

DIVIDING AN ANGLE INTO EQUAL PARTS

STEPHEN BUCKLEY AND DESMOND MACHALE

Department of Mathematics, University College, Cork

The problem of finding a geometric construction in a finite number of steps to trisect an arbitrary angle using straightedge and compasses alone is a very old one, originally proposed by the mathematicians of ancient Greece. It was only with modern algebraic techniques in the nineteenth century that it was shown conclusively that no such construction can exist. This fact has in no way deterred a legion of crank angle-trisectors from presenting their alleged solutions to the problem! However some angles (even non-constructible ones) can be trisected: we'll see later, for example, that given an angle $\pi/7$ it **can** be trisected. We can also consider the more general problem:

For which natural numbers $n > 2$ does there exist a geometric construction in a finite number of steps to divide an arbitrary angle into n equal parts, using straightedge and compasses alone?

For convenience, let T be the set of all such natural numbers n . Many textbooks on algebra contain a proof that $3 \notin T$. The proof usually depends on the fact that the angle 60° is not trisectible because the number $\cos 20^\circ$ is not constructible (see [3] and [4]). The angle 60° is chosen because of the convenient fact that $\cos 60^\circ$ is rational. Another way of looking at this proof is that a regular polygon with 18 sides is not constructible. In fact, we can easily decide which integers n belong to T once we know the integers for which a regular n -sided polygon is constructible. The following classical theorem is quoted in [1] and [5].

For $n > 2$ a regular n -sided polygon is constructible by straight-edge and compass alone if and only if n is either of the form 2^k for an integer k or of the form $2^m p_1 \cdots p_r$ for integer m and distinct primes p_1, \dots, p_r of the form $2^{2^s} + 1$ (the 'Fermat primes').

Before proceeding, note the following elementary facts about T , which the reader will be able to confirm with a little thought:

- (*) If $n \in T$ and $k \mid n$, then $k \in T$.
- (**) If $n \in T$ then $n^2 \in T$.

We are now able to show that T consists just of the powers of 2. In other words an arbitrary angle can be bisected over and over again but, apart from that, no other general construction to ' n -sect' an angle exists.

THEOREM. For positive integer n there exists a construction by straightedge and compasses to divide an arbitrary angle into n equal parts if and only if n is a power of 2.

Proof. The "if" part of the theorem is easy, being merely the above observation that the angle can be repeatedly bisected.

Suppose now that $n \in T$ and that $n > 2$. Then there must exist a construction for dividing the angle 360° into n equal parts. The angle $(360/n)^\circ$ is the exterior angle of a regular n -sided polygon and so such a polygon can be constructed. Hence, by the result quoted above, n must have the form 2^k for an integer k or the form $2^m p_1 \cdots p_r$ where p_1, \dots, p_r are distinct Fermat primes. If n is not a power of 2 then, for some Fermat prime p , we have $p \mid n$. Hence, by (*), $p \in T$ and, by (**), $p^2 \in T$. It is therefore possible to construct a regular p^2 -sided polygon. That contradicts the result quoted above and so n must be a power of 2, as claimed.

Let us coin the phrase “ α is n -sectible” if it is possible to divide the given angle α into n equal parts using straightedge and compasses alone. In the proof of the theorem the concepts of constructibility and n -sectibility are closely intertwined, but there is no inherent connection between the two concepts. Recall for example that an arbitrary angle, whether constructible or not, can always be bisected. Also, for particular values of α and n , the angle α can be used to construct α/n . We proceed to discuss the four *a priori* possibilities and show that all four possibilities do in fact occur.

(a) Constructible and n -sectible. Here we merely require an angle α and a positive integer n such that both α and α/n are constructible. This is trivially achieved by taking α to be n times a constructible angle (e.g. $\alpha = n\pi/2^k$ where $2^k > n$).

(b) Constructible but not n -sectible. Here we merely require α to be constructible but α/n not to be. For example 60° is constructible but not trisectible (since 20° is not constructible). We now outline the fact that if p is an odd prime and m an integer with $1 \leq m < p$, then $\alpha = \cos^{-1}(m/p)$ is constructible but $\alpha/3$ is not. (Indeed very similar arguments show that α/n fails to be constructible for any n which is not a power of 2.)

Clearly m/p is constructible and hence so is $\alpha = \cos^{-1}(m/p)$. On the other hand $\alpha/3$ is constructible if and only if $x = \cos(\alpha/3)$ is constructible. But

$$\frac{m}{p} = \cos \alpha = 4 \cos^3 \left(\frac{\alpha}{3} \right) - 3 \cos \left(\frac{\alpha}{3} \right) = 4x^3 - 3x$$

and so x satisfies the polynomial equation

$$4px^3 - 3px - m = 0.$$

Eisenstein’s irreducibility criterion ([2], Theorem 3.10.2) shows that this polynomial is irreducible and it follows from Theorem 5.4.1 of [2] that, as the degree of this polynomial is not a power of 2, then x is not constructible.

(With $\alpha/3$ replaced by α/n one gets a polynomial of the form

$$2^{n-1}px^n + pq_{n-1}x^{n-1} + \cdots + pq_1x + (pq_0 - m) = 0$$

for some integers q_0, \dots, q_{n-1} and similar reasoning shows that α/n is not constructible.)

(c) Not constructible but n -sectible. The angle $\pi/7$ (radians) is not constructible but, as well as being bisectible etc., it is also trisectible since

$$\frac{\pi}{21} = \frac{\pi}{3} - 2\frac{\pi}{7}$$

which is a combination of the constructible angle $\pi/3$ and the given angle $\pi/7$. Many other angles which are not constructible but trisectible (or, more generally, n -sectible) can be found in this way. For example if $p > 3$ is a prime but not a Fermat prime then $\pi/p2^k$ is not constructible. Also there exist integers s, t with $3s + tp = 1$. Therefore

$$\begin{array}{ccccc} \frac{\pi}{p2^k} & = & \frac{(3s + tp)}{p2^k}\pi & = & \frac{3s\pi}{p2^k} + \frac{t\pi}{2^k} \\ \text{not} & & & & \text{constructible} \\ \text{constructible} & & & & \end{array}$$

and so $3\pi/p2^k$ is not constructible. However if we are *given* the angle $3\pi/p2^k$ then the above relation shows that $\pi/p2^k$ can be constructed from it. Hence $3\pi/p2^k$ is not constructible but it is trisectible. (Similar arguments work for any n which is not a power of 2.)

Not all examples need be rational multiples of π . As an alternative method consider the continuous function $f : [0, \pi/6] \rightarrow [0, \sqrt{3}/2]$ given by $f(\theta) = \cos \theta - \cos 3\theta$. If p is a prime and m a positive integer with $m/p < \sqrt{3}/2$ then, by the continuity of f , $m/p = f(\theta)$ for some θ . Hence

$$\frac{m}{p} = \cos \theta - \cos 3\theta = 4 \cos \theta - 4 \cos^3 \theta$$

and

$$4px^3 - 4px + m = 0,$$

where $x = \cos \theta$. As in (b) it follows that x (and θ) are not constructible. Also, since

$$\cos \theta = \frac{m}{p} + \cos 3\theta$$

it follows that $\cos 3\theta$ (and 3θ) are not constructible. But if we are *given* 3θ then the above relationship shows that we can construct θ . Hence 3θ is not constructible but it is trisectible.

(d) Neither constructible nor n -sectible. This is perhaps the most interesting and the most difficult case to consider as no contradiction follows from showing that α/n is not constructible. One needs to move into the theory of field extensions, extending

the field \mathbb{Q} by the addition of $\cos \alpha$. By such methods it is possible to prove that if p is an odd prime and m an integer with $1 \leq m < p$ then $\frac{1}{3} \cos^{-1}(m/p)$ is n -sectible if and only if n is a power of 2. So choosing m and p so that $\frac{1}{3} \cos^{-1}(m/p)$ is not constructible ensures, for example, that it is neither constructible nor trisectible.

Finally, we leave the reader with a number of questions for investigation.

- (i) Which angles with a rational cosine are trisectible? Note that $\cos^{-1}((3p^2 - 4)/p^3)$ is trisectible for any prime p .
- (ii) Find a trisectible angle $\cos^{-1}(a/b)$ with a, b positive integers and $a + b$ as small as possible.
- (iii) Discuss the problem of n -secting the angle $\sin^{-1}(m/p)$.

REFERENCES

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