Groups with $\Pr(G) = 1/3$

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Abstract. We find all isoclinism families of groups with commuting probability 1/3.

1. Introduction

We define the *commuting probability* of a finite group G to be

$$\Pr(G) := \frac{|\{(x, y) \in G \times G \ : \ xy = yx\}|}{|G|^2}, \tag{1}$$

where $|S|$ denotes cardinality of a set S. Much has been written on this concept: see for instance [5], [13], [8], [16], [21], [14], [4], [7], [3], and [11]. In particular, $\Pr(G)$ is an isoclinism invariant [14, Lemma 2.4], so to understand which groups have a given commuting probability t, it suffices to find all isoclinism families with commuting probability t. If $\mathcal{F}$ is an isoclinism family containing a finite group G, we call $\mathcal{F}$ a *finite family* and define $\Pr(\mathcal{F})$ to be $\Pr(G)$.

Lescot [14] found all families $\mathcal{F}$ with $\Pr(\mathcal{F}) \geq 1/2$, and the results of Rusin [21] allow one to determine all $\mathcal{F}$ with $\Pr(\mathcal{F}) > 11/32$. In this paper, we find all finite families $\mathcal{F}$ for which $\Pr(\mathcal{F}) = 1/3$.

Theorem 1. *There are precisely three finite families $\mathcal{F}$ with $\Pr(\mathcal{F}) = 1/3$. Each has a unique stem group, namely the alternating group A_4, the dihedral group D_9, and the generalized dihedral group $\mathrm{Dih}(C_3 \times C_3)$.*

We chose the value 1/3 above because it is an interesting cutoff value. Barry, MacHale, and Ní Shé [2] showed that a finite group G is supersolvable if $\Pr(G) > 1/3$, and noted that this lower bound is minimal because the alternating group A_4 is not supersolvable and satisfies $\Pr(A_4) = 1/3$. Lescot, Nguyen, and Yang [15, Corollary 2] showed that for these properties, A_4 is essentially unique: if $\Pr(G) \geq 1/3$ and G is not supersolvable, then G is isoclinic to A_4.

After some background material in Section 2, we prove Theorem 1 in Section 3.

2. Preliminaries

Throughout the remainder of the paper, G is always a group. Our notation is standard, except for the following: $[G, x]$ is the subset $\{[y, x] \mid y \in G\}$ of G, and $C_H(x)$ is the *centralizer* of $x \in G$ in the subgroup $H \leq G$ (this is the subgroup of H consisting of all $h \in H$ that commute with x). A group is said to be *capable* if it is isomorphic to $G/Z(G)$ for some group G.

We also use fairly standard notation for a few well-known groups. C_n is the cyclic group of order n and, if $n \geq 3$, S_n is the symmetric group of order $n!$, A_n is the alternating group of order $n!/2$, D_n is the dihedral group of order $2n$, Q_2 is the quaternion group of order 8, and $\mathrm{Dih}(N)$ is the generalized dihedral group of order $2|N|$ that has a subgroup isomorphic to N; specifically, $\mathrm{Dih}(N) := N \rtimes_\phi C_2$, where $\phi(x)$ is the inversion automorphism of N for the generator x of C_2. For every other explicitly mentioned group, we use the GAP ID: specifically, $\mathrm{Gp}(n,m)$ denotes the group with GAP ID (n,m). Some computations in this paper were made using GAP; for more on GAP, see [6].

Isoclinism is an equivalence relation for groups that was introduced by Hall [10], and is widely used in the group theory literature. For the definition of isoclinism, we use $G^{\times 2}$ as an alternative notation for the set $G \times G$; here, the group structure of $G \times G$ is irrelevant. If $\phi : G \to H$, then $\phi^{\times 2} : G^{\times 2} \to H^{\times 2}$ is the natural product map.

Observation 2. Since $[g_1, g_2] = [g_1z_1, g_2z_2]$ for all $g_1, g_2 \in G$ and all $z_1, z_2 \in Z(G)$, the commutator map induces a natural map

$$\begin{aligned} \kappa_G : (G/Z(G))^{\times 2} &\to G' \\ (g_1Z(G), g_2Z(G)) &\mapsto [g_1, g_2]\,. \end{aligned}$$

Definition 3. A pair of groups, G and H, are said to be *isoclinic* if there are isomorphisms $\phi : G/Z(G) \to H/Z(H)$ and $\psi : G' \to H'$ such that $\psi([a,b]) = [a',b']$ whenever $\phi(aZ(G)) = a'Z(H)$ and $\phi(bZ(G)) = b'Z(H)$. Equivalently, the diagram in Figure 1 commutes, where κ_G, κ_H are as in Observation 2.

We call (ϕ, ψ) an *isoclinism from G to H*, and write $G \sim H$ if G is isoclinic to H. The isoclinism equivalence class containing a given group G is called the *family of G*.

$$\begin{tikzcd}
(G/Z(G))^{\times 2} \arrow[r, "\phi^{\times 2}", "\simeq"'] \arrow[d, "\kappa_G"'] & (H/Z(H))^{\times 2} \arrow[d, "\kappa_H"] \\
G' \arrow[r, "\psi", "\simeq"'] & H'
\end{tikzcd}$$

Figure 1. Isoclinism

As mentioned in the introduction, a *finite family* is a family $\mathcal{F}$ that contains a finite group G, and $\mathrm{Pr}(\mathcal{F}) := \mathrm{Pr}(G)$ is well-defined because $\mathrm{Pr}(\cdot)$ is a family invariant (when restricted to finite groups).

As proved by Hall [10], every family $\mathcal{F}$ has one or more *stem groups*, meaning groups G in the family satisfying $Z(G) \leq G'$. In a finite family $\mathcal{F}$, stem groups are precisely the groups of minimal order in $\mathcal{F}$, and all other finite groups in $\mathcal{F}$

have order a multiple of $|G|$. Consequently, a finite family $\mathcal{F}$ contains groups of odd order if and only if its stem groups have odd order, in which case we call $\mathcal{F}$ an *odd family.*

The next lemma is well known [13]. It is also a straightforward corollary of the degree equation $|G| = \sum_{i=1}^{k} d_i^2$ (where the numbers d_i are the degrees of the irreducible complex representations of G), and the fact that the number of linear complex characters of G equals $(G : G')$.

Lemma 4. *Let p be the smallest prime divisor of $|G|$. Then*

$$\Pr(G) \le \frac{1}{p^2}\left(1 + \frac{p^2-1}{|G'|}\right), \tag{2}$$

with equality if and only if all nonlinear irreducible complex representations of G are of degree p.

We will need the following pair of results of Rusin, which both concern p-groups of nilpotency class at most 2. The first result is a rewording of [21, Proposition 2], while the second follows from the first result and [21, Theorem 1].

Proposition 5. *If G is a p-group for some prime p, with $G' \le Z(G)$ and G' cyclic, then $G/Z(G)$ can be written as a direct product $A \times A$, where A is an abelian p-group. Furthermore, the maximal orders of elements in A and G' are the same.*

Theorem 6. *Suppose G is a p-group for some prime p, with $G' \le Z(G)$. Then*

$$\Pr(G) = \frac{1}{|G'|}\left(1 + \frac{p-1}{p}\sum_{K}\frac{(G' : K)}{(G : K^*)}\right), \tag{3}$$

where the sum is over all $K \le G'$ for which G'/K is a nontrivial cyclic group, and K^* denotes the subgroup of G consisting of all $x \in G$ such that $[G, x] \subset K$. In particular, if $G' \simeq C_p$ and so $G/Z(G) \simeq C_p^{2k}$ for some $k \in \mathbb{N}$, then

$$\Pr(G) = \frac{p^{2k} + p - 1}{p^{2k+1}}. \tag{4}$$

We next state a few other results from the literature that we need.

Theorem 7 ([20, Theorem 5])**.** *If G is a finite group with $Z(G) = \{1\}$, then $C_G(G') \le G'$.*

Theorem 8 ([12, Theorem 4.9])**.** *If $G' \simeq C_{2^n}$, then G is nilpotent of class at most $n+1$.*

Theorem 9 ([15, Corollary 2])**.** *If G is a finite group with $\Pr(G) \ge 1/3$, then either G is supersolvable or G is isoclinic to A_4.*

Lemma 10 ([2, Lemma 3.13])**.** *If $G' \simeq C_2 \times C_2$, then either*

(a) G is nilpotent, or

(b) $G/Z(G) \simeq A_4$, $G' \cap Z(G) = \{1\}$, and $\Pr(G) = 1/3$.

Theorem 11 ([1, Theorem 3])**.** *A group G has all its irreducible complex representations of degree at most 2 if and only if one of the following conditions holds:*

(a) G is abelian.
(b) G has an abelian subgroup H of index 2.
(c) $G/Z(G)$ is an abelian group of order 8.

Lemma 12 ([9, Corollary 3.2])**.** *A finite Abelian group is capable if and only if its two largest invariants coincide, i.e. if and only if it has the form $C_n \times C_n \times H$ for some $n \in \mathbb{N}$, where all elements in H have order dividing n.*

3. Proofs

Before we prove our main result, we state and prove three preparatory lemmas.

Lemma 13. *If G is a finite group with $Z(G) = \{1\}$, then $|G|$ is a divisor of $|Z(G')| \cdot |\operatorname{Aut} G'|$.*

Proof. Since G' is a normal subgroup of G, conjugation of G' by elements of G defines a homomorphism $\phi : G \to \operatorname{Aut} G'$. Thus $|G| \le |\operatorname{Aut} G'| \cdot |\ker \phi|$. Since $\ker \phi = C_G(G')$, the result follows from Theorem 7. □

Lemma 14. *Suppose G is a finite nilpotent stem group with $\Pr(G) = m/n$ for some coprime positive integers m and n. Then the prime factors of $|G|$ are the same as those of n.*

Proof. We assume without loss of generality that G is not abelian. It is readily verified that $\Pr(H \times K) = \Pr(H)\Pr(K)$, so $\Pr(G)$ is the product of $\Pr(P)$ as P ranges over all Sylow subgroups of G.

It follows readily from the fact that $Z(G) \le G'$ that none of the Sylow subgroups of G is abelian. For a nonabelian p-group, it is clear that $\Pr(G)$ is of the form a/b, where a, b are coprime positive integers and $b = p^k$ for some $k \in \mathbb{N}$. Consequently, the prime factors of n are precisely the prime factors of $|G|$, as required. □

The last of our three lemmas gives the commuting probability of all nonabelian groups that have an abelian subgroup of index 2. A more general version of this result was proved by Ní Shé [19, Theorem 1.3.6], but we include a proof for completeness.

Lemma 15. *Suppose G is a finite non-abelian group containing an abelian subgroup H of index 2. Then $Z(G)$ is a proper subgroup of H, and $\Pr(G) = (n+3)/4n$, where $n = (H : Z(G))$.*

Proof. H is a normal subgroup of G by virtue of the fact that it has index 2 in G. Let us choose $x \in G \setminus H$. Since G is nonabelian, x must fail to commute with some $u \in H$. Each $g \in G$ can be written uniquely in the form hx^i for some $h \in H$ and $i \in \{0, 1\}$. It is clear that for all $h \in H$, hx fails to commute with u, and so $Z(G) \subset H \setminus \{u\}$. Thus $Z(G)$ is a proper subgroup of H.

Let us write $m := |Z(G)|$, and so $|G| = 2mn$. Recall that the number of conjugates of an element $y \in G$ equals $(G : C_G(y))$. In particular, y has only a single conjugate if $y \in Z(G)$, or exactly two conjugates if $y \in H \setminus Z(G)$. We claim that y has exactly n conjugates when $y \in G \setminus H$. Assuming this claim, we see that the number of conjugacy classes in G is

$$k(G) := |Z(G)| + \frac{|H| - |Z(G)|}{2} + \frac{|G| - |H|}{n} = m + \frac{mn - m}{2} + \frac{mn}{n} = \frac{(n+3)m}{2}\,.$$

In view of the well known equation $\Pr(G) = k(G)/|G|$, the result follows.

To prove the claim, we need to verify that $|C_G(y)| = 2m$ whenever $y = gx$ for some $g \in H$. Suppose y commutes with $h \in H$. Then $x^{-1}g^{-1}hgx = h$. But $g^{-1}hg = h$, so we have $x^{-1}hx = h$. This holds exactly when $h \in C_H(x) = Z(G)$. We conclude that $|C_G(y) \cap H| = |Z(G)| = m$.

Suppose next that y commutes with $v := hx$ for some $h \in H$, and so $[hx, gx] = 1$. Now $[hx, gx] = x^{-1}wx$, where $w := h^{-1}x^{-1}g^{-1}hxg$, so we must have $w = 1$. Because H is normal in G, we see that $c := x^{-1}g^{-1}hx \in H$, and so

$$w = h^{-1}cg = h^{-1}gc = [d, x] = 1\,,$$

where $d := g^{-1}h \in H$. Thus y commutes with v if and only if $g^{-1}h \in Z(G)$, and so there are $|Z(G)| = m$ elements $v \in G \setminus H$ that commute with y. Since we already know that $|C_G(y) \cap H| = m$, the claim follows. $\square$

In the following proof, we consider isomorphic groups to be the same whenever it is convenient.

Proof of Theorem 1. Suppose first that G is a stem group of odd order with $\Pr(G) = 1/3$. The right-hand side of (2) is a decreasing function of p, so

$$\frac{1}{3} \le \frac{1}{3^2}\left(1 + \frac{3^2 - 1}{|G'|}\right)\,,$$

and this inequality can be rewritten as $|G'| \le 4$. Since G is of odd order and nonabelian, we must have $|G'| = 3$ and, since G is a stem group, it follows that either $Z(G) = G'$ or $Z(G) = \{1\}$.

If $Z(G) = G'$, then G is nilpotent, and by Lemma 14, G is a 3-group. But then (4) implies that $\Pr(G) > 1/3$, giving a contradiction. Suppose instead that $Z(G) = \{1\}$. By Lemma 13, $|G| \le |G'| \cdot |\operatorname{Aut} G'| = 3 \cdot 2$. But all groups of odd order at most 6 are abelian, giving a contradiction. This rules out all odd families.

Thus G must be an even order stem group with $\Pr(G) = 1/3$. However, the equation $\Pr(G) = 1/3$ also forces $|G|$ to be divisible by 3, so $|G|$ must in fact be divisible by 6. Applying Lemma 14 with $(m, n) = (1, 3)$, we see that G cannot be nilpotent, and so $Z(G)$ must be a proper subgroup of G'.

By appealing again to (2), we see that

$$\frac{1}{3} \le \frac{1}{2^2}\left(1 + \frac{2^2 - 1}{|G'|}\right)\,,$$

and so $|G'| \le 9$.

We examine separately the various possibilities for the isomorphism class of G'. First, consider the case where $|G'|$ is a prime p. Since $Z(G)$ is a proper subgroup of G', we must have $Z(G) = \{1\}$. By Lemma 13, $|G| \le p(p-1)$. On the other hand, $|G|$ is divisible both by 6 and by $p = |G'|$. This gives a contradiction when $p \in \{2, 5\}$. For $p = 3$, we need only check nonabelian groups of order 6. There is only one of these, namely S_3, and $\mathrm{Pr}(S_3) = 1/2$, so we get a contradiction.

Finally for $p = 7$, it suffices to examine nonabelian groups of order 42. There are five of these, but three ($S_3 \times C_7$, $D_7 \times C_3$, and $H \times C_2$, where H is the nonabelian group of order 21) are decomposable and can be eliminated because they have a nontrivial center. The remaining two groups are ruled out quickly using GAP: $\mathrm{Pr}(\mathrm{Gp}(42,1)) = 1/6$ and $\mathrm{Pr}(\mathrm{Gp}(42,5)) = 2/7$.

Suppose instead that $|G'| = 4$. Since G is not nilpotent, we rule out $G' = C_4$ by Theorem 8, and so we must have $G' \simeq C_2 \times C_2$. Since G is not nilpotent and $Z(G) \le G'$, it follows from Lemma 10 that $G \simeq A_4$. By Theorem 9, *we may assume from now on that G is supersolvable.*

We next consider $|G'| = 6$. According to [17, p. 126], there is no group G with $G' \simeq S_3$, so we only have to examine $G' \simeq C_6$. Now G' contains a unique element t of order 2, and G' is normal in G so $x^{-1}tx$ must equal x for all $x \in G$. Thus $t \in Z(G)$. Since also $Z(G)$ is a proper normal subgroup of G', it follows that $Z(G) \simeq C_2$. Now

$$(G/Z(G))' = G'Z(G)/Z(G) = G'/Z(G) \simeq C_3$$

and $Z(G/Z(G)) = \{1\}$ (since otherwise $G/Z(G)$, and hence G, is nilpotent), and so by Lemma 13, $|G/Z(G)| \le 3 \cdot 2$. Thus $|G| \le 12$ and $|G|$ is divisible by 6. We have already ruled out groups of order 6. No group of order 12 has a derived subgroup of order 6, so we have eliminated all groups with $|G'| = 6$.

We next consider $|G'| = 8$. There are five groups of order 8. By Theorem 8, we can eliminate $G' \simeq C_8$. According to [17, p. 125], no group has $G' \simeq D_4$ and, according to [2, Remark 3.3], $\mathrm{Pr}(G) < 1/3$ if $G' \simeq Q_2$. (Alternatively, by [12, Theorem 7], there is no supersolvable group with $G' \simeq Q_2$.) There remain the possibilities that $G' \simeq C_4 \times C_2$ or $G' \simeq C_2 \times C_2 \times C_2$.

Suppose first that $G' \simeq C_4 \times C_2$. If c is an element of order 4 in G', then all automorphisms of G' fix c^2, so c^2 must be central in G. Thus $|Z(G)|$ must equal either 2 or 4. If $|Z(G)| = 2$, then as before $(G/Z(G))' \simeq C_2 \times C_2$. Lemma 10 now implies that either $G/Z(G)$ is nilpotent or its central quotient group is isomorphic to A_4. Both of these lead to contradictions: the first because it implies that G is nilpotent, and the second because A_4 is not supersolvable, so $G/Z(G)$ and G also fail to be supersolvable.

Suppose instead that $G' \simeq C_2 \times C_2 \times C_2$. We cannot have $Z(G) \simeq C_2 \times C_2$, since then $(G/Z(G))' = G'/Z(G) \simeq C_2$, and G would be nilpotent by Theorem 8 as before. If $Z(G) \simeq C_2$, then $(G/Z(G))' \simeq C_2 \times C_2$, and we rule this out as in the previous paragraph.

Finally, suppose $Z(G) = \{1\}$. By Lemma 13, we see that $|G|$ is a divisor of $8 \cdot (7 \cdot 6 \cdot 4) = 1344$ and also $|G|$ is a multiple of both 6 and $|G'| = 8$. Thus $|G|$ is a multiple of 24. A search in GAP reveals that there are no groups G satisfying $|G'| = 8$ and $Z(G) = \{1\}$, with any of the eight orders dividing 1344 and divisible by 24.

Finally, we consider the case $|G'| = 9$. By the condition for equality in (2), we see that all nonlinear characters have degree 2. We now appeal to Theorem 11. Two of the possibilities there cannot arise in our setting: G cannot be abelian, and $G/Z(G)$ cannot be an abelian group of order 8, since then G would be nilpotent of class 2.

The only remaining possibility is that G has an abelian subgroup H of index 2. By Lemma 15, $(H : Z(G)) = n$, where

$$\frac{1}{3} = \frac{n+3}{4n}\,.$$

We conclude that $n = 9$ and so $|G/Z(G)| = 18$. Since $Z(G)$ must be of order 1 or 3, we need to check groups of orders 18 and 54.

There are three nonabelian groups of order 18. One of them, $S_3 \times C_3$, has commuting probability 1/2. The other two, D_9 and $\mathrm{Dih}(C_3 \times C_3)$, have commuting probability 1/3, as required. Finally, GAP reveals that there are just two groups G of order 54 with $|G'| = 9$ and $Z(G) = \{1\}$, and in both cases $\Pr(G) = 5/27 \neq 1/3$.

We have shown that there are exactly three stem groups G with $\Pr(G) = 1/3$. No two of these groups are isoclinic because they have non-isomorphic derived groups: $(A_4)' \simeq C_2 \times C_2$, $(D_9)' \simeq C_9$, and $(\mathrm{Dih}(C_3 \times C_3))' \simeq C_3 \times C_3$. □

Finally, we give a corollary concerning another well-known indicator of commutativity of a finite group that is somewhat analogous to $\Pr(G)$, namely the *character degree sum ratio*

$$f(G) := \frac{1}{|G|} \sum_{i=1}^{k(G)} d_i\,,$$

where $d_1, \ldots, d_{k(G)}$ are the degrees of the irreducible complex representations of G.

Corollary 16. *If G is a finite group with $\Pr(G) = 1/3$, then $f(G)$ is either $1/2$ or $5/9$.*

Proof. Since $f(G)$ is an isoclinism invariant (for which, see [18, Theorem 12]), it suffices to compute $f(G)$ for each of the stem groups G in Theorem 1. Now A_4 has three characters of degree one, and a single irreducible character of degree 3, so $f(A_4) = (3 \cdot 1 + 3)/12 = 1/2$. Both D_9 and $\mathrm{Dih}(C_3 \times C_3)$ have two characters of degree one and four irreducible characters of degree two, so

$$f(D_9) = f(\mathrm{Dih}(C_3 \times C_3)) = (2 \cdot 1 + 4 \cdot 2)/18 = 5/9\,. \qquad \square$$

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