BRANCHING RULES FOR SPECHT MODULES

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1. Introduction

Let n be a positive integer and let Σ_n be the symmetric group of degree n. For any field F and any partition λ of n, the Specht module S_F^λ is defined to be the submodule of the permutation module $(1_{\Sigma_\lambda}) \uparrow^{S_n}$ spanned by all λ-polytabloids, where Σ_λ is the Young subgroup associated to λ. Specht modules play a central role in the representation theory of the symmetric group. This is because in characteristic 0, the Specht modules are the simple $F\Sigma_n$-modules, while in characteristic p the heads of the Specht modules S_F^λ such that λ is p-regular are the simple $F\Sigma_n$-modules. When the field F has charactersitic 0, the structure of the restriction of S_F^λ to Σ_{n-1} is given by the Classical Branching Rule, which states that $S_F^\lambda{\downarrow}_{\Sigma_{n-1}}$ is a direct sum $\bigoplus_\mu S_F^\mu$, where μ runs through all partitions of $n-1$ obtained from λ by removing node from its Young diagram. In 1971, Peel [5] gave a version of this theorem for characteristic p. He showed that there is a series of submodules such that the successive quotients are the Specht modules S_F^μ, where μ runs through the same set. Nevertheless, the structure of the restriction $S_F^\lambda \downarrow_{\Sigma_{n-1}}$ is not well understood. For example, the problem of finding a composition series is open and very difficult. See Kleshchev [3] for more information on the restrictions of irreducible Σ_n-modules to Σ_{n-1}.

In this paper, we find the indecomposable components of $S_F^\lambda \downarrow_{\Sigma_{n-1}}$, when the characteristic of F is not 2. These are given by Theorem 3.4, which states that if B is a block idempotent of $F\Sigma_{n-1}$, then $S_F^\lambda \downarrow_{\Sigma_{n-1}} B$ is 0 or indecomposable. We also prove the analogous theorem for the induced module $S_F^\lambda \uparrow^{\Sigma_{n+1}}$. The two proofs are almost identical. In [1] we will give a complete description of the endomorphism ring of $S_F^\lambda \downarrow_{\Sigma_{n-1}}$, and also that of $S_F^\lambda \uparrow^{\Sigma_{n+1}}$.

The assumption that char $F \neq 2$ in Theorm 3.4 cannot be dropped – in characteristic 2 there are decomposable Specht modules, and these can easily be used to construct examples where block components of $S^\lambda{\downarrow}_{\Sigma_{n-1}}$ or $S^\lambda{\uparrow}_{\Sigma_{n+1}}$ are decomposable.

2. Minimal polynomial of the sum of all transpositions acting on the restriction and induction of a Specht module

Throughout this paper n is a fixed positive integer, $\lambda = [\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l > 0]$ is a fixed partition of n and m is the number of different nonzero parts of λ. We orient the Young diagram $[\lambda]$ left to right and top to bottom. This means that longer rows are above shorter rows, and longer columns are to the left of shorter columns; also, the *first* row is the one at the top and the *first* column is the one

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at the left. The (i, j) node is in the ith row and the jth column. We will use $\widehat{n}$ to denote the set $\{1, \ldots, n\}$ and let Σ_n denote the group of permutations of $\widehat{n}$. Permutations and homomorphisms will generally act on the right. The *Murphy element* L_n is the sum of all transpositions in Σ_n that are not in Σ_{n-1}. We use E_n to denote the sum of all transpositions in Σ_n. So E_n is the first elementary symmetric function in the Murphy elements.

Let F be any field and let S^λ denote the Specht module, defined over F, corresponding to λ. We use the notation

$$\begin{array}{ll} \mathcal{R} & \text{for the restricted module } S^\lambda\downarrow_{\Sigma_{n-1}} \text{ and} \\ \mathcal{I} & \text{for the induced module } S^\lambda\uparrow^{\Sigma_{n+1}}. \end{array}$$

In this section we compute the minimal polynomial of E_{n-1} acting on $\mathcal{R}$ and the minimal polynomial of E_{n+1} acting on $\mathcal{I}$.

A λ-*tableau* is a bijective map $t : [\lambda] \to \widehat{n}$. The value of t at a node (r, c) is denoted by t_{rc}. The group Σ_n acts on λ-tableaux by functional composition; $(t\pi)_{rc} = t_{rc}\pi$, for each $\pi \in \Sigma_n$.

We regard a λ-*tabloid* as an ordered partition $\mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_l)$ of $\widehat{n}$ such that the cardinality of $\mathcal{P}_u$ is λ_u, for $u = 1, \ldots, l$. Each λ-tableau t determines the λ-tabloid $\{t\}$ whose u-th part is the set of entries in the u-th row of t. If s is a λ-tableau, then $\{t\} = \{s\}$ if and only if $s = t\pi$, for some π in the row stabilizer R_t of t. We denote the column stabilizer of t by C_t. The *polytabloid* e_t is the following element of M^λ:

$$e_t := \sum_{\pi \in C_t} \operatorname{sgn} \pi \, \{t\pi\}.$$

It is shown in [2] that the polytabloids span the Specht module S^λ.

Adapting the notation of [2], let $(r_1, c_1), \ldots, (r_m, c_m)$ be the removable nodes of $[\lambda]$, ordered so that $r_1 < \ldots < r_m$ and $c_1 > \ldots > c_m$. Set $r_0 = 0 = c_{m+1}$. The addable nodes of $[\lambda]$ are the $(m+1)$ nodes $(r_u + 1, c_{u+1} + 1)$, for $u = 0, \ldots, m$. We use $\lambda\downarrow_u$ to denote the partition of $n-1$ obtained by decreasing the r_u-th part of λ by 1, for $u \in \widehat{m}$. In addition, we use $\lambda\uparrow^u$ to denote the partition of $n+1$ obtained by increasing the $(r_u + 1)$-th part of λ by 1, for $u \in \widehat{m+1}$.

We need special notation for certain subsets of entries in t. For any $u \in \widehat{m}$, let $H_u(t)$ be the set of entries in the union of the top r_u rows of t, and let $V_u(t)$ be the set of entries in the union of columns of t numbered from $c_{u+1} + 1$ to c_u (inclusive). Clearly $H_1(t) \subset \ldots \subset H_m(t)$, while $V_m(t), \ldots, V_1(t)$ form a partition of t. If $u, v \in \widehat{m}$ then $V_u(t) \subseteq H_v(t)$ if and only if $u \leq v$. As $H_u(t)$ depends only on the rows of t, we may define $H_u(\{t\}) := H(t)$.

By Theorem 9.3 in [2], $\mathcal{R}$ has a Specht series

$$0 = \mathcal{R}_0 \subset \mathcal{R}_1 \subset \mathcal{R}_2 \subset \ldots \subset \mathcal{R}_m = \mathcal{R},$$

with $\mathcal{R}_u/\mathcal{R}_{u-1} \cong S^{\lambda\downarrow_u}$, for $u \in \widehat{m}$. James' description of $\mathcal{R}$, and the Garnir relations, show that e_t lies in $\mathcal{R}_u \backslash \mathcal{R}_{u-1}$ if $n \in V_u(t) \backslash H_{u-1}(t)$ Also, by 17.14 in [2], the module $\mathcal{I}$ has a Specht series

$$\mathcal{I} = \mathcal{I}_1 \supset \mathcal{I}_2 \supset \ldots \supset \mathcal{I}_{m+1} \supset \mathcal{I}_{m+2} = 0,$$

with $\mathcal{I}_u/\mathcal{I}_{u+1} \cong S^{\lambda\uparrow^u}$, for $u \in \widehat{m+1}$. Moreover, James shows that each factor $\mathcal{I}/\mathcal{I}_{u+1}$ is isomorphic to a submodule of the permutation module $M^{\lambda\uparrow^u}$.

Lemma 2.1. *Suppose that the $F\Sigma_n$-module M has a Specht series*

$$0 = M_0 \subset M_1 \subset \ldots \subset M_m = M.$$

Let $z \in Z(F\Sigma_n)$ and let $u \in \widehat{m}$. Then the map $M_u/M_{u-1} \to M_u/M_{u-1}$ given by multiplication by z, equals z_u times the identity, for some $z_u \in F$.

Proof. If char $F = 0$, then M_u/M_{u-1} is an irreducible $F\Sigma_n$-module (a Specht module), and the conclusion is obvious. If char $F = p$ is positive, then M_u/M_{u-1} is the p-modular reduction of an irreducible module defined over a suitable discrete valuation ring of characteristic 0. The conclusion follows in this case from the characteristic zero case. □

This lemma allows us to give the following upper bound on the degrees of the minimal polynomials of E_{n-1} and E_{n+1}.

Corollary 2.2. *The minimal polynomial of E_{n-1} acting on $\mathcal{R}$ has degree at most m, while the minimal polynomial of E_{n+1} acting on $\mathcal{I}$ has degree at most $m+1$.*

Proof. Let $u \in \widehat{m}$. Lemma 2.1 shows that $\mathcal{R}_u(E_{n-1} - z_u) \subseteq M_{u-1}$, for some scalar z_u. It follows from a simple inductive argument that $\mathcal{R}\prod_{u=1}^{m}(E_{n-1} - z_u) = 0$. A similar argument deals with the action of E_{n+1} on $\mathcal{I}$. □

It will turn out that the polynomials given in the proof of Corollary 2.2 are the minimal polymials we are seeking. Before we prove this, we will identify the scalars z_u in terms of Young diagrams.

The *residue* of a node (r, c) is the scalar $(c-r)1_F$. If F is a symmetric polynomial, let $F(\lambda)$ denote the evaluation of F on the multiset of residues of the nodes in $[\lambda]$. In particular, if E denotes the first elementary polynomial in n variables, then $E(\lambda)$ is the sum of the residues of the nodes in $[\lambda]$. An easy calculation shows that $E(\lambda) = \sum_{i=1}^{l} \frac{1}{2}\lambda_i(\lambda_i + 1 - 2i)1_F$. The next lemma is a special case of a more general result proved by G. E. Murphy [4, (3.3)]: first elementary symmetric function can be replaced by any symmetric function in n variables.

Lemma 2.3. *E_n acts as the scalar $E(\lambda)$ on S^λ.*

Proof. Let t be a λ-tableau, let $(r, c) \in [\lambda]$ and let $i = t_{rc}$. Fix $1 \leq c_1 < c$. Then by a simple Garnir relation, $e_t \sum_j (i, j) = e_t$, where j runs over all entries in the c_1-th column of t. Also $e_t(i, j) = -e_t$, for each entry j above i in column c of t. It follows that

$$e_t \sum_j (i, j) = (c - r)e_t,$$

where j runs over those elements of $\widehat{n}$ that lie strictly to the left of i, or above i in the same column of t. If we sum over all $(r, c) \in [\lambda]$, each transposition (i, j) occurs exactly once on the left hand side, while the coefficient of e_t on the right hand side is $E(\lambda)$. □

We next describe the induced module $\mathcal{I}$. Suppose that $u \in \widehat{m+1}$. Let T be a $\lambda{\uparrow}^u$-tableau, and let t denote the restriction of T to $[\lambda]$. Then the (λ, T)-polytabloid e_T^λ is the following element of $M^{\lambda\uparrow^u}$:

$$e_T^\lambda := \sum_{\pi \in C_t} \operatorname{sgn}\pi\, \{T\pi\}.$$

In Section 17 of [2] it is shown that when $u = m+1$, the corresponding (λ, T)-polytabloids span an $F\Sigma_{n+1}$-submodule of $M^{\lambda\uparrow^{m+1}}$ that is isomorphic to the induced module $\mathcal{I}$. We will always work with this copy of $\mathcal{I}$.

When we are showing that the polynomials given in the proof of 2.2 are minimal, it will be more convenient to look at the action of the Murphy elements L_n and L_{n-1} rather than E_{n-1} and E_{n+1}. The following lemma provides a link between these actions.

Lemma 2.4. *Let t be a λ-tableau and let T be the $\lambda{\uparrow}^{m+1}$-tableau whose restriction to $[\lambda]$ is t. Suppose that $f(x) \in F[x]$. Then*

$$\begin{aligned} e_t\, f(E_{n-1}) &= e_t\, f(E(\lambda) - L_n); \\ e_T^{\lambda}\, f(E_{n+1}) &= e_T^{\lambda}\, f(E(\lambda) + L_{n+1}). \end{aligned}$$

Proof. Lemma 2.3 shows that E_n acts as the scalar $E(\lambda)$ on $\mathcal{R}$. The first statement then follows from $E_{n-1} = E_n - L_n$.

Let V be the subspace of $M^{\lambda\uparrow^{m+1}}$ that is spanned by all e_U^{λ} such that U is a $\lambda{\uparrow}^{m+1}$-tableau with $n+1$ in the unique entry of its last row. Then V is a direct summand of the restriction of $\mathcal{I}$ to Σ_n that is isomorphic to S^{λ}. Since $e_T^{\lambda} \in V$, Lemma 2.3 implies that $e_T^{\lambda} E_n = E(\lambda) e_T^{\lambda}$. The second statement now follows from $E_{n+1} = E_n + L_{n+1}$, and the fact that $E_n L_{n+1} = L_{n+1} E_n$. □

When we are showing that the polynomials given in the proof of 2.2 are minimal, we will want to show that there is a λ-tableau t such that the vectors $\{e_t L_n^i \mid 0 \leq i \leq m-1\}$ are linearly independent. This will be accomplished using the following technical lemma concerning the action of L_n on $\mathcal{R}$.

Lemma 2.5. *Let t be a λ-tableau such that $n \in V_m(t) \backslash H_{m-1}(t)$. For each $u \in \widehat{m-1}$, choose $x_u \in V_u(t) \backslash H_{u-1}(t)$. Define the λ-tableau $s := t(n, x_{m-1}, x_{m-2}, \ldots, x_1)$. Let i be a positive integer. Then the coefficient of $\{s\}$ in the expansion of $e_t L_n^i$ into tabloids is*

$$\begin{aligned} &0, \quad \text{if } 0 \leq i \leq m-2; \\ &1, \quad \text{if } i = m-1. \end{aligned}$$

Proof. Clearly $L_n^i = \sum (w_i, n)(w_{i-1}, n) \ldots (w_1, n)$, where $(w_1, \ldots, w_i)$ ranges over all functions $\widehat{i} \to \widehat{n-1}$. Let $(y_1, \ldots, y_i)$ be a function $\widehat{i} \to \widehat{n-1}$, let $\theta = (y_i, n)(y_{i-1}, n) \ldots (y_1, n)$, and assume that $\{s\}$ appears with nonzero coefficient in the expansion of $e_t \theta$. We have two goals:

(a) to show that $i = m-1$.
(b) to show that the cyclic permutations $(y_1, \ldots, y_{m-1})$ and $(x_1, \ldots, x_{m-1})$ are equal.

The second part of the lemma follows easily from this second goal, as we now show. In the sum $\sum e_t(w_i n) \ldots (w_1 n)$, $\{s\}$ can appear in only one term, namely $e_t(x_{m-1}, n) \ldots (x_1, n)$. Since this term is equal to $e_t(n, x_{m-1}, x_{m-2}, \ldots, x_1) = e_s$, $\{s\}$ appears with coefficient 1.

Since $e_t \theta = e_{t\theta}$, there exists π in the column stabilizer of $t\theta$ such that $\{s\} = \{t\,\theta\,\pi\}$. Let $u \in \widehat{m-1}$. Then by construction $x_u \in V_{u+1}(s) \backslash H_u(s)$; since $\{s\} = \{t\theta\pi\}$, it follows that $x_u \notin H_u(t\,\theta\,\pi)$. As π^{-1} is a column permutation of $t\theta$, we have $x_u \in V_{u+1}(t\theta) \cup \ldots \cup V_m(t\theta)$. Thus

$$\forall u \in \widehat{m-1}, \quad x_u \theta^{-1} \in V_{u+1}(t) \cup \ldots \cup V_m(t). \tag{1}$$

In particular, θ does not fix any of the $m-1$ distinct symbols $x_1, \ldots, x_{m-1} \in \widehat{n-1}$.

In this paragraph, we will show that θ does not fix n. Assume that θ does fix n. If the symbols in the list $y_1, \ldots, y_i$ were distinct, θ would be the cycle

$(y_i, y_{i-1}, \ldots, y_1, n)$; since θ fixes n, it follows that there is some repetition in the list $y_1, \ldots, y_i$. Since $\theta = (y_i, n)(y_{i-1}, n) \ldots (y_1, n)$ and θ fixes n, the only symbols potentially moved by θ are on the list $y_1, \ldots, y_i$. Since this list contains a repeat, θ moves at most $i-1$ symbols. The previous paragraph shows that θ moves at least $m-1$ symbols. Therefore $m \leq i$. But by hypothesis $i \leq m-1$. This contradiction shows that θ moves n.

We now know that θ moves all the m symbols in $\{x_1, \ldots, x_{m-1}, n\}$. Since $\theta = (y_i, n)(y_{i-1}, n) \ldots (y_1, n)$, θ can only move symbols on the list $y_1, y_2, \ldots, y_i, n$. By hypothesis, $i \leq m-1$. It follows that $i = m-1$, which is goal (a). It also follows that the sets $\{x_1, \ldots, x_{m-1}\}$ and $\{y_1, \ldots, y_{m-1}\}$ coincide and that the elements on the list $y_1, y_2, \ldots, y_{m-1}$ are distinct. Hence θ is equal to the m-cycle $(y_{m-1}, y_{m-2}, \ldots, y_1, n)$. In particular, $y_{m-1}\theta^{-1} = n$. From (1) applied with $u = m-1$, $x_{m-1}\theta^{-1} = n$. (This is because n is the only symbol moved by θ that is in $V_m(t)$.) Hence $y_{m-1} = x_{m-1}$. From this fact and (1) applied with $u = m-2$, it follows that $x_{m-2}\theta^{-1} = x_{m-1}$. Hence $y_{m-2} = x_{m-2}$. Continuing in this way, by reverse induction on u, it follows that for all $u \in \widehat{m} - 1$, $y_u = x_u$. This gives goal (b) above, and completes the proof. □

The corresponding result for the action of L_{n+1} on $\mathcal{I}$ is:

Lemma 2.6. *Let t be a λ-tableau and let T be the $\lambda{\uparrow}^{m+1}$-tableau whose restriction to $[\lambda]$ is t. For each $u \in \widehat{m}$, choose $x_u \in V_u(t) \backslash H_{u-1}(t)$. Define the $\lambda{\uparrow}^{m+1}$-tableau $S := T\,(n+1, x_m, x_{m-1}, \ldots, x_1)$. Let i be a positive integer. Then the multiplicity of $\{S\}$ in the expansion of $e_T^{\lambda} L_{n+1}^i$ into tabloids is*

$$\begin{aligned} &0, \quad \text{if } 0 \leq i \leq m-1; \\ &1, \quad \text{if } i = m. \end{aligned}$$

Proof. Clearly $L_{n+1}^i = \sum (w_i, n+1)(w_{i-1}, n+1) \ldots (w_1, n+1)$, where $(w_1, \ldots, w_i)$ ranges over all functions $\widehat{i} \to \widehat{n}$. Let $(y_1, \ldots, y_i)$ be a function $\widehat{i} \to \widehat{n}$, let $\theta = (y_i, n+1)(y_{i-1}, n+1) \ldots (y_1, n+1)$, and assume that $\{S\}$ appears with nonzero multiplicity in the expansion of $e_T^{\lambda}\theta$ as a linear combination of tabloids. Then there exists π in the column stabilizer of $t\theta$ such that $\{S\} = \{T\,\theta\,\pi\}$.

As π fixes the single entry in the last row of $T\theta$, and x_m occupies this node in S, it follows that $(n+1)\theta = x_m$. Let $u \in \widehat{m-1}$ and let s denote the restriction of S to λ. Then $x_u \in V_{u+1}(s) \backslash H_u(s)$, whence $x_u \notin H_u(t\,\theta\,\pi)$. As π^{-1} is a column permutation of $t\theta$, we have $x_u \in V_{u+1}(t\theta) \cup \ldots \cup V_m(t\theta)$. Thus

$$x_u\theta^{-1} \in V_{u+1}(t) \cup \ldots \cup V_m(t). \tag{2}$$

In particular, θ does not fix x_u.

From its definition, θ moves at most $i+1$ elements of $\widehat{n+1}$. But θ does not fix any of the $m+1$ distinct symbols $n+1, x_m, \ldots, x_1$, and $i \leq m$. So we must have $i = m$. This, and (2), implies that $x_u\theta^{-1} \in \{x_{u+1}, \ldots, x_m\}$. Reverse induction on u shows that $x_u\theta^{-1} = x_{u+1}$. Thus θ coincides with the $(m+1)$-cycle $(n+1, x_m, x_{m-1}, \ldots, x_2, x_1)$. We conclude that $x_u = y_u$, for $u \in \widehat{m}$. This shows that θ occurs with multiplicity 1 in the expansion of L_{n+1}^m as a linear combination of group elements, whence $\{S\}$ appears with multiplicity 1 in the expansion of $e_T^{\lambda} L_{n+1}^m$ as a linear combination of tabloids in $M^{\lambda{\uparrow}^{m+1}}$. □

We can now prove the main result of this section.

Theorem 2.7. *The minimal polynomial of E_{n-1} acting on $\mathcal{R}$ is*

$$\prod_{u=1}^{m}(x - E(\lambda{\downarrow_u})),$$

while the minimal polynomial of E_{n+1} acting on $\mathcal{I}$ is

$$\prod_{u=1}^{m+1}(x - E(\lambda{\uparrow^u})).$$

Proof. First, we will prove the result on $\mathcal{R}$. Let t be as in Lemma 2.5. Then Lemma 2.5 implies that the set of vectors $\{e_t L_n^i \mid 0 \leq i \leq m-1\}$ is linearly independent. It follows from Lemma 2.4 that the set $\{e_t E_{n-1}^i \mid 0 \leq i \leq m-1\}$ is linearly independent. So the minimal polynomial of E_{n-1} has degree at least m. But Lemma 2.3 and the proof of Corollary 2.2 show that $\mathcal{R}\prod_{u=1}^{m}(E_{n-1} - E(\lambda{\downarrow_u})) = 0$.

The result on $\mathcal{I}$ follows from an identical argument using Lemma 2.6 in place of Lemma 2.5. □

3. The indecomposable components of the restriction and induction of a Specht module.

In this section we compute the indecomposable components of $\mathcal{R}$ and $\mathcal{I}$, when the characteristic of F is not 2. It is convenient to consider an $F\Sigma_n$-module M that shares the following properties in common with $\mathcal{R}$ and $\mathcal{I}$:

1. M has a Specht series
$$0 = M_0 \subset M_1 \subset \ldots \subset M_m = M,$$
such that $M_u/M_{u-1} \cong S^{\lambda_u}$, where λ_u is a partition of n, for each $u \in \widehat{m}$.
2. The labelling partitions satisfy $\lambda_1 \lhd \ldots \lhd \lambda_m$.
3. There exists $z \in Z(F\Sigma_n)$ such that the minimal polynomial of z acting on M has degree m.

Looking at the proof of Corollary 2.2, we see that z has minimal polynomial $\prod_{u=1}^{m}(x - z_u)$, where z acts as the scalar z_u on the Specht factor M_u/M_{u-1}.

Lemma 3.1. *There exists $\tau \in M$ such that $\tau\prod_{i=u+1}^{m}(z - z_i)$ lies in $M_u \backslash M_{u-1}$, for each $u \in \widehat{m}$.*

Proof. The hypothesis on the degree of the minimal polynomial of z implies that there exists $\tau \in M$ such that τz^{m-1} does not lie in the span of the vectors $\{\tau, \tau z, \ldots, \tau z^{m-2}\}$. Set $\tau_u = \tau\prod_{i=u+1}^{m}(z - z_i)$. Repeated application of Lemma 2.1 shows that $\tau_u \in M_u$.

Suppose that $\tau_u \in M_{u-1}$. Then Lemma 2.1 implies that $\tau_u\prod_{i=1}^{u-1}(z - z_u) \subseteq M_{u-1}\prod_{i=1}^{u-1}(z - z_u) = 0$. Thus $\tau\prod_{i=1, i\neq u}^{m}(z - z_i) = 0$. This contradicts our choice of τ. So $\tau_u \notin M_{u-1}$, which completes the proof. □

We now consider the endomorphism ring of M.

Lemma 3.2. *Suppose that char $F \neq 2$. Then:*

(i) *if $\theta \in \mathrm{End}_{F\Sigma_n}(M)$ and $u \in \widehat{m}$, then $M_u\theta \subseteq M_u$, and there is a well-defined Σ_n-endomorphism $\theta_u : M_u/M_{u-1} \to M_u/M_{u-1}$ given by $(v + M_{u-1})\theta_u = v\theta + M_{u-1}$;*

(ii) *the map $\Phi : \mathrm{End}_{F\Sigma_n}(M) \to \bigoplus_u \mathrm{End}_{F\Sigma_n}(M_u)$ such that $(\theta)\Phi = (\theta_1, \ldots, \theta_m)$, for each $\theta \in \mathrm{End}_{F\Sigma_n}(M)$, is an algebra homomorphism;*

(iii) *the kernel of* Φ *is the Jacobson radical of* $\mathrm{End}_{F\Sigma_n}(M)$.

Proof. First, we prove Part (i). By induction, we may assume that $M_{u-1}\theta \subseteq M_{u-1}$. Suppose that $M_u\theta \not\subseteq M_u$. Choose v so that $m \geq v > u$ and v is maximal so that $M_u\theta \not\subseteq M_{v-1}$. Then $M_u\theta \subseteq M_v$, and applying θ to elements of M_u induces a well-defined nonzero Σ_n-homomorphism

$$M_u/M_{u-1} \to M_v/M_{u-1} \twoheadrightarrow M_v/M_{v-1}.$$

But $\lambda_u \lhd \lambda_v$. This, and the fact that char $F \neq 2$, contradicts 13.17 of [2]. Thus indeed $M_u\theta \subseteq M_u$. This shows in particular that M_{u-1} is in the kernel of the map $M_u \to M_u/M_{u-1}$ given by the restriction of θ followed by projection. So θ_u is well-defined. This proves Part (i).

It is immediate from the definition of θ_u that Φ is an algebra homomorphism. As char $F \neq 2$, the only Σ_n-endomorphisms of M_u/M_{u-1} are scalar multiples of the identity, by 13.17 of [2]. It follows that the codomain of Φ is commutative and semisimple. Any element of the kernel must send M_u to M_{u-1}; therefore the kernel of Φ is nilpotent. This completes the proof of Parts (ii) and (iii) □

We now compute the indecomposable summands of M.

Proposition 3.3. *Assume that char* $F \neq 2$. *Let* B *be a block idempotent of* $F\Sigma_n$. *Then the* $F\Sigma_n$*-module* MB *is* 0 *or indecomposable.*

Proof. Assume that $MB \neq 0$. Let A be the algebra $\mathrm{End}_{F\Sigma_n}(MB)$. Identify the algebra A in the natural way with a direct summand of the algebra $\mathrm{End}_{F\Sigma_n}(M)$. We will use the notation and results from Lemma 3.2 throughout this proof. Our goal is to show that $A/J(A)$ has dimension 1 over F.

Suppose then that $\theta \in A$. Let w be maximal such that the Specht module M_w/M_{w-1} belongs to B. Our task is to show that if $\theta_w = 0$, then $\theta_u = 0$ for all u such that M_u/M_{u-1} belongs to B. (The proposition follows easily from this. Let ϕ be in A. Then there is a scalar c such that the map ϕ_w is c times the identity. Let $\theta = \phi - c1_A$. Then $\theta_w = 0$. Since θ_u is also 0 for all u with M_u/M_{u-1} belonging to B, it follows from the last part of Lemma 3.2 that $\theta \in J(A)$. Hence $A/J(A)$ has dimension 1.)

Now assume that $\theta_w = 0$, and let u be an integer such that M_u/M_{u-1} belongs to B. Let $\tau \in M$ be as in Lemma 3.1, set $\tau_u := \tau\prod_{i=u+1}^{m}(z - z_i)$, and set $\tau_w := \tau\prod_{i=w+1}^{m}(z - z_i)$. The lemma states that $\tau_u \in M_u\backslash M_{u-1}$ and $\tau_w \in M_w\backslash M_{w-1}$. Since $u \leq w$, we have

$$\begin{aligned}
\tau_u\theta &= \left(\tau_w \prod_{i=u+1}^{w} (z - z_i)\right)\theta \\
&= \tau_w\theta \prod_{i=u+1}^{w} (z - z_i), \quad \text{as } z \text{ is in the centre of } \mathrm{End}_{F\Sigma_n}(M), \\
&\in M_{w-1} \prod_{i=u+1}^{w} (z - z_i), \quad \text{as } \theta_w = 0 \text{ implies that } \tau_w\theta \in M_{w-1} \\
&= \left(M_{w-1} \prod_{i=u+1}^{w-1} (z - z_i)\right)(z - z_w) \\
&\subseteq M_u(z - z_w), \quad \text{using Lemma 2.1 repeatedly.}
\end{aligned}$$

Now M_u/M_{u-1} and M_w/M_{w-1} both belong to B. So $z_u = z_w$, since both scalars are equal to the image of z under the central character of B. Lemma 2.1 and the last inclusion displayed above then show that $\tau_u\theta \in M_{u-1}$. But $\tau_u \notin M_{u-1}$, as proved in Lemma 3.1, and $\mathrm{End}_{F\Sigma_n}(M_u/M_{u-1})$ is one-dimensional, by 13.17 of [2]. We conclude that $\theta_u = 0$, as required. □

We have now done all the work to prove the main result of this paper.

Theorem 3.4. *Assume that char $F \neq 2$. Let b be a block idempotent of $F\Sigma_{n-1}$. Then the $F\Sigma_{n-1}$-module $(S^\lambda{\downarrow}_{S_{n-1}})\, b$ is 0 or indecomposable. Let B be a block idempotent of $F\Sigma_{n+1}$. Then the $F\Sigma_{n+1}$-module $(S^\lambda{\uparrow}^{S_{n+1}})\, B$ is 0 or indecomposable.*

Proof. We know that $\mathcal{R}$ and $\mathcal{I}$ satisfty properties 1. and 2. of M. That they also satisfy property 3. is a consequence of Theorem 2.7. The result now follows from Proposition 3.3. □

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