Course abstracts

Positive Harmonic functions on nonsmooth domains

Hiroaki Aikawa, Hokkaido University

This course is organised along the following lines.

1. *Overview*
   The Martin boundaries for general domains, the boundary Harnack principle, nonsmooth domains (interior conditions and exterior conditions), the common property between John domains and domains with the capacity density condition.
2. *Uniform domains and the uniform boundary Harnack principle*
   By the box argument we prove the uniform boundary Harnack principle; as a result we identify the Martin boundary.
3. *John domains and the weak boundary Harnack principle*
   We introduce the notion of a system of local reference points and prove the weak boundary Harnack principle by the Domar argument. We observe that the number of minimal Martin boundary points over a Euclidean boundary point is finite.
4. *Equivalence between the Carleson estimate and the boundary Harnack principle*
   Carleson estimate and the global (local) boundary Harnack principle if we define these notions appropriately. We establish the global boundary Harnack principle for Hölder domains and domains with the quasihyperbolic boundary condition.

Absolute continuity properties of quasiconformal mappings

Pekka Koskela, University of Jyväskylä

The metric definition of quasiconformality is simple to state but it is hard to draw immediate consequences from it. Nevertheless, these mappings turn out to be absolutely continuous on almost all curves even in a rather general setting. In the Euclidean setting, they are are quasisymmetric and preserve the class of null sets for the Lebesgue measure. This also holds on Ahlfors regular metric spaces that support a suitable Poincaré inequality.

We will give a self-contained proof of the absolute continuity properties of quasiconformal mappings on almost all lines parallel to coordinate axes, that allows for extension to the metric setting. We will also sketch the proofs for quasisymmetry and for preservation of the Lebesgue null sets.

Dyadic Martingales and some applications to Analysis

Artur Nicolau, Universitat Autònoma de Barcelona

The main purpose of the course is to study certain problems in Classical Analysis using Dyadic Martingale techniques. We will present Fatou type results, L^p estimates and versions of the Law of the Iterated Logarithm (LIL), relating the growth of a dyadic martingale and the size of its quadratic variation. These results tell that the asymptotic behaviour of a dyadic martingale is governed by the size of its quadratic variation. The continuous versions of these results are classical Theorems due to A. Calderón, E. Stein, A. Zygmund and others as well as some more recent results by R. Bauelos, I. Klemes and C. Moore. In the eighties N. Makarov proved a series of deep results on boundary behaviour of conformal maps and metric properties of harmonic measure as consequences of results on dyadic martingales. We will review some of these results. We will also apply these discrete techniques to study three other questions in Geometric Function Theory:

- To what extent can a doubling measure be singular with respect to Lebesgue measure?
- What is the analogue of the Lusin Area Function for analytic self mappings of the disc?
- How large is the set of points where a function in the Zygmund class is differentiable?

Quasisymmetric Maps–a Geometrical View

Jang-Mei Wu, University of Illinois at Urbana-Champaign

Quasisymmetric maps are natural generalizations of conformal maps and quasiconformal maps to metric spaces. The theory of such maps has recently found applications in geometric analysis, complex dynamics and geometric group theory. However a large number of fundamental questions remain unanswered, for example

- Quasisymmetric parametrization of a metric space by a Euclidean space,
- Factorization of quasisymmetric maps into maps of small distortion,
- Extension of quasisymmetric maps to the ambient spaces.

In this course, we will start from the definition, explain geometric properties of these maps and give a survey of known results. We then discuss open questions. In order to reduce some of the technical issues, we bring in a large number of examples from geometric topology to illustrate the underlying problems in dimensions 3, 4 and 5. The course is intended to be self-contained.

Talk abstracts

Functional inequalities and Hamilton-Jacobi equations in geodesic spaces

Zoltan Balogh, University of Bern

We study the connection between the p–Talagrand inequality and the q–logarithmic Sololev inequality for conjugate exponents $p \geq 2$, $q \leq 2$ in proper geodesic metric spaces. By means of a general Hamilton–Jacobi semigroup we prove that these are equivalent, and moreover equivalent to the hypercontractivity of the Hamilton–Jacobi semigroup. Our results generalize those of Lott and Villani. They can be applied to deduce the p-Talagrand inequality in the sub-Riemannian setting of the Heisenberg group.

Finely continuously differentiable functions

Jana Björn, University of Linköping

We define quasiminimizers of the p-energy integral as more robust generalizations of p-harmonic and harmonic functions. They share many useful properties with p-harmonic functions, such as maximum principle, Harnack inequality and some interior regularity. Other properties are much less understood or even fail, as shown by some counterexamples.

Hyperbolic convexity and conformal reflections

Edward Crane, University of Bristol

A classical result of Jorgensen from the 1950s says that a Euclidean disc is a hyperbolically convex subset of any simply-connected hyperbolic plane domain that contains it. This result has been generalized in various directions by Minda and Solynin. We give an application of these ideas to the problem of finding the best constant in the Hayman-Wu theorem. Jorgensen's theorem prompted us to look for a conformally invariant characterization of a Euclidean disc in a hyperbolic plane domain. We give a criterion in terms of the existence of a conformal reflection; a related criterion for hyperbolic convexity follows. If time allows, we will discuss multiply-connected and quasiconformal analogues of the criterion.

Asymptotic Teichmuller space

Alastair Fletcher, University of Glasgow

We will briefly discuss the asymptotic Teichmuller space of a Riemann surface M, how it is modelled on a certain Banach space $Q(M)/Q_0(M)$, how these Banach spaces are all isomorphic and conclude that all asymptotic Teichmuller spaces which are not one point, are all locally bi-Lipschitz equivalent.

Finely continuously differentiable functions

Stephen Gardiner, University College Dublin

The fine topology on Euclidean space $\mathbb{R}^n$ is the coarsest topology which renders all superharmonic functions continuous. Since about 1970, Fuglede and others have developed the theory of finely harmonic functions on finely open subsets of $\mathbb{R}^n$, and finely holomorphic functions on finely open subsets of the complex plane $\mathbb{C}$. This talk will present an explicit description of functions that are continuously differentiable with respect to the fine topology on $\mathbb{R}^n$.

A remark on the harmonic doubling condition

Steffen Junge, Norwegian University of Science and Technology

Let D be a bounded Jordan domain in $\mathbb{R}^2$. Jerison and Kenig proved that a necessary and sufficient condition for D to be a quasidisk is that D and D^* satisfy the harmonic doubling condition. Their proof of sufficiency is not written out, and the treatment in the book by Garnett and Marshall also has some shortcomings. Based on personal communication with John B. Garnett we have been able to fill in all the details giving a somewhat new and efficient proof of Jerison and Kenig's result.

Equidistribution of the Fekete points on the sphere

Joaquim Ortega-Cerdà, Universitat de Barcelona

The Fekete points are the points that maximize a Vandermonde-type determinant that appears in the polynomial Lagrange interpolation formula. They are well suited points for interpolation formulas and numerical integration. We prove the asymptotic equidistribution of the Fekete points in the sphere. The way we proceed is by showing their connection with other array of points, the Marcinkiewicz-Zygmund arrays and the interpolating arrays, that have been studied recently. This is joint work with J. Marzo.

Complex hyperbolic lattices

John Parker, University of Durham

A complex hyperbolic lattice is a discrete group of Bergman isometries of the unit complex ball with quotient of finite volume. Some lattices arise from arithmetic but it is known that in complex dimensions 2 and 3 there are examples of non-arithmetic complex hyperbolic lattices. In higher dimensions this is unknown and is an important open question. In this talk I will give a gentle introduction to the topic that focuses on a particular class of examples, namely lattices constructed from equilateral triangle groups.

Harmonic functions on compact sets

Tony Perkins, Syracuse University

In our talk we will discuss the space of harmonic functions on compact sets defined as uniform limits of harmonic functions on neighborhoods. Two problems are of our interest. The first one is the description of this space similar to the description of this space on closures of regular domains. The second one is the extension of the Szpilrajn-Cole-Ransford result describing Suslin functions with the subaveraging property relative to Jensen measures. As we will explain both problems require an introduction of a new metric on the compact set and a new notion of capacity.

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p-harmonic measure in simply connected domains

Pietro Poggi-Corradini, Kansas State University

In joint work with John Lewis and Kaj Nyström, we extend to simply-connected domains Makarov-type results about the Hausdorff dimension of p-harmonic measure pioneered by Lewis and Lewis-Bennewitz in the context of quasidisks.

The key to our analysis is a gradient estimate in terms of the distance to the boundary and constants that only depend on p. This is achieved by studying the conformal map from the unit disk to the simply connected domain to construct good quasicurves from an interior point to the boundary.

Harmonic fibrations of maximal functions

Evgeny Poletsky, Syracuse University

A maximal function is plurisubharmonic and satisfies the homogeneous Monge-Ampere equation. If it is smooth then Bedford and Kalka proved that the domain of the function can be foliated by complex curves with harmonic restrictions of the function to the curves. However, Duval and Levenberg built a continuous maximal function where such curves do not exist on a set of full measure. We will discuss how to replace curves with compact sets with harmonic restrictions.

Superharmonicity and harmonicity properties of condenser energy under dilation

Stamatis Pouliasis, Aristotle University of Thessaloniki

Consider a domain D in $\mathbb{C}$ and a compact subset K of D such that $D \setminus K$ is connected. If $z \in \mathbb{C}$, the dilation of D is the set $zD = \{zw : zw \in D\}$. If $E(z)$ is the energy of the condenser with plates K and $\mathbb{C} \setminus zD$, for all z such that $K \subset zD$, then E is superharmonic. E is harmonic if and only if $\mathrm{Int}(K)$, D are approximately concentric disks. If R is another domain which contains K we consider the case where the Green equilibrium measures of K with respect to D and R are the same.

Positive harmonic functions on Denjoy-type domains

Joanna Pres, University College Dublin

This talk concerns positive harmonic functions on domains that are complementary to a subset of a cylindrical surface. It characterizes those domains that admit minimal harmonic functions with exponential growth. Such domains can be regarded as cylindrical analogues of Denjoy domains, which have been widely studied. Our results have applications (via inversion) to the study of irregular boundary points and approximation properties of positive harmonic functions. (Joint work with Marius Ghergu)

Eigenvalues and isoperimetric inequalities

Jesse Ratzkin, University College Cork

I will discuss some lower bounds for the first Dirichlet eigenvalue of the Laplacian for bounded domains in cones. These bounds arise from weighted isoperimetric inequalities. Time allowing, I will also discuss applications.

On random quasiconformal maps

Eero Saksman, University of Helsinki

We will consider some results on random quasi-conformal maps. These include homogenization of qc-maps (joint with K. Astala, S. Rohde, and T. Tao) and SLE-inspired random weldings of Peter Jones type (joint with K. Astala, P. Jones and A. Kupiainen).

Continued fractions and Hausdorff dimension

Ian Short, University of Cambridge

We borrow ideas from the theory of Kleinian groups to examine the Hausdorff dimension of sets of divergence of sequences of Moebius transformations related to continued fractions.

On the growth properties and range of functions in Moebius-invariant spaces

Dragan Vukotić, Universidad Autónoma de Madrid & ICMAT

We will try to present a self-contained account of some known and some new growth properties of the functions in analytic (diagonal) Besov spaces, including the Dirichlet space, showing that such estimates are quite sharp. We will also consider the images of the unit disk under univalent functions in these spaces and their geometric properties. Various applications will be mentioned. This talk will be based on recent joint works with S.M. Buckley and also with J.J. Donaire and D. Girela, possibly also with other coauthors.
