \ldots, t_n) \in (0,\infty)^n$, $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ and $M(\lambda) = \mathrm{m}(\lambda_1^2, \ldots, \lambda_n^2)$, $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$.

- We define

$$\mathcal{M}_\alpha(t,u) = \int_{(0,\infty)^n} \prod_{j=1}^{n} \lambda_j^{-iu_j-1} m_\alpha(t,\lambda) d\lambda,$$

  with $t = (t_1, \ldots, t_n) \in (0,\infty)^n$ and $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$.

- Let $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$. We write $L^{i\beta}$ to refer us to the operator T_{m_β} where $\mathrm{m}_\beta(\lambda_1, \ldots, \lambda_n) = \prod_{j=1}^{n} \lambda_j^{i\beta}$, $\lambda = (\lambda_1, \ldots, \lambda_n) \in (0,\infty)^n$, that is,

$$L^{i\beta} f = \sum_{k_1,\ldots,k_n=0}^{\infty} \prod_{j=1}^{n} \lambda_{k_j}^{i\beta_j} c_k(f) h_k, \;\; f \in L^2(\mathbb{R}^n).$$

- The operator $L^{i\beta} \otimes \mathrm{Id}_{\mathbb{B}}$ is defined in the usual way on $L^2(\mathbb{R}^n) \otimes \mathbb{B}$.
- $L^{i\beta} \otimes \mathrm{Id}_{\mathbb{B}}$ can be extended to $L^p(\mathbb{R}^n, \mathbb{B})$ as a bounded operator from $L^p(\mathbb{R}^n, \mathbb{B})$ into itself, for every $1 < p < \infty$, provided that $\mathbb{B}$ is a UMD Banach space. (Betancor, Castro, Curbelo, Fariña, and Rodríguez-Mesa, Complex Anal. Op. Th., 2011).

Theorem

Let $\mathbb{B}$ be a UMD Banach space with the property (α) and $1 < p < \infty$. Suppose that m is a bounded Borel measurable function on $(0, \infty)^n$, such that for some $\gamma \in \mathbb{N}^n$,

$$\int_{\mathbb{R}^n} \sup_{t \in (0,\infty)^n} |\mathcal{M}_\gamma(t,u)| \| L^{i\frac{u}{2}} \|_{L^p(\mathbb{R}^n,\mathbb{B}) \to L^p(\mathbb{R}^n,\mathbb{B})} du < \infty.$$

Then, the multiplier operator T_{m} is bounded from $L^p(\mathbb{R}^n, \mathbb{B})$ into itself.

- We need to use intermediate UMD spaces in the following theorem to get a suitable estimate for the operator norm $\| L^{i\gamma} \|_{L^p(\mathbb{R}^n,\mathbb{B}) \to L^p(\mathbb{R}^n,\mathbb{B})}$, $\gamma \in \mathbb{N}^n$ and $1 < p < \infty$.

- We consider a class of UMD Banach spaces, called intermediate UMD spaces, that includes all known examples of UMD spaces.
- We say that $\mathbb{B}$ is an intermediate UMD space when $\mathbb{B}$ is isomorphic to a closed subquotient of a complex interpolation space $[X, Q]_\theta$, where $\theta \in (0,1)$, X is a UMD Banach space and Q is a Hilbert space.
- This class of UMD spaces has been used recently by Berkson and Gillespie, Hytönen, Hytönen and Lacey, and Taggart.
- It is, as far as we know, an open problem if every UMD space is an intermediate UMD space. This question was posed by Rubio de Francia.

- For every $\psi \in (0,\pi)$ we denote by Γ_ψ the $\mathbb{C}^n$-sector

$$\Gamma_\psi = \{(z_1,\ldots,z_n) \in \mathbb{C}^n\colon |\text{Arg}\, z_j| < \psi,\ j = 1,\ldots,n\}.$$

Theorem

Suppose that $\mathbb{B}$ is isomorphic to a closed subquotient of $[X,Q]_\theta$, where $\theta \in (0,1)$, X is a UMD Banach space and Q is a Hilbert space and that $\mathbb{B}$ has the property (α). If m is a bounded holomorphic function in Γ_ψ for some $\psi > \frac{\pi}{4}$, then the multiplier operator T_{m} can be extended to $L^p(\mathbb{R}^n,\mathbb{B})$ as a bounded operator from $L^p(\mathbb{R}^n,\mathbb{B})$ into itself, provided that $\left|\frac{2}{p} - 1\right| < \theta$.

Proof

- Assume that $\mathbb{B}$ is isomorphic to a closed subquotient of a complex interpolation space $[\mathcal{H}, X]_\theta$. If $\left|\frac{2}{p}-1\right| < \theta$ and $0 \le \Omega < \frac{\pi}{2}(1-\theta)$, there exists $\omega < \frac{\pi}{2} - \Omega$ such that
$$\|H^{i\alpha}\|_{L^q(\mathbb{R},\mathbb{B})\to L^q(\mathbb{R},\mathbb{B})} \le Ce^{\omega|\alpha|}, \quad \alpha \in \mathbb{R},$$
Hence, if $u = (u_1, \dots, u_n) \in \mathbb{R}^n$ we conclude that
$$\|L^{iu}\|_{L^q(\mathbb{R},\mathbb{B})} \le Ce^{\omega \sum_{i=1}^n |u_i|},$$
where C does not depend on u.
- Working coordinate to coordinate we can obtain that
$$\sup_{t\in(0,\infty)^n} |\mathcal{M}_\gamma(t,u)| \le C \prod_{j=1}^n (1+|u_j|)|\Gamma(\gamma_j - iu_j)|, \quad u = (u_1, \dots, u_n) \in \mathbb{R}^n.$$
Then, if $\left|\frac{2}{p}-1\right| < \theta$ it follows that
$$\int_{\mathbb{R}^n} \sup_{t\in(0,\infty)^n} |\mathcal{M}_\gamma(t,u)| \|L^{i\frac{u}{2}}\|_{L^p(\mathbb{R}^n,\mathbb{B})\to L^p(\mathbb{R}^n,\mathbb{B})} du < \infty.$$
We deduce that $T_{\rm m}$ is bounded from $L^p(\mathbb{R}^n, \mathbb{B})$ into itself provided that $\left|\frac{2}{p}-1\right| < \theta$.

- Bongioanni and Torrea (Proc. Indian Acad. Sci. Math. Sci., 116 (3) (2006), 337-360, and Studia Math., 192 (2) (2009), 147-172) studied Sobolev spaces in the Hermite scalar setting.

- The Hermite operator $\tilde{H}$ admits the following factorization

$$\tilde{H} = \frac{1}{2}\sum_{j=1}^{n}(A_j A_{-j} + A_{-j} A_j),$$

where $A_j = \frac{d}{dx_j} + x_j$ and $A_{-j} = -\frac{d}{dx_j} + x_j$, $j = 1, \ldots, n$ (creation and the annihilation operators).

- The creation operator and the annihilation operator play an important role in the harmonic analysis for the Hermite operator.

- Let $\mathbb{B}$ be a Banach space, $1 < p < \infty$ and $\ell \in \mathbb{N} \setminus \{0\}$.
- Hermite Sobolev space $W^p_{H,\ell}(\mathbb{R}^n, \mathbb{B})$. Let $j_i \in \mathbb{Z}$, $1 \leq |j_i| \leq n$, $1 \leq i \leq m \leq \ell$

$$W^p_{H,\ell}(\mathbb{R}^n, \mathbb{B}) = \{ f \in L^p(\mathbb{R}^n, \mathbb{B}) \colon A_{j_1} \cdots A_{j_m} f \in L^p(\mathbb{R}^n, \mathbb{B}) \}.$$

$$\|f\|_{W^p_{H,\ell}(\mathbb{R}^n,\mathbb{B})} = \|f\|_{L^p(\mathbb{R}^n,\mathbb{B})} + \sum_{\substack{j_i \in \mathbb{Z},\ 1 \leq |j_i| \leq n \\ i=1,\dots,m \leq \ell}} \|A_{j_1} \cdots A_{j_m} f\|_{L^p(\mathbb{R}^n,\mathbb{B})}.$$

- Hermite Sobolev space $\tilde{W}^p_{H,\ell}(\mathbb{R}^n, \mathbb{B})$. Let $j_i \in \mathbb{N}$, $1 \leq j_i \leq n$, $1 \leq i \leq m \leq \ell$

$$\tilde{W}^p_{H,\ell}(\mathbb{R}^n, \mathbb{B}) = \{ f \in L^p(\mathbb{R}^n, \mathbb{B}) \colon A_{-j_1} \cdots A_{-j_m} f \in L^p(\mathbb{R}^n, \mathbb{B}) \}.$$

$$\|f\|_{\tilde{W}^p_{H,\ell}(\mathbb{R}^n,\mathbb{B})} = \|f\|_{L^p(\mathbb{R}^n,\mathbb{B})} + \sum_{\substack{j_i \in \mathbb{N},\ 1 \leq j_i \leq n \\ i=1,\dots,m \leq \ell}} \|A_{-j_1} \cdots A_{-j_m} f\|_{L^p(\mathbb{R}^n,\mathbb{B})}.$$

- Let $\beta > 0$. The $-\beta$-power $H^{-\beta}$ of the Hermite operator is defined by

$$H^{-\beta} f = \sum_{k \in \mathbb{N}^n} \frac{c_k(f)}{(2|k|+n)^{\beta}} h_k, \quad f \in L^p(\mathbb{R}^n, \mathbb{B}), \; 1 < p < \infty.$$

- $H^{-\beta}$ is bounded, positive and one to one from $L^p(\mathbb{R}^n, \mathbb{B})$ into itself, for every $1 < p < \infty$.
- The potential space

$$L^p_{H,\beta}(\mathbb{R}^n, \mathbb{B}) = \{ f \in L^p(\mathbb{R}^n, \mathbb{B}) : \; f = H^{-\beta} g, \text{ for some } g \in L^p(\mathbb{R}^n, \mathbb{B}) \}.$$

$$\|f\|_{L^p_{H,\beta}(\mathbb{R}^n, \mathbb{B})} = \|g\|_{L^p(\mathbb{R}^n, \mathbb{B})}$$

Theorem

Suppose that $\mathbb{B}$ is isomorphic to a closed subquotient of $[X, Q]_\theta$ where $\theta \in (0,1)$, X is a UMD Banach space, and Q is a Hilbert space, and that $\mathbb{B}$ has the property (α). Then, for every $\ell \in \mathbb{N} \setminus \{0\}$,

$$\tilde{W}^p_{H,\ell}(\mathbb{R}^n, \mathbb{B}) = W^p_{H,\ell}(\mathbb{R}^n, \mathbb{B}) = L^p_{H,\ell}(\mathbb{R}^n, \mathbb{B}), \text{ provided that } \left| \frac{2}{p} - 1 \right| < \theta.$$

Proof

- The Hermite Riesz transform $R_{m,j}$ is defined as follows

$$R_{m,j}f = A_1^{m_1} \cdots A_j^{m_j} A_{-(j+1)}^{m_{j+1}} \cdots A_{-n}^{m_n} H^{-\frac{|m|}{2}} f, \;\; f \in \mathfrak{F}_H.$$

where $m = \sum_{j=1}^n m_j$. $\mathfrak{F}_H$ denotes the linear space span$\{h_k\}_{k \in \mathbb{N}^n}$.

Proposition

Suppose that $\mathbb{B}$ is isomorphic to a closed subquotient of $[X, Q]_\theta$, where $\theta \in (0,1)$, X is a UMD Banach space and Q is a Hilbert space and that $\mathbb{B}$ has the property (α) and $1 < p < \infty$ being $\left|\frac{2}{p} - 1\right| < \theta$. If $m = (m_1, \ldots, m_n) \in \mathbb{N}^n$ and $j = (j_1, \ldots, j_n) \in \mathbb{Z}^n$, being $|j_i| = i$, $i = 1, \ldots, n$, the Hermite Riesz transform R_m^j defined by $R_m^j f = \prod_{i=1}^n A_{j_i}^{m_i} H^{-\frac{|m|}{2}} f, \;\; f \in \mathfrak{F}_H$, where $|m| = \sum_{i=1}^n m_i$, can be extended to $L^p(\mathbb{R}^n, \mathbb{B})$ as a bounded operator from $L^p(\mathbb{R}^n, \mathbb{B})$ into itself.

- If $j \in \mathbb{Z}$, $1 \leq |j| \leq n$, we consider the Hermite Riesz transform R_j^ℓ by

$$R_j^\ell f = A_j^\ell H^{-\frac{\ell}{2}} f, \;\; f \in \mathfrak{F}_H,$$

  and we define the operator $\tau_\ell = \sum_{j=1}^n R_j^\ell R_{-j}^\ell$.
- τ_ℓ is bounded from $L^p(\mathbb{R}^n, \mathbb{B})$ into itself.
- We consider the function

$$\mathrm{m}(z_1, \dots, z_n) = \frac{(z_1 + \cdots + z_n + 2)^{\frac{\ell}{2}} (z_1 + \cdots + z_n)^{\frac{\ell}{2}}}{\sum_{j=1}^n \prod_{m=1}^{\ell} (\frac{z_j}{2} + m - \frac{1}{2})}, \;\; z = (z_1, \dots, z_n) \in \Gamma_{\frac{\pi}{2}}.$$

  The multiplier T_m associated with m is bounded from $L^p(\mathbb{R}^n, \mathbb{B})$ into itself.
- Hence, we obtain

$$\|g\|_{L^p(\mathbb{R}^n,\mathbb{B})} \leq \|T_\mathrm{m} \tau_\ell g\|_{L^p(\mathbb{R}^n,\mathbb{B})} \leq C \sum_{j=1}^n \|A_{-j}^\ell H^{-\frac{\ell}{2}} g\|_{L^p(\mathbb{R}^n,\mathbb{B})}, \;\; g \in \mathfrak{F}_H.$$

  Then,

$$\|f\|_{L^p_{H,\ell}(\mathbb{R}^n,\mathbb{B})} \leq C \|f\|_{W^p_{H,\ell}(\mathbb{R}^n,\mathbb{B})}, \;\; f \in \mathfrak{F}_H.$$

  We conclude that $W^p_{H,\ell}(\mathbb{R}^n, \mathbb{B})$ is continuously contained in $L^p_{H,\ell}(\mathbb{R}^n, \mathbb{B})$.

- Let $\mathbb{B}$ be a Banach space and $1 < p < \infty$. Suppose that $\beta > 0$ and $k \in \mathbb{N} \setminus \{0\}$. We consider the Littlewood-Paley operator $G^H_{P,\beta,k;\mathbb{B}}$ defined by

$$G^H_{P,\beta,k;\mathbb{B}}(f)(t,x) = t^\beta \partial_t^k P_t^H(f)(x,t), \quad x \in \mathbb{R}^n \text{ and } t > 0,$$

  for every $f \in L^p(\mathbb{R}^n, \mathbb{B})$.

-

$$F^H_{\beta,k}(\mathbb{R}^n, \mathbb{B}) = \{f \in L^p(\mathbb{R}^n, \mathbb{B}) : \; G^H_{P,\beta,k;\mathbb{B}}(f) \in L^p(\mathbb{R}^n, \gamma(\mathcal{H}^n, \mathbb{B}))\}.$$

$$\|f\|_{F^H_{\beta,k}(\mathbb{R}^n,\mathbb{B})} = \|f\|_{L^p(\mathbb{R}^n,\mathbb{B})} + \|G^H_{P,\beta,k;\mathbb{B}}(f)\|_{L^p(\mathbb{R}^n,\gamma(\mathcal{H}^n,\mathbb{B}))}.$$

- Note that the space $F^H_{\beta,k}(\mathbb{R}^n, \mathbb{B})$ can be considered as a Triebel-Lizorkin type space in the Hermite setting.

Theorem

Let $\mathbb{B}$ be a UMD Banach space and $1 < p < \infty$. Suppose that $\beta > 0$ and $k \in \mathbb{N}$ is such that $k > \beta$. Then, $L^p_{H,\beta}(\mathbb{R}^n, \mathbb{B}) = F^H_{k-\beta,k}(\mathbb{R}^n, \mathbb{B})$.

Proof

- We have that
$$G^H_{P,k-\beta;\mathbb{B}}(f) = G^H_{P,\beta,k;\mathbb{B}}(H^{-\frac{\beta}{2}}f), \;\; f \in \mathfrak{F}_H \otimes \mathbb{B}.$$

- Then it follows that
$$\frac{1}{C}\|f\|_{L^p_{H,\frac{\beta}{2}}(\mathbb{R}^n,\mathbb{B})} \leq \|G^H_{P,\beta,k;\mathbb{B}}(f)\|_{L^p(\mathbb{R}^n,\gamma(\mathcal{H}^1,\mathbb{B}))} \leq C\|f\|_{L^p_{H,\frac{\beta}{2}}(\mathbb{R}^n,\mathbb{B})}, \;\; f \in \mathfrak{F}_H \otimes \mathbb{B}.$$

- Suppose now that $f \in L^p(\mathbb{R}^n,\mathbb{B})$ and $G^H_{P,\beta,k;\mathbb{B}}(f) \in L^p(\mathbb{R}^n,\gamma(\mathcal{H}^1,\mathbb{B}))$. We are going to see that $f \in L^p_{H,\frac{\beta}{2}}(\mathbb{R}^n,\mathbb{B})$. Let $\delta > 0$. We define
$$F_\delta = \sum_{k\in\mathbb{N}^n}(2|k|+n)^{\frac{\beta}{2}}e^{-\delta(2|k|+n)^{\frac{1}{2}}}c^H_k(f)h_k.$$
$F_\delta \in L^p(\mathbb{R}^n,\mathbb{B})$. Since $H^{-\frac{\beta}{2}}F_\delta = P^H_\delta(f) \in L^p(\mathbb{R}^n,\mathcal{H})$, $P^H_\delta(f) \in L^p_{H,\frac{\beta}{2}}(\mathbb{R}^n,\mathbb{B})$ and $\|P^H_\delta(f)\|_{L^p_{H,\frac{\beta}{2}}(\mathbb{R}^n,\mathbb{B})} = \|F_\delta\|_{L^p(\mathbb{R}^n,\mathbb{B})}$. Hence
$$\|F_\delta\|_{L^p(\mathbb{R}^n,\mathbb{B})} \leq C\|G^H_{P,\beta,k;\mathbb{B}}(P^H_\delta(f))\|_{L^p(\mathbb{R}^n,\gamma(\mathcal{H}^1,\mathbb{B}))}.$$

- The set $\{F_\delta\}_{\delta>0}$ is bounded in $L^p(\mathbb{R}^n, \mathbb{B})$.
- By Banach-Alaoglu's Theorem there exists a decreasing sequence $\{\delta_k\}_{k\in\mathbb{N}\setminus\{0\}} \subset (0,\infty)$ and $F \in L^p(\mathbb{R}^n, \mathbb{B})$ such that $\delta_k \to 0$, as $k \to \infty$,

$$\int_{\mathbb{R}^n} \langle g(x), F_{\delta_k}(x)\rangle_{\mathbb{B}^*,\mathbb{B}} dx \to \int_{\mathbb{R}^n} \langle g(x), F(x)\rangle_{\mathbb{B}^*,\mathbb{B}} dx, \text{ as } k \to \infty,$$

for every $g \in L^{p'}(\mathbb{R}^n, \mathbb{B}^*)$, and $\|F\|_{L^p(\mathbb{R}^n,\mathbb{B})} \leq C\|G^H_{P,\beta,k;\mathbb{B}}(f)\|_{L^p(\mathbb{R}^n,\gamma(\mathcal{H}^1,\mathbb{B}))}$.

- Taking into account that $H^{-\frac{\beta}{2}}$ is a bounded operator from $L^p(\mathbb{R}^n, \mathbb{B})$ into itself, we get, for every $g \in L^{p'}(\mathbb{R}^n, \mathbb{B}^*)$,

$$\int_{\mathbb{R}^n} \langle g(x), P^H_{\delta_k}(f)(x)\rangle_{\mathbb{B}^*,\mathbb{B}} dx \to \int_{\mathbb{R}^n} \langle g(x), H^{-\frac{\beta}{2}}(F)(x)\rangle_{\mathbb{B}^*,\mathbb{B}} dx, \text{ as } k \to \infty.$$

Since $P^H_{\delta_k}(f) \to f$, as $k \to \infty$, in $L^p(\mathbb{R}^n, \mathbb{B})$, we conclude that $f = H^{-\frac{\beta}{2}}(F)$. Hence, $f \in L^p_{H,\frac{\beta}{2}}(\mathbb{R}^n, \mathbb{B})$ and

$$\|f\|_{L^p_{H,\frac{\beta}{2}}(\mathbb{R}^n,\mathbb{B})} \leq C\|G^H_{P,\beta,k;\mathbb{B}}(f)\|_{L^p(\mathbb{R}^n,\gamma(\mathcal{H}^1,\mathbb{B}))}.$$

"Caminante no hay camino"

Caminante, son tus huellas
el camino y nada más;
Caminante, no hay camino,
se hace camino al andar.

Al andar se hace el camino,
y al volver la vista atrás
se ve la senda que nunca
se ha de volver a pisar.

Caminante no hay camino
sino estelas en la mar.

Antonio Machado (1875 – 1939)