REAL SUBPAIRS AND FROBENIUS-SCHUR INDICATORS OF CHARACTERS IN 2-BLOCKS

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ABSTRACT. Let B be a real 2-block of a finite group G. A defect couple of B is a certain pair (D, E) of 2-subgroups of G, such that D a defect group of B, and $D \leq E$. The block B is principal if $E = D$; otherwise $[E : D] = 2$. We show that (D, E) determines which B-subpairs are real.

The involution module of G arises from the conjugation action of G on its involutions. We outline how (D, E) influences the vertices of components of the involution module that belong to B.

These results allow us to enumerate the Frobenius-Schur indicators of the irreducible characters in B, when B has a dihedral defect group. The answer depends both on the decomposition matrix of B and on a defect couple of B. We also determine the vertices of the components of the involution module of B.

Key words and phrases. Block, Dihedral Defect Group, Frobenius-Schur Indicator, Extended Defect Group, Subpairs.

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1. Introduction

Throughout this paper G is a finite group. We adopt the standard notation and results for the representation theory of G, as expounded in [23]. In particular $\mathcal{O}$ is a complete discrete valuation ring of characteristic 0 with residue field $k := \mathcal{O}/J(\mathcal{O})$ of characteristic $p > 0$. Mostly, but not exclusively, $p = 2$. We assume that $\mathrm{frak}(\mathcal{O})$ and k are splitting fields for all subgroups of G. We write $\mathrm{N}(X) = \mathrm{N}_G(X)$ for the normalizer and $\mathrm{C}(X) = \mathrm{C}_G(X)$ for the centralizer of a subset X of a G-set. If n is a natural number, we use $\nu(n)$ for the highest power of 2 that divides n, and $\mathfrak{S}_n$ to denote the symmetric group of degree n.

Let B be a p-block of G. There is a corresponding primitive idempotent e_B of the centre $Z(kG)$ of the group algebra kG and a k-algebra map $\omega_B : Z(kG) \to k$ such that $\omega_B(e_B) = 1_k$. Attached to B is a set $\mathrm{Irr}(B)$ of irreducible characters and a set $\mathrm{IBr}(B)$ of irreducible Brauer characters of G. As is standard, $k(B) = |\,\mathrm{Irr}(B)|$ and $l(B) = |\,\mathrm{IBr}(B)|$.

The notion of vertex of a module, admissible block and induced block are as in [23]. Suppose that M is an indecomposable module for $H \leq G$ and that $\mathrm{C}(V) \leq H$, for some vertex V of M. Then M belongs to an admissible p-block b of H. *Green's vertex theorem* [14, (3.7a)] states that if L is an indecomposable kG-module such that M is a component of the restriction $L{\downarrow_H}$ then L belongs to b^G.

A *B-subsection* is a pair (x, b), where x is a p-element of G and b is a block of the centralizer $\mathrm{C}(x)$ of x in G such that $b^G = B$. If $\theta \in \mathrm{IBr}(b)$, we call (x, θ) a *column* of B. For $\chi \in \mathrm{Irr}(B)$ and y a p'-element in $\mathrm{C}(x)$ we have $\chi(xy) = \sum_{(x,b)} \sum_{\theta \in \mathrm{IBr}(b)} d^{(x)}_{\chi\theta}\theta(y)$, where $d^{(x)}_{\chi\theta}$ are algebraic integers called *generalized decomposition numbers*. The group G acts by conjugation on subsections and columns; $(x, \theta)^g = (x^g, \theta^g)$, for all $g \in G$. We say that (x, θ) is *real* if $(x, \theta)^g = (x^{-1}, \overline{\theta})$, for some $g \in G$. Now identify conjugate B-subsections and columns. Brauer [1] showed:

$$k(B) = \sum_{(x,b)} l(b), \quad \text{where } (x, b) \text{ ranges over the } B\text{-subsections.}$$

Lemma 1.1. *The number of real irreducible characters in B equals the number of real columns in B.*

Proof. Let $\chi \in \mathrm{Irr}(B)$ and let (x, θ) be a column of B. Then

$$d^{(x)}_{\chi\theta} = \sum \langle \chi \downarrow_{\mathrm{C}_G(x)}, \psi \rangle \frac{\psi(x)}{\psi(1)} d_{\psi\theta},$$

where ψ runs over the irreducible characters of $\mathrm{C}_G(x)$. Thus

$$d^{(x^{-1})}_{\overline{\chi}\overline{\theta}} = \sum \langle \overline{\chi} \downarrow_{\mathrm{C}_G(x)}, \overline{\psi}\rangle \frac{\overline{\psi}(x^{-1})}{\overline{\psi}(1)} d_{\overline{\psi}\overline{\theta}} = d^{(x)}_{\chi\theta}.$$

The generalized decomposition matrix $[d^{(x)}_{\chi\theta}]$ is nonsingular. It then follows from the previous paragraph and Brauer's permutation lemma that the number of $\chi \in \mathrm{Irr}(B)$ with $\chi = \overline{\chi}$ equals the number of Brauer subsections (x, θ) such that (x, θ) is conjugate to $(x^{-1}, \overline{\theta})$. $\square$

The *extended centralizer* of $g \in G$ is $\mathrm{C}^*(g) = \mathrm{C}^*_G(g) := \mathrm{N}(\{g, g^{-1}\})$. So $[\mathrm{C}^*(g) : \mathrm{C}(g)] \leq 2$. We use $\mathrm{cl}(g) = \mathrm{cl}_G(g)$ to denote the G-conjugacy class of g.

Now let $\mathrm{char}(k) = 2$ and let B be a real 2-block of G. As noted by R. Gow [12], there is a real conjugacy class $\mathcal{C}$ of G, necessarily 2-regular, so that $\mathcal{C}$ is contained in the support of e_B, and $\omega_B(\mathcal{C}^+) \neq 0$. Here $\mathcal{C}^+ = \sum_{c \in C} c$ in $\mathrm{Z}(kG)$. Any such $\mathcal{C}$ is called a *real defect class* of B.

Definition 1.2. *A* defect couple *of a real* 2*-block* B *is a pair* (D, E)*, where* E *is a Sylow* 2*-subgroup of* $\mathrm{C}^*(c)$*,* $D = \mathrm{C}_E(c)$ *is a Sylow* 2*-subgroup of* $\mathrm{C}(c)$*, and* c *is an element of a real defect class of* B*.*

As D is a defect group of B, it is unique up to G-conjugacy. Gow [13] showed that E is also unique up to G-conjugacy. He referred to E as an *extended defect group* of B. In fact, the pair (D, E) is unique up to G-conjugacy. This can be deduced from Proposition 14 in [21].

Note that if β is an admissible real 2-block of $H \leq G$ with defect couple (D_1, E_1), then by [22, 2.1] there is a defect group D_2 of β^G such that $D_1 \leq D_2$ and (D_2, D_2E_1) is a defect couple of β^G.

The Frobenius-Schur (FS)-indicator $\epsilon(\chi)$ of a character χ of G is the average value of $\chi(g^2)$, as g ranges over G. If χ is irreducible, it is known that $\epsilon(\chi) = 0, +1, -1$, depending on whether χ is non-real, afforded by a real representation, or real-valued but not afforded by a real representation, respectively. The following is Corollary 2.5 in [22]:

Lemma 1.3. *Let* (D, E) *be a defect couple of* B *and let* $x \in D$*. Then*

$$\sum_{\chi \in \mathrm{Irr}(B)} \epsilon(\chi)\chi(x) \geq 0,$$

with > 0 *if and only if* $E = D\langle e\rangle$*, where* e^2 *is* G*-conjugate to* x*.*

The columns of B, weighted by FS-indicators, are locally determined, as Brauer has shown:

Lemma 1.4. *Let (x,θ) be a column of B, with $\theta \in \mathrm{Ibr}(b)$. Then*

$$\sum_{\chi\in\mathrm{Irr}(B)} \epsilon(\chi)d^{(x)}_{\chi,\theta} = \sum_{\psi\in\mathrm{Irr}(b)} \epsilon(\psi)d^{(x)}_{\psi,\theta}.$$

The value of these sums is non-negative if $l(b) = 1$.

Proof. The equality is Theorem (4A) of [2]. This holds even if B is a p-block, with p odd. Suppose that $l(b) = 1$. Then

$$\sum_{\psi\in\mathrm{Irr}(b)} \epsilon(\psi)d^{(x)}_{\psi,\theta}\theta(1) = \sum_{\psi\in\mathrm{Irr}(b)} \epsilon(\psi)\psi(x) \geq 0, \quad \text{by Lemma 1.3.}$$

□

The following elementary lemma is useful for computing these sums:

Lemma 1.5. *Let m be an odd integer. Then for each integer $d > 0$ with $|m| < 2^d$, there is a* unique *d-tuple $(\varepsilon_0, \ldots, \varepsilon_{d-1})$ of signs such that*

$$\sum_{j=0}^{d-1} \varepsilon_j 2^j = m.$$

We will use $[M : I]$ to denote the multiplicity with which an irreducible module I occurs in a composition series of a module M. Then $[M : I]$ is well-defined, by the Krull-Schmidt theorem.

Lemma 1.6. *Let M be a self-dual non-semisimple module and let I be a self-dual irreducible submodule of M. Then $[M : I] \geq 2$.*

Proof. We have $I \subseteq \mathrm{soc}(M) \subseteq \mathrm{rad}(M)$ and, by duality $I \subseteq M/\mathrm{rad}(M)$. □

The set of involutions in G, and the identity element, form the G-set

$$\mathfrak{I} = \mathfrak{I}_G := \{g \in G \mid g^2 = 1_G\},$$

under conjugation. Suppose that $\mathrm{char}(k) = 2$. Then we call the permutation module $k\mathfrak{I}$ the *involution module* of G. The involution module of B is the sum $k\mathfrak{I}B$ of all submodules of $k\mathfrak{I}$ that belong to B. We showed in [21] that $k\mathfrak{I}B \neq 0$ if and only if B is *strongly real* i.e. B is real and has a defect couple $(D, D\langle t\rangle)$, for some $t \in \mathfrak{I}$.

If $N \trianglelefteq G$ and $X \subseteq G$, we use $\overline{X} = \{Nx \mid x \in X\}$ to denote the image of X in the quotient group $\overline{G} := G/N$.

Lemma 1.7. *Let $Q \trianglelefteq G$ be a 2-group and let (D, E) be a defect couple of B. Then there is a real 2-block $\overline{B}$ of $\overline{G}$, that is dominated by B, and that has defect couple $(\overline{D}, \overline{E})$.*

Proof. Write $\overline{x} \in k\overline{G}$ for the image of $x \in kG$, under the natural k-algebra projection $kG \to k\overline{G}$. Let $\overline{B}_1, \ldots, \overline{B}_r$ be the blocks of $\overline{G}$ that are dominated by B. The dual blocks $\overline{B}_i^o$ are also dominated by B.

Let $\mathcal{C}$ be a real defect class of B. Then $\overline{\mathcal{C}}$ is a conjugacy class of $\overline{G}$ and $\overline{\mathcal{C}}^+$ appears in the support of $\overline{e_B} = \sum e_{\overline{B}_i}$, but not in the support of $e_{\overline{B}_i} + e^o_{\overline{B}_i}$, for any i. It follows that we may choose i such that $\overline{B} := \overline{B}_i$ is real and $\overline{\mathcal{C}}^+$ appears in the support of $e_{\overline{B}}$. As moreover $\omega_{\overline{B}}(\overline{\mathcal{C}}^+) = \omega_B(\mathcal{C}^+) \neq 0$, we deduce that $\overline{\mathcal{C}}$ is a real defect class of $\overline{B}$. Choose $c \in \mathcal{C}$ such that E is a Sylow 2-subgroup of $\mathrm{C}^*(c)$ and $D = E \cap \mathrm{C}(c)$. Then c is the unique element of odd order in $\overline{c}$. So $\overline{\mathrm{C}^*_G(c)} = \mathrm{C}^*_{\overline{G}}(\overline{c})$. It follows that $(\overline{D}, \overline{E})$ is a defect couple of $\overline{B}$. ☐

In the context of the lemma, it is known that if $G = Q\,\mathrm{C}(Q)$, then $\overline{B}$ is the only block of $\overline{G}$ dominated by B.

Suppose that $N \trianglelefteq G$ and b is a block of N. Then $\mathrm{N}(b)$ denotes the stabilizer of b in G, and $\mathrm{N}^*(b)$ the stabilizer of $\{b, b^o\}$ in G, all under the conjugation action of G on the blocks of N. So $\mathrm{N}(b) \leq \mathrm{N}^*(b)$ and $[\mathrm{N}^*(b) : \mathrm{N}(b)] \leq 2$. We call $\mathrm{N}^*(b)$ the *extended stabilizer* of b.

Lemma 1.8. *Let $N \trianglelefteq G$ and let b be an admissible 2-block of N such that b^G is real and $G = \mathrm{N}^*(b)$. Let (D, E) be a defect couple of b^G.*

(i) *If $b = b^o$ then $(D \cap N, E \cap N)$ is a defect couple of b.*
(ii) *If $b \neq b^o$, then $G = E\,\mathrm{N}(b)$ and $D \cap N$ is a defect group of b.*
(iii) *Let $F \leq E$ with $E = DF$. Then $(D \cap NF, E \cap NF)$ is contained in a defect couple of the real block b^{NF}.*

Proof. Write $e_b = \sum_{n \in N} \beta_n n$, where $\beta_n \in k$, for each n. As b is admissible, each real defect class $\mathcal{C}$ of b^G is contained in N. Let $c \in \mathcal{C}$ have defect couple (D, E) in G.

Assume the hypothesis of (i). Then b is G-invariant and hence e_b is the primitive idempotent in $Z(kG)$ corresponding to b^G. Decompose $\mathcal{C} = \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_t$ into a union of conjugacy classes of N, where $c \in \mathcal{C}_1$. Now G permutes the $\mathcal{C}_i$ transitively among themselves. It follows that $\omega_b(\mathcal{C}_i^+) = \omega_b(\mathcal{C}_1^+)$, for all i. So $0 \neq \omega_{b^G}(\mathcal{C}^+) = t\omega_b(\mathcal{C}_1^+)$. In particular $\omega_b(\mathcal{C}_1^+) \neq 0$. So $\mathcal{C}_1$ is a real defect class of b. Then (i) follows from the fact that $E \cap N$ is a Sylow 2-subgroup of $\mathrm{C}_N(c)$ and $D \cap N = (E \cap N) \cap \mathrm{C}_N(c)$.

Suppose in addition the hypothesis of (iii). Now $F \leq \mathrm{C}^*(c)$ and $\mathcal{C}_1$ is a real class of N. It follows that $\mathcal{C}_1$ is a real conjugacy class of NF. But e_b is the block idempotent of b^{NF}. So $\mathcal{C}_1$ is a real defect class of b^{NF}. As $(D \cap NF, E \cap NF)$ is contained in a defect couple of c in NF, it is also contained in a defect couple of b^{NF}.

Assume the hypothesis of (ii). Then $[G : \mathrm{N}(b)] = 2$ and $e_b + e_b^o$ is the primitive idempotent in $Z(kG)$ corresponding to b^G. Regard e_b as the block idempotent of a non-real block of $\mathrm{N}(b)$. As c appears with nonzero multiplicity in $e_b + e_b^o$, we have $\beta_c + \beta_{c^{-1}} \neq 0$. But β is a class function of $\mathrm{N}(b)$. So c is not conjugate to c^{-1} in $\mathrm{N}(b)$, and in particular $G = E\,\mathrm{N}(b)$. It is a standard fact that $D \cap N$ is a defect group of b.

Suppose in addition the hypothesis of (iii). Then b and b^o are conjugate in NF. So b^{NF} is a real block. We may write $\mathcal{C} = (\mathcal{C}_1 \cup \ldots \cup \mathcal{C}_t) \cup (\mathcal{C}_1^o \cup \ldots \cup \mathcal{C}_t^o)$ into a union of non-real conjugacy classes of N, where $\mathcal{C}_1 \cup \ldots \cup \mathcal{C}_t$ is a single conjugacy class of $\mathrm{N}(b)$. As before $\omega_b(\mathcal{C}_i^+) = \omega_b(\mathcal{C}_1^+)$ and $\omega_b(\mathcal{C}_i^{o+}) = \omega_b(\mathcal{C}_1^{o+})$, for all i. Thus $0 \neq \omega_{b^G}(\mathcal{C}^+) = t\omega_b((\mathcal{C}_1 \cup \mathcal{C}_1^o)^+)$. Now $\mathcal{C}_1 \cup \mathcal{C}_1^o$ is a real conjugacy class of NF, and $e_b + e_b^o$ is the block idempotent of b^{NF}. We deduce that $\mathcal{C}_1 \cup \mathcal{C}_1^o$ is a real defect class of b^{NF}. As $(D \cap NF, E \cap NF)$ is contained in a defect couple of c in NF, it is contained in a defect couple of b^{NF}. $\square$

We now describe the rest of the paper. In Sections 2 and 3 our only assumptions are that $p = 2$, B is a real 2-block of G, and (D, E) is a defect couple of B. We prove a number of results on the Brauer category of B in Section 2. Our main result, Theorem 2.7, shows how (D, E) determines the reality of each B-subpair.

Suppose that T is a conjugacy class of involutions in G. Let kTB denote the sum of the components of the associated permutation module that belong to B. In Section 3 we show how (D, E) influences the vertices of the components of kTB: Theorem 3.2 gives a connection between these vertices and the centralizers in D of elements of $T \cap E$.

From Section 4 onwards, we assume that B is a real 2-block that has a dihedral defect group D. We first enumerate the possible isomorphism types of an extension E of D with $[E : D] = 2$. Table 1 describes the E-conjugacy classes in $E \backslash D$ for each of these types.

In Section 5, we determine, for each type of E, the real subpairs of B (Lemma 5.5), and the real irreducible characters in B that have positive height (Theorem 5.7).

Section 6 reviews known results on the Morita equivalence classes of 2-blocks with dihedral defect groups. In particular, Lemma 6.1 partially describes the generalized decomposition matrices.

In Sections 7 through 11 we determine the FS-indicators of the irreducible characters in B, and the vertices of the components of $k\mathfrak{I}B$, according to the isomorphism type of the extended defect group E, and the Morita equivalence class of B.

Our results for blocks with dihedral defect group are summarized in Table 2. We believe that the methods of this paper could be applied to all blocks with tame representation type (i.e. dihedral, semidihedral or quaternion defect group) in order to determine the Frobenius-Schur indicators of the irreducible characters and the vertices of components of the involution module.

2. Real subpairs

The purpose of this section is to show how a defect couple of a real 2-block determines the reality of its subpairs. We assume that B is a real 2-block of G, and let (D, E) be a defect couple of B.

A *B-subpair* consists of (Q, b_Q) where Q is a 2-subgroup of G and b_Q is a 2-block of $Q\,\mathrm{C}(Q)$ such that $b_Q^G = B$. By abuse of notation $\mathrm{N}(b_Q)$ and $\mathrm{N}^*(b_Q)$ denote the stabilizer and extended stabilizer of b_Q in $\mathrm{N}(Q)$. A B-subpair of the form (D, b_D) is known as a *Sylow B-subpair* (or a *root* of B). Brauer showed that all Sylow B-subpairs (D, b_D) are $\mathrm{N}(D)$-conjugate, and also that $[\mathrm{N}(b_D) : \mathrm{C}(D)D]$ is odd.

The set of B-subpairs is a poset. Containment $\leq$ is generated by normal containment: $(Q, b_Q) \trianglelefteq (R, b_R)$ if $R \leq \mathrm{N}(b_Q)$ and $b_R^{R\,\mathrm{C}(Q)} = b_Q^{R\,\mathrm{C}(Q)}$. Given a B-subpair (Q, b_Q), there is a Sylow B-subpair (D, b_D) such that $(Q, b_Q) \leq (D, b_D)$. For each Sylow B-subpair (D, b_D) and $Q \leq D$, there is a unique B-subpair (Q, b_Q) such that $(Q, b_Q) \leq (D, b_D)$. We will refer to this as the *uniqueness* property of subpairs.

Lemma 2.1. *Suppose that (Q, b_Q) and (R, b_R) are B-subpairs such that b_R is real and $(Q, b_Q) \leq (R, b_R)$. Then b_Q is real.*

Proof. We have $(Q, b_Q) \leq (R, b_R)$, and $(Q, b_Q^o) \leq (R, b_R^o) = (R, b_R)$. So $b_Q^o = b_Q$, by the uniqueness property of subpairs. □

We now examine the relationship between the extended defect groups of B and the extended stabilizers of its Brauer subpairs.

Lemma 2.2. *The following hold:*

(i) *If (D, b_D) is a Sylow B-subpair, then there is a defect couple (D, E) of B such that $b_D^{E\,\mathrm{C}(D)}$ is real with defect couple (D, E).*
(ii) *If (D, E) is a defect couple of B then there is a Sylow B-subpair (D, b_D) such that $b_D^{E\,\mathrm{C}(D)}$ is real with defect couple (D, E).*

Proof. Suppose first that b_D is real, with defect couple (D, E). Then (D, E) is a defect couple of B. Moreover, $E\,\mathrm{C}(D) = D\,\mathrm{C}(D)$. Suppose then that b_D is not real. As (D, b_D) and (D, b_D^o) are Sylow B-subpair, they are conjugate in $\mathrm{N}(D)$, and hence in $\mathrm{N}^*(b_D)$. Set $b = b_D^{\mathrm{N}^*(b_D)}$. Then each defect couple (D, E) of b is a defect couple of B. Lemma 1.8 implies that $b_D^{E\,\mathrm{C}(D)}$ is real with defect couple (D, E).

Now let (D, E) be a defect couple for B, and let (D, b_D) be any Sylow B-subpair. By part (i) there exists a defect couple (D, E_1) for B such that $b_D^{E\,\mathrm{C}(D)}$ is real with defect couple (D, E_1). But the defect couples of B are conjugate in G. So there exists $x \in G$ such that $(D, E) = (D, E_1)^x$. Now replace b_D^x by b_D. Then (D, b_D) a Sylow B-subpair such that $b_D^{E\,\mathrm{C}(D)}$ is real with defect couple (D, E). $\square$

A *B-subtriple* consists of (Q, b_Q, R) where:

- (Q, b_Q) is a B-subpair;
- $R \leq \mathrm{N}^*(b_Q)$ and $b_Q^* := b_Q^{R\,\mathrm{C}(Q)}$ is real;
- R is an extended defect group of b_Q^*.

We say that this is a *Sylow B-subtriple* if Q is a defect group of B. For two B-subtriples an inclusion $(Q, b_Q, R) \leq (S, b_S, T)$ occurs if $(Q, b_Q) \leq (S, b_S)$, in the sense of B-subpairs, and $T = RS$. We write $(Q, b_Q, R) \trianglelefteq (S, b_S, T)$ if in addition $Q \trianglelefteq S$. Note that if (Q, b_Q) is a real B-subpair and if R is an extended defect group of b_Q, then $b_Q^* = b_Q$ and (Q, b_Q, R) is a B-subtriple.

Lemma 2.3. *Suppose that (D, b_D, E) is a Sylow B-subtriple and that $Q \leq D$. Then $b_Q^o = b_{Q^e}^{e^{-1}}$, for all $e \in E$ such that $E = D\langle e\rangle$.*

Proof. We have $D^e = D$ and $b_D^e = b_D^o$. So $(Q^e, b_Q^{eo}) \leq (D^e, b_D^{eo}) = (D, b_D)$. Then $b_{Q^e} = b_Q^{eo}$, by the uniqueness property of subpairs. The lemma follows from this. $\square$

Lemma 2.4. *All Sylow B-subtriples are conjugate in G.*

Proof. Let (D_1, b_1, E_1) and (D_2, b_2, E_2) be Sylow B-subtriples. Then (D_i, b_i) are Sylow B-subpairs. So $(D_2, b_2)^x = (D_1, b_1)$, for some $x \in G$. Now (D_1, b_1, E_2^x) is a Sylow B-subtriple. In particular (D_1, E_2^x) is a defect couple of b_1^* regarded as a block of $\mathrm{N}^*(b_1)$. But (D_1, E_1) is another defect couple of this block. So there exists $y \in \mathrm{N}^*(b_1)$ such that $(D_1, E_2^x)^y = (D_1, E_1)$. Now $b_1^y = b_1$ or b_1^o. In the former case set $z := 1$. Suppose we have the latter case. Then there exists $z \in E_1\,\mathrm{C}(D_1)$ such that $(b_1^o)^z = b_1$. Then $(D_2, b_2, E_2)^g = (D_1, b_1, E_1)$, with $g = xyz$. $\square$

Let (Q, b_Q, R) be a B-subtriple. So b_Q^* is a real block of $R\,\mathrm{C}(Q)$. We can choose a defect couple (S, T) of b_Q^*, regarded as a block of $\mathrm{N}^*(b_Q)$, such that $T = RS$. Then S is a defect group of b_Q, regarded as a block of $\mathrm{N}(b_Q)$. Lemma 2.2 implies that there is a Sylow b_Q^*-subpair (S, b_S) in $N^*(b_Q)$ such that $b_S^{T\,\mathrm{C}(S)}$ is a real block with defect couple (S, T). Moreover, $b_S^{S\,\mathrm{C}(Q)}$ covers b_Q or b_Q^o, so by switching to b_S^o, if necessary, we may assume that $(Q, b_Q) \leq (S, b_S)$. It follows that (S, b_S, T) is a

B-subtriple and $(Q, b_Q, R) \leq (S, b_S, T)$. We call any such B-subtriple a *normalizer-subtriple* of (Q, b_Q, R).

Set $(Q_1, b_1, R_1) := (Q, b_Q, R)$ and inductively choose $(Q_{i+1}, b_{i+1}, R_{i+1})$ to be a normalizer subtriple of (Q_i, b_i, R_i), for $i = 1, 2, \ldots$. Since (Q_{i+1}, b_{i+1}) is a normaizer-subpair of (Q_i, b_i), it follows that the sequence $(Q_1, b_1, R_1) \trianglelefteq (Q_2, b_2, R_2) \trianglelefteq (Q_3, b_3, R_3) \trianglelefteq \ldots$ terminates at a Sylow B-subtriple (D, b_D, E), where (D, E) is a defect couple of B. In this case $E = R_1 D$. We say that Q is *really well-placed in* D with respect to (D, b_D, E) if such a sequence exists.

Lemma 2.5. *Let (D, b_D, E) be a Sylow B-subtriple and let (Q, b_Q, R) be any B-subtriple such that $(Q, b_Q) \leq (D, b_D)$. Then there exists $g \in G$ such that Q^g is really well-placed in D with respect to (D, b_D, E) and $(Q, b_Q, R)^g \leq (D, b_D, E)$.*

Proof. By the discussion above, there is a Sylow B-subtriple (D_1, b_1, E_1) that contains (Q, b_Q, R), such that Q is *really well-placed in* D_1 with respect to (D_1, b_1, E_1). By Lemma 2.4 there exists $g \in G$ such that $(D_1, b_1, E_1)^g = (D, b_D, E)$. The lemma follows from this. □

Our next Lemma is concerned with strongly real B-subpairs.

Lemma 2.6. *Let (D, b_D, E) be a Sylow B-subtriple and let $(Q, b_Q) \leq (D, b_D)$ be such that $E = D\,\mathrm{C}_E(Q)$. Then b_Q is real and has a defect couple containing $(Q\,\mathrm{C}_D(Q), Q\,\mathrm{C}_E(Q))$. In particular, if $E = D\langle t\rangle$, for some $t \in \mathfrak{I}_{\mathrm{C}(Q)}$, then b_Q is strongly real.*

Proof. If $Q = D$, the result follows from Lemma 1.8. So we may assume that $Q < D$. Set $b_Q, D_0 := Q$ and inductively $D_i := \mathrm{N}_D(D_{i-1})$ and $E_i := \mathrm{N}_E(D_{i-1})$, for $i > 0$. Then $[E_i : D_i] \leq 2$ and $E = DE_i$. For each positive i there is a unique subpair $(Q, b_Q) \leq (D_i, b_i) \leq (D, b)$. Let t be the smallest positive integer such that $D_t = D$ and $E_t = E$.

We use downwards induction on i to prove that $b_i^* := b_i^{E_i\,\mathrm{C}(D_i)}$ is real with a defect couple containing (D_i, E_i). The base case $i = t$ follows from the hypothesis. Assume the result for $i+1$. Set $b_{i+1}^{**} := b_{i+1}^{E_{i+1}\mathrm{C}(D_i)}$. Then $b_{i+1}^{**} = (b_{i+1}^*)^{E_{i+1}\mathrm{C}(D_i)}$. So b_{i+1}^{**} is real with a defect couple containing (D_{i+1}, E_{i+1}). Part (iii) of Lemma 1.8, applied with $N = D_i\,\mathrm{C}(D_i)$, $b = b_i$, $G = E_{i+1}\,\mathrm{C}(D_i)$ and $F = E_i$, gives the inductive step.

The previous paragraph shows that b_1^* is a real block of $\mathrm{N}_E(Q)\,\mathrm{C}(Q)$ that covers the block b_Q of $Q\,\mathrm{C}(Q)$. We apply part (iii) of Lemma 1.8, with $N = Q\,\mathrm{C}(Q)$, $b = b_Q$, $G = \mathrm{N}_E(Q)\,\mathrm{C}(Q)$ and $F = \mathrm{C}_E(Q)$. Then $NF = Q\,\mathrm{C}(Q)$. So b_Q is a real block with a defect couple containing $(Q\,\mathrm{C}_D(Q), Q\,\mathrm{C}_E(Q))$. □

We use $\sim$ for G-conjugacy. Our main result on real subpairs is:

Theorem 2.7. *Let (D, b_D, E) be a Sylow B-subtriple. Set*

$$\mathcal{R}(D, b_D, E) := \{(Q, b_Q) \leq (D, b_D) \mid E = D\,\mathrm{C}_E(Q)\}.$$

Then a B-subpair (R, b_R) is real if and only if $(R, b_R) \sim (Q, b_Q)$, for some $(Q, b_Q) \in \mathcal{R}(D, b_D, E)$. Moreover, (R, b_R) is strongly real if and only if $(R, b_R) \sim (Q, b_Q)$, where $E = D\langle t\rangle$, for some $t \in \mathfrak{I}_{\mathrm{C}(Q)}$.

Proof. Suppose first that $(Q, b_Q) \in \mathcal{R}(D, b_D, E)$. Then by Lemma 2.2 we can choose $e \in \mathrm{C}_E(Q)$ such that $b_D^{oe} = b_D$. Then $(Q, b_Q^o) = (Q, b_Q^o)^e \leq (D, b_D^o)^e = (D, b_D)$. But (Q, b_Q) is the unique B-subpair involving Q and contained in (D, b_D). So $(Q, b_Q^o) = (Q, b_Q)$, whence b_Q is real. Now suppose that $x \in G$ and $(Q, b_Q)^x \leq (D, b_D)$. Then $(b_Q^x)^o = (b_Q^o)^x = b_Q^x$. So $(Q, b_Q)^x$ is a real B-subpair contained in (D, b_D). If in addition $E = D\langle t\rangle$, for some $t \in \mathfrak{I}_{\mathrm{C}(Q)}$, then b_Q is strongly real, by Lemma 2.6. This completes the 'if' part of the theorem.

The 'only if' statement follows from Lemma 2.5. □

3. The involution module

In this section we survey known facts about the involution module $k\mathfrak{I}$ of G, and prove a number of new results about the vertices of its components. As in the previous section, B is a real 2-block of G that has defect couple (D, E). Fix a conjugacy class $T \subseteq \mathfrak{I}$ of G.

Suppose that I is an irreducible B-module, with Brauer character θ. The projective character of G associated with θ is $\Phi = \sum_{\chi \in \mathrm{Irr}(B)} d_{\chi\theta}\,\chi$. Here the $d_{\chi\theta}$ are non-negative integers called *decomposition numbers*. Lemma 1 of [24] implies that

$$\epsilon(\Phi) = [k\mathfrak{I} : I]. \tag{1}$$

Now suppose that I is also self-dual and projective. Then $D = \langle 1_G\rangle$ and Φ is the unique irreducible character in B. So $[k\mathfrak{I} : I] = \epsilon(\Phi) = +1$. Conversely, by [20], each projective component of $k\mathfrak{I}$ is self-dual and irreducible and belongs to a real 2-block with a trivial defect group.

Let M be a component of kTB and let V be a vertex of M. As M has a trivial vertex, the Green correspondent P of M with respect to $(G, V, \mathrm{N}(V))$ is a component of $k\,\mathrm{C}_T(V)$. In particular, P is projective as $\mathrm{N}(V)/V$-module. Clifford theory shows that $P{\downarrow_{V\,\mathrm{C}(V)}} = m \sum R^n$, where $m > 0$, R is indecomposable with vertex V and n ranges over a set of representatives for the cosets of the stabilizer of R in $\mathrm{N}(V)$. Moreover, R is projective as $V\mathrm{C}(V)/V$-module, and is a component of $k\,\mathrm{C}_T(V){\downarrow_{V\,\mathrm{C}(V)}}$. Let b_V be the block of $V\mathrm{C}(V)$ that contains R. Then (V, b_V) is a B-subpair and P belongs to $b_V^{\mathrm{N}(V)}$. Given V, R is uniquely determined up to $\mathrm{N}(V)$-conjugacy. We call (V, b_V) a *vertex B-subpair* and R a b_V*-root* of M. These notions are set out in a more general context in [17].

Lemma 3.1. *Suppose that D is non-trivial and (V, b_V) is a vertex B-subpair of a component of kTB. Then there is an involution $v \in V$, and a B subsection (v, b), such that some component of $k\,\mathrm{C}_T(v)b$ has vertex b-subpair (V, b_V).*

Proof. The hypothesis on D, and the discussion prior to the lemma, implies that $V \neq \{1_G\}$. Choose an involution $v \in Z(V)$. Then $V\mathrm{C}(V) \leq \mathrm{C}(v)$. Let M be a component of kTB that has vertex B-subpair (V, b_V). Then some component of $M{\downarrow_{\mathrm{C}(v)}}$ has vertex subpair (V, b_V). Green's vertex theorem implies that this module belongs to a block b such that (v, b) is a B-subsection. □

Our main result in this section sharpens Theorem 1.5 of [22]:

Theorem 3.2. *The following hold:*

(i) *If $kTB \neq 0$ and M is a component of kTB, then there exists $t \in T$ such that $E = D\langle t\rangle$ and $\mathrm{C}_D(t)$ contains a vertex of M.*
(ii) *Suppose that $E = D\langle t\rangle$, with $t \in T$, but $\mathrm{C}_D(t) \not\leq_G \mathrm{C}_D(s)$ for any $s \in T \cap Dt$. Then some component of kTB has vertex $\mathrm{C}_D(t)$.*

Proof of (i). For both parts we use induction on $|D|$. The base case $|D| = 1$, is dealt with by Theorem 19 of [21]. So we may assume that $|D| > 1$. Let V be a vertex of M and set $H := V\,\mathrm{C}(V)$. Then $V \neq \{1_G\}$. Let (V, b) be a vertex B-subpair and let R be a b-root of M. So R is a component of $k\,\mathrm{C}_T(V)b$. Choose $t \in \mathrm{C}_T(V)$ such that R is a component of $k\,\mathrm{cl}_H(t)$.

Set $\overline{H} := H/Z$, where $Z \leq \mathrm{Z}(V)$ has order 2. By Lemma 1.7, there is a unique block $\overline{b}$ of $\overline{H}$ that is dominated by b. Now $\mathrm{N}_H(\overline{t})$ is the preimage of $\mathrm{C}_{\overline{H}}(\overline{t})$ in H and $[\mathrm{N}_H(\overline{t}) : \mathrm{C}_H(t)] \leq 2$. So either $k_{\mathrm{N}_H(\overline{t})}{\uparrow^H} = k\,\mathrm{cl}_H(t)$ or there is a short exact sequence of kH-modules

$$0 \to k_{\mathrm{N}_H(\overline{t})}{\uparrow^H} \to k\,\mathrm{cl}_H(t) \to k_{\mathrm{N}_H(\overline{t})}{\uparrow^H} \to 0.$$

In any event each composition factor of R is a composition factor of $k\,\mathrm{cl}_{\overline{H}}(\overline{t})\overline{b}$. As $\overline{b}$ has a smaller defect group than B, the inductive hypothesis implies that $\overline{b}$ has a defect couple of the form $(\overline{D_1}, \overline{D_1}\langle\overline{t}\rangle)$. Lemma 1.7 now shows that b has a defect couple $(D_1, D_1\langle t\rangle)$. As $b^G = B$, we may assume that D is chosen so that $D_1 \leq D$ and $(D, D\langle t\rangle)$ is a defect couple of B. Thus also $V \leq \mathrm{C}_D(t)$. □

Proof of (ii). Set $V := \mathrm{C}_D(t)$ and $H := V\,\mathrm{C}(V)$. Let $\mathrm{Br}_V\colon \mathrm{Z}(kG) \to \mathrm{Z}(k\,\mathrm{C}(V))$ be the Brauer homomorphism with respect to V. Then $\mathrm{Br}_V(e_B) = \sum e_b$, where b ranges over the 2-blocks of H such that

$b^G = B$. Note that $(b^o)^G = b^G$, as B is real. Choose c in a real defect class of B such that $D\langle t\rangle$ is a Sylow 2-subgroup of $\mathrm{C}^*(c)$ and $c^t = c^{-1}$. In particular $c, t \in H$. Set $\mathcal{C} := \mathrm{cl}_H(c)$. Then $\mathcal{C}^+$ is a real conjugacy class of H that appears in $\mathrm{Br}_V(e_B)$. As $e_b + e_b^o$ is supported on the non-real classes of H, for each block b of H, it follows that there exists a real block b of H such that $\mathcal{C}^+$ appears in e_b and $b^G = B$. Theorem 3.3 of [19] implies that b has a defect couple of the form $(D_1, D_1\langle t\rangle)$.

Lemma 1.7 implies that b dominates a real block $\overline{b}$ of $\overline{H} := H/V$ such that $\overline{b}$ has defect couple $(\overline{D_1}, \overline{D_1}\langle\overline{t}\rangle)$. As $|\overline{D_1}| < |D_1| \leq |D|$, the inductive hypothesis implies that $k\mathcal{C}l_{\overline{H}}(\overline{t})\overline{b} \neq 0$. The inflation of this module to H is a quotient module of $k\mathcal{C}l_H(t)b$. So $k\mathcal{C}l_H(t)b \neq 0$.

Let M_1 be a component of $k\mathcal{C}l_H(t)b$. Then V is contained in each vertex of M_1. Let M be a component of kT such that M_1 is a component of $M{\downarrow_H}$. Then M belongs to B and V is contained in some vertex V_1 of M. Now from part (i) of this theorem $V_1 \leq_G \mathrm{C}_D(s)$, for some $s \in T \cap Dt$. Since $\mathrm{C}_D(t) \leq V_1 \leq_G \mathrm{C}_D(s)$, it follows from the hypothesis that $V = V_1$ is a vertex of M. □

Corollary 3.3. *Suppose that t is an involution in $\mathrm{O}_2(G)$. Then every component of $k_{\mathrm{C}_G(t)}{\uparrow^G}$ belongs to the principal 2-block of G.*

Proof. Let B be a 2-block of G such that $k_{\mathrm{C}_G(t)}{\uparrow^G}B \neq 0$. Then each defect group D of B contains $\mathrm{O}_2(G)$. In particular $t \in D$. But then $D = D\langle t\rangle$ is an extended defect group of B. It follows from this that B has a real defect class in $\mathfrak{I}$. Since a defect class is necessarily 2-regular, the class $\{1_G\}$ is a defect class of B. As B is real, it follows from this that B is the principal 2-block of G. □

We take the opportunity to correct an error in the hypothesis of Lemma 2.11 of [22]. We also strengthen the conclusion. Fortunately, we only used that lemma in its correct form.

Lemma 3.4. *Suppose that $D \trianglelefteq G$ and $E = D \times \langle e\rangle$. Then there is a self-dual irreducible B-module I, such that I has vertex D and*

$$\bigoplus_{i\geq 0} I \otimes \mathrm{rad}^i(k\mathfrak{I}_{Z(D)})/\,\mathrm{rad}^{i+1}(k\mathfrak{I}_{Z(D)})$$

is the sum of all components of $k\mathfrak{I}B$ that have vertex D.

Proof. By hypothesis $e^2 = 1$ and $e \in \mathrm{C}(D)$. This is what we needed, and used, in the proof of Lemma 2.11 of [22]. So that proof gives all but the last statement.

We make a very general remark. Let H be a finite group, and let M be an indecomposable kH-module that has vertex V and source k_V. Set

fM as the Green correspondent of M with respect to $(H, V, \mathrm{N}_H(V))$. Suppose that $V \leq W \leq \mathrm{N}_H(V)$. Then $fM{\downarrow_W}$ is the sum of all components of $M{\downarrow_W}$ that have vertex V.

We can now prove the last statement of the lemma. Adopt the notation of Lemma 2.11 of [22]. We apply the previous paragraph with $H = G \wr \Sigma$, $M = B$, $V = \Delta D \times \Sigma$ and $W = \Delta G \times \Sigma$. Then by Lemma 2.9 of [22], $B(V){\downarrow_{\Delta G}}$ can be identified with the sum of all components of $k\mathfrak{I}B^{\mathrm{Fr}}$ that have vertex D. We get the equivalent result for $k\mathfrak{I}B$ using Lemma 2.8 of [22]. □

4. Extensions of dihedral 2-groups

We need to describe the degree 2-extensions of D, where D is a dihedral group of order 2^d. Fix a presentation $D = \langle s, t \mid s^{2^{d-1}}, t^2, (st)^2 \rangle$. We shall adopt the following notation for subgroups of D. The maximal cyclic subgroup $S := \langle s \rangle$ of D has order 2^{d-1} and the centre $\mathrm{Z}(D) = \langle s_1 \rangle$ has order 2. Also $X_1 := \langle t \rangle$ and $Y_1 := \langle st \rangle$ are subgroups of order 2. Set $s_i := s^{2^{d-1-i}}$, for $i = 1, \ldots, d-1$. Then $S_i := \langle s_i \rangle$ is a cyclic group of order 2^i. Moreover, $X_2 := \langle s_1, t \rangle$ and $Y_2 := \langle s_1, st \rangle$ are Klein-four groups, while $X_i := \langle s_{i-1}, t \rangle$ and $Y_i := \langle s_{i-1}, st \rangle$ are dihedral groups of order 2^i, for $i \geq 3$. In particular, X_{d-1} and Y_{d-1} are maximal subgroups of G that are dihedral of order 2^{d-1}.

Proposition 4.1. *There are* 4 *isomorphism classes of degree* 2 *extensions of* D_8. *There are* 5 *isomorphism classes of degree* 2 *extensions* $E = D\langle e \rangle$ *of* $D \cong D_{2^d}$, *for* $d \geq 4$. *These are:*

(a) $E = D \times \langle e \rangle$ *where* $e \in \mathrm{C}(D)$ *and* $e^2 = 1$;
(b) $E = D\langle e \rangle$ *where* $e \in \mathrm{C}(D)$ *and* $e^2 = s_1$;
(c) $E = D_{2^{d+1}}$, *a dihedral group of order* 2^{d+1};
(d) $E = SD_{2^{d+1}}$, *a semi-dihedral group of order* 2^{d+1};
(e) $E = D \rtimes \langle e \rangle$ *where* $e^2 = 1$, $s^e = s_1 s, t^e = t$;

Proof. Set $q = 2^{d-1}$. The additive group of the ring $\mathbb{Z}_q$ of integers modulo q is cyclic, while its group of units $\mathbb{U}_q$ is isomorphic to the direct product $\mathbb{Z}_2 \times \mathbb{Z}_{2^{d-3}}$. The $\mathbb{Z}_2$ factor is generated by -1 and the involutions in $\mathbb{U}_q$ are $1, -1, 2^{d-2} - 1, 2^{d-2} + 1$. Form the semidirect product $\mathbb{U}_q \ltimes \mathbb{Z}_q$. So $\mathbb{U}_q$ acts by multiplication on $\mathbb{Z}_q$. We define an action of $\mathbb{U}_q \ltimes \mathbb{Z}_q$ on D via

$$s^{(a,b)} := s^a, \quad t^{(a,b)} := s^{-b} t, \quad \text{for } a \in \mathbb{U}_q, b \in \mathbb{Z}_q.$$

In this way $\mathbb{U}_q \ltimes \mathbb{Z}_q$ can be identified with the automorphism group of D. The group of inner automorphisms of D is $I := \langle -1 \rangle \ltimes 2\mathbb{Z}_q$. So the outer automorphism group is isomorphic to $\mathbb{Z}_{2^{d-3}} \times \mathbb{Z}_2$. It follows that the image of E in the automorphism group of D is generated, modulo I by one of $(1,0), (-1,1), (2^{d-2}+1,0)$ or $(2^{d-2}-1,1)$. We set the image of e to be one of these four elements, in turn. The action of e on D determines the coset $e^2Z(D)$. As $Z(D)$ has order 2, we get two possibilities for E in each case.

Suppose that the image of e is $(1,0)$. Then $e^2 \in \mathrm{Z}(D)$ and E is the group of type (a) if $e^2 = 1$, or the group of type (b) if $e^2 = s_1$.

Suppose that the image of e is $(-1,1)$. Then $e^2 \in \{1, s_1\}$ and E is a dihedral group if $e^2 = 1$, or a semidihedral group, if $e^2 = s_1$.

From now on assume that $d \geq 4$. Suppose that the image of e is $(2^{d-2}+1,0)$. Then $e^2 = 1$ or s_1. In either case we obtain a group which is isomorphic to one of type (e).

We claim that the image of e cannot equal $(2^{d-2}-1,1)$. For suppose otherwise. Then $s^e = s_1s^{-1}$. So e inverts $s_2 \in \langle s^2 \rangle$. On the other hand $e^2 = s_2$ or s_2^{-1}. So e centralizes s_2. This contradiction proves our claim. □

Corollary 4.2. *$E\!:\!D$ is non-split if and only if E is semi-dihedral. In all cases E/D' splits over D/D'.*

Proof. The first statement follows from the third column of Table 1, below. For $E \backslash D$ contains no involution if and only if E is of type (d). The second statement follows from the fact that if E is semi-dihedral then $D/D' \cong \mathbb{Z}_2^2$ and $E/D' \cong D_8$. □

Corollary 4.3. *Let B be a 2-block with a dihedral defect group. Then all real height zero irreducible characters in B have FS-indicator $+1$.*

Proof. This follows from Theorem 5.6 of [12] and the Corollary above. □

From now on, $\mathrm{char}(k) = 2$ and B is a real 2-block that has a defect group $D \cong D_{2^d}$. R. Brauer [3] showed that B has $2^{d-2}+3$ irreducible characters. Four of these, denoted $\chi_1, \chi_2, \chi_3, \chi_4$, have height 0. The remaining $2^{d-2}-1$ irreducible characters have height 1, and fall into $d-2$ disjoint families $F_0, \ldots, F_{d-3}$. The family F_i consists of 2^i irreducible characters, all of whom are 2-conjugate (i.e. conjugate via Galois automorphisms that fix all 2-power roots of unity).

Choose $\chi^{(j)} \in F_j$, and set $\epsilon^{(j)} = \epsilon(\chi^{(j)})$, for $j = 0, \ldots, d-3$. Note that $\epsilon^{(j)}$ does not depend on the choice of $\chi^{(j)}$ in F_j, as Galois conjugation preserves FS-indicators. Also set $\epsilon_i = \epsilon(\chi_i)$, for $i = 1, 2, 3, 4$.

We also fix a defect couple (D, E) of B. If B is principal, then $E = D$ and we abuse notation by saying that E is of type (a). Otherwise $[E : D] = 2$ and E is of type (a),(b),(c),(d) or (e), as in Proposition 4.1.

We give representatives x for the E conjugacy classes in $E\backslash D$ in the table below. We also indicate when x is an involution:

E	$x \in E\backslash D$	$o(x) = 2$	$\mathrm{C}_D(x)$
(a) or (b)	e	(a)	D
	s_1e	(a)	D
	$s^ie, {\scriptstyle 1\leq i\leq 2^{d-2}-1}$	s_2e if (b)	S
	te	(a)	X_2
	ste	(a)	Y_2
(c) or (d)	$s^ite, {\scriptstyle 0\leq i\leq 2^{d-2}-1}$	never	S
	e	(c)	S_1
(e)	e	(e)	X_{d-1}
	s_2e	never	Y_{d-1}
	$s^ie, {\scriptstyle 1\leq i\leq 2^{d-3}-1}$	never	S_{d-2}
	te	(e)	X_2
	ss_2te	never	Y_2

Table 1

5. Local structure and Real Characters

We first enumerate the real subpairs and the real irreducible characters of height 1, in a 2-block with dihedral defect group. Fix a sylow B-subpair (D, b_D), such that $E \leq \mathrm{N}^*(b_D)$. So (D, E) is a defect couple of $b_D^{E\,\mathrm{C}(D)}$.

For each $Q \leq D$, there is a unique subpair $(Q, b_Q) \leq (D, b_D)$. The notation b_Q is compatible with that used in [3]. We use $\sim$ or $\sim_G$ to denote G-conjugacy of subpairs.

Lemma 5.1. *Let $Q \leq D$, $Q \not\leq S$ with $|Q| = 2^i$. If $Q \leq X_{d-1}$, then $(Q, b_Q) \sim_D (X_i, b_{X_i})$. If $Q \leq Y_{d-1}$, then $(Q, b_Q) \sim_D (Y_i, b_{Y_i})$. Along with these, the following generate all G-conjugacies among the B-subpairs contained in (D, b_D):*

(i) *if $N(b_{X_2})/\,\mathrm{C}(X_2) \cong \mathfrak{S}_3$ then $(\langle t\rangle, b_{\langle t\rangle}) \sim (S_1, b_{S_1})$.*
(ii) *if $N(b_{Y_2})/\,\mathrm{C}(Y_2) \cong \mathfrak{S}_3$ then $(\langle st\rangle, b_{\langle st\rangle}) \sim (S_1, b_{S_1})$.*

Following Brauer, case (aa) is the simultaneous occurrence of (i) and (ii); case (ab) is the occurrence of (i) but not (ii); case (ba) is the occurrence of (ii) but not (i); case (bb) is the occurrence of neither (i) nor (ii). Then B has 3, 2 or 1 irreducible modules according as case (aa), (ab) (or (ba)) or (bb) occurs. Moreover, B is a nilpotent block in case (bb).

Lemma 5.2. *If E is of type (c) or (d) then $l(B) \neq 2$.*

Proof. There exists $f \in E \backslash D$ such that $X_2^f = Y_2$. So $\mathrm{C}(X_2)^f = \mathrm{C}(Y_2)$. But $(X_2, b_{X_2})^f = (Y_2, b_{Y_2}^o)$, by Lemma 2.3. Thus $\mathrm{N}(b_{X_2})^f = \mathrm{N}(b_{Y_2}^o) = \mathrm{N}(b_{Y_2})$. The conclusion now follows, as $N(b_{X_2})/\mathrm{C}(X_2) \cong N(b_{Y_2})/\mathrm{C}(Y_2)$. □

The following three lemmas can be proved by careful applications of Theorem 2.7, using the information in Table 1. We omit the proofs.

Lemma 5.3. *(D, b_D) is real if and only if E is of type (a) or (b).*

Lemma 5.4. *(S_1, b_{S_1}) is real. Moreover, it is strongly real if and only if E is not of type (d).*

Lemma 5.5. *Suppose that $1 < Q < D$. Then (Q, b_Q) is real if and only if one of the following holds:*

- *E is of type (a) or (b).*
- *E is of type (c) or (d) and $(Q, b_Q) \sim (S_i, b_{Y_i})$ for some i.*
- *E is of type (e) and $(Q, b_Q) \neq (S, b_S)$.*

Of these, (Q, b_Q) is strongly real in the following cases:

- *E is of type (a).*
- *E is of type (b) and $(Q, b_Q) \sim (S_i, b_{S_i})$, for some i.*
- *E is of type (c) and $(Q, b_Q) \sim (S_1, b_{S_1})$.*
- *E is of type (e) and $(Q, b_Q) \sim (S_i, b_{S_i})$ or (X_i, b_{X_i}) for some i.*

For $d \in D$ we set $b_d := b_{\langle d \rangle}$. So $(\langle d \rangle, b_d) \leq (D, b_D)$ is a B-subsection. According to [3], if $d \neq 1$ then b_d has a unique irreducible Brauer character θ. We use the notation $d_{\chi\theta}^{(d)}$ for the generalized decomposition number associated with $\chi \in \mathrm{Irr}(B)$ i.e. we supress the dependence of θ on d. The associated B-column is (d, θ).

Lemma 5.6. *The following generate all D-conjugacies among B-subsections:*

- *$(d, b_d) \sim (d^{-1}, b_d)$, for all $d \in D$.*
- *if $d \in X_{n-1} \backslash S_1$ then $(d, b_d) \sim (t, b_t)$.*
- *if $d \in Y_{n-1} \backslash S_1$ then $(d, b_d) \sim (st, b_{st})$.*

Together with D-conjugacies, the following generate all G-conjugacies among B-subsections:

(i) *if $N(b_{X_2})/\mathrm{C}(X_2) \cong \mathfrak{S}_3$ then $(t, b_t) \sim (s_1, b_{s_1})$.*
(ii) *if $N(b_{Y_2})/\mathrm{C}(Y_2) \cong \mathfrak{S}_3$ then $(st, b_{st}) \sim (s_1, b_{s_1})$.*

Proof. Suppose that $d \in S \backslash S_1$. Then $D \leq \mathrm{N}(b_x)$ and $d^t = d^{-1}$. So $(d, b_d)^t = (d^{-1}, b_d)$. The first statement follows from this. This was already proved by Brauer in [3, (4.16)]. The next two statements follow from the discussion before Lemma 5.1. The remaining statements follow from Proposition (4A) of [3]. □

Brauer showed in [3] that B has $2^{d-2}+3$ columns. Exactly 5 of these are 2-rational, namely those of the form (d,θ), with $d = 1, s_1, s_2, t$ or st and $\theta \in \mathrm{IBr}(\mathrm{C}(d))$. There are $d-2$ families, with representatives $\{(s_i,\theta) \mid i = 2,\dots,d-1\}$. The family of (s_i,θ) contains the 2^{i-2} columns $\{(s_i^r,\theta) \mid 1 \leq r \leq 2^{i-1}-1,\ r \text{ odd}\}$, which form a single 2-conjugate orbit. Recall the notation $\chi_i, \chi^{(i)}$ for the irreducible characters in B. The 5 irreducible characters which are 2-rational are $\chi_1, \chi_2, \chi_3, \chi_4$, and $\chi^{(0)}$.

We can now prove the main result of this section.

Theorem 5.7. *The number of real* 2*-rational irreducible characters in* B *equals the number of real* 2*-rational columns in* B*. For* $d \geq 4$*, all irreducible characters in* $F_0,\dots,F_{d-4}$ *are real, while all irreducible characters in* F_{d-3} *are real if and only if* E *is not of type (e).*

Proof. The first statement is trivial if $d = 3$. So we assume that $d \geq 4$.

Let m be the largest odd divisor of $|G|$ and let ω be a primitive $(2^{d-1}m)$-th root of unity. Then $\omega = \omega_2\omega_{2'}$, where ω_2 is a primitive 2^{d-1}-th root of unity and $\omega_{2'}$ is a primitive m-th root of unity. Let $\gamma, \sigma \in \mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})$ be such that $\omega^\gamma = \omega_2^5\omega_{2'}$ and $\omega^\sigma = \overline{\omega}$. Then γ has order 2^{d-3} and σ is an involution. Set $\tau := \gamma^{2^{d-4}}$. So τ is the unique involution in $\langle\gamma\rangle$.

Set $\mathcal{G} := \langle\gamma\rangle \times \langle\sigma\rangle$ and let $\alpha \in \mathcal{G}$. There is an integer (α) such that $\omega^\alpha = \omega^{(\alpha)}$. For $g \in G$, set $g^\alpha := g^{(\alpha)}$. If θ is a character (ordinary or Brauer) of a subgroup of G, define θ^α by $\theta^\alpha(g) := \theta(g^\alpha)$, for all g in the domain of θ. Then set $(d,\theta)^\alpha := (d^\alpha, \theta^\alpha)$. In this way $\mathcal{G}$ acts on the ordinary irreducible characters and on the columns of B. Note that a character is 2-rational if it is fixed by γ, and real if it is fixed by σ.

It can be checked that $d^{(x^\alpha)}_{\chi^\alpha\theta^\alpha} = d^{(x)}_{\chi\theta}$, for all $\alpha \in \mathcal{G}$. As the generalized decomposition matrix is non-singular, it follows that the characters of $\mathcal{G}$ acting on the irreducible characters and on the columns coincide. We cannot use Brauer's permutation lemma to deduce that the permutation actions are isomorphic, as $\mathcal{G}$ is not cyclic. However, we can prove this fact using a careful case-by-case analysis.

Let $i = 1,\dots,d-1$. Lemmas 5.5 and 5.6 imply that (s_i,θ) is not real if and only if E is of type (e) and $i = d-1$. In particular the 2-rational columns (s_1,θ) and (s_2,θ) are necessarily real. Moreover, B has at least one real irreducible Brauer character (this holds for any real block).

We distinguish 5 cases:

Case (I) is that E is not of type (e) and all 2-rational columns are real. Then all columns are real. We conclude from Theorem 5.7 that all irreducible characters in B are real.

Case (II) is that E is not of type (e) and two 2-rational columns are nonreal. Then there are two nonreal columns. We deduce from Theorem 5.7 that there are two nonreal irreducible characters. Consider the action of $\langle\gamma\sigma\rangle$ on columns and on irreducible characters. The stabilizer of a nonreal column is $\langle\gamma^2\rangle$. The remaining 2-rational columns are fixed by $\gamma\sigma$. For $i = 3, \dots, d-1$, the stabilizer of (s_i, θ) is $\langle\gamma^{2^{i-2}}\rangle$. We deduce that exactly two orbits on the columns have stabilizer $\langle\gamma^2\rangle$. It then follows from Brauer's permutation lemma that two orbits on the irreducible characters have stabilizer $\langle\gamma^2\rangle$. A 2-rational orbit has stabilizer $\langle\gamma^2\rangle$ if and only if it is nonreal. There is at most one such orbit. Suppose that a family F_i has stabilizer $\langle\gamma^2\rangle$, for $i \geq 2$. Then $|F_i| \leq [\langle\gamma\rangle : \langle\gamma^2\rangle] = 2$. So $i = 2$. We conclude that there is a pair of nonreal 2-rational characters in B. Moreover, F_1 has stabilizer $\langle\gamma^2\rangle$, whence it is real.

Case (III) is that E is of type (e) and all 2-rational columns are real. Then there are exactly 2^{d-3} nonreal columns, namely the 2-conjugates of (s, θ). We deduce from Theorem 5.7 that there are 2^{d-3} nonreal irreducible characters. As $\langle\sigma\rangle$ acts nontrivially on the 2-conjugates of (s, θ), this column has stabilizer $\langle\tau\sigma\rangle$ in $\mathcal{G}$. All other columns are fixed by $\tau\sigma$. So by Brauer's permutation lemma all irreducible characters are $\langle\sigma\tau\rangle$-invariant. Let $\chi \in F_{d-3}$. Then $\chi^\sigma = \chi^\tau \neq \chi$. So F_{d-3} is a nonreal family of characters. This accounts for all nonreal irreducible characters in B.

Case (IV) is that $d = 4$, E is of type (e) and exactly two 2-rational columns are nonreal. Thus B has two nonreal 2-rational columns and 2 nonreal 2-irrational columns. It then follows from Theorem 5.7 that B has 4 nonreal irreducible characters. As B has 4 irreducible characters of height 0 and 3 of height 1, it follows that B has a pair of nonreal characters of height 0 and another pair of height 1. The latter belong to F_1.

Case (V) is that $d > 4$, E is of type (e) and exactly two 2-rational columns are nonreal. So B has $2^{d-3} + 2$ nonreal columns and hence $2^{d-3} + 2$ nonreal irreducible characters. We consider the action of $\langle\gamma\sigma\rangle$ on columns and on irreducible characters. Just as in case (II), there are two orbits on the columns whose stabilizer is $\langle\gamma^2\rangle$ (as $d > 4$, the stabilizer $\langle\tau\sigma\rangle$ of (s_{d-1}, θ) in $\langle\gamma\sigma\rangle$ has trivial intersection with $\langle\gamma\sigma\rangle$). Again just as in case (II), we conclude that there is a pair of nonreal

2-rational characters in B. Suppose that some character in F_i has a trivial stabilizer. Then $|F_i| \geq |\langle\gamma\sigma\rangle| = 2^{d-3}$. So $F_i = F_{d-3}$. Let $\chi \in F_i$. Then $\chi^\sigma = \chi^{\gamma^{-1}} \neq \chi$. So F_{d-3} contains 2^{d-3} nonreal characters. As we have accounted for all $2^{d-3}+2$ nonreal irreducible characters in B, the proof is complete. □

6. Morita equivalence classes

Karin Erdmann classified the possible blocks with dihedral defect group, up to morita equivalence, in [9]. Her results are summarized in the table on pp294–296. It is not known if each of the possible 8 Morita equivalence classes occur for all dihedral groups.

Let B be a 2-block with a dihedral defect group D, where $|D| = 2^d$. Then by [3] $k(B) = 2^{d-2}+3$ and $l(B) \leq 3$. If $l(B) = 1$ then B is a nilpotent block. In particular B is Morita equivalent to kD. This corresponds to Brauer's case (bb). The case $l(B) = 3$ corresponds to Brauer's case (aa). Then there are three possible Morita equivalence classes. The principal 2-blocks of the groups $\mathrm{L}_2(q)$ with $q \equiv 1$ or 3 mod 4, and the principal block of the alternating group $\mathfrak{A}_7$ provide examples of each class. As Erdmann remarks on p293 of [9], her proof in [10] that $d = 3$ for the last class is erroneous. On the other hand, there are no known examples (at least to this author) of such blocks with $d > 3$.

Finally we consider when $l(B) = 2$, which corresponds to Brauer's cases (ab) and (ba). Then there are quivers Q_1, Q_2, a parameter $c = 0, 1$, and ideals $I_i(c) \leq Q_i$, $i = 1, 2$ inside the ideal of paths of length ≥ 2, such that B is Morita equivalent to exactly one of the 4 algebras $kQ_i/I_i(c)$. By [11], the case $kQ_2/I(0)$ occurs for the principal blocks of certain quotients of the unitary groups $\mathrm{GU}_2(q)$, when $q \equiv 3$ mod 4. By a new result of [16], both $kQ_1/I_1(1)$ and $kQ_2/I_2(1)$ occur for the principal blocks of $\mathrm{PGL}_2(q)$, corresponding to $q \equiv 1$ or -1 mod 8, respectively. Note that in these cases $d \geq 4$. It turns out that in all cases the decomposition matrix of B depends on Q_i, but not on c. For this reason, we will refer to B as being of type $\mathrm{PGL}_2(q)$, with $q \equiv 1$ mod 4, for Q_1, or $q \equiv 3$ mod 4, for Q_2.

The papers [3], [4], [5], [6] and [10] give more information on the decomposition matrices and modules of these blocks. Derived equivalences between the block algebras are discussed in [15] and [18].

We obtain the form of the generalized decomposition matrices given in the next lemma using [3], [5] and [10]. In particular, we make use of the parameters $\delta_1, \delta_2, \delta_3, \delta_4$ introduced in Proposition (6C) of [3], and the further information given by Theorem 5 and Propositions (6H) and (6I) of that paper. The δ_i are evaluated when $l(B) = 2$ in [5] (implicitly) and when $l(B) = 3$ in [10]. Entries that we don't care to specify are denoted by $*$.

Lemma 6.1. *For $d \geq 3$, a block B with a dihedral defect group of order 2^d has one of 6 possible generalized decomposition matrices. Allowing i to range over $0, \ldots, d-4$, and j over $2, \ldots, d-2$, there are signs $\varepsilon, \varepsilon', \varepsilon_j$ such that this matrix has the form:*

(i) *B is nilpotent and has one irreducible module M_1.*

	M_1	s_1	t	st	s_j	s
χ_1	1	ε	1	ε'	ε_j	1
χ_2	1	ε	-1	$-\varepsilon'$	ε_j	1
χ_3	1	ε	1	$-\varepsilon'$	ε_j	-1
χ_4	1	ε	-1	ε'	ε_j	-1
$\chi^{(i)}$	2	2ε	0	0	$*$	$*$
$\chi^{(d-3)}$	2	-2ε	0	0	$*$	$*$

(ii) *B has two irreducible modules M_1, M_2 and is of type* $\mathrm{PGL}(2, q)$ *with $q \equiv 1$ mod 4.*

	M_1	M_2	s_1	t	s_j	s
χ_1	1	0	ε	1	ε_j	1
χ_2	1	1	ε	-1	ε_j	1
χ_3	1	0	ε	-1	ε_j	-1
χ_4	1	1	ε	1	ε_j	-1
$\chi^{(i)}$	2	1	2ε	0	$*$	$*$
$\chi^{(d-3)}$	2	1	-2ε	0	$*$	$*$

(iii) *B has two irreducible modules M_1, M_2 and is of type* $\mathrm{PGL}(2, q)$ *with $q \equiv 3$ mod 4.*

	M_1	M_2	s_1	t	s_j	s
χ_1	1	0	$-\varepsilon$	1	$-\varepsilon_j$	-1
χ_2	1	1	ε	-1	ε_j	1
χ_3	1	0	$-\varepsilon$	-1	$-\varepsilon_j$	1
χ_4	1	1	ε	1	ε_j	-1
$\chi^{(i)}$	0	1	2ε	0	$*$	$*$
$\chi^{(d-3)}$	0	1	-2ε	0	$*$	$*$

(iv) *B has 3 irreducible modules M_1, M_2, M_3. If $d = 3$, then B is Morita equivalent to the principal block of A_7.*

	M_1	M_2	M_3	s_1	s_j	s
χ_1	1	0	0	$-\varepsilon$	$-\varepsilon_j$	-1
χ_2	1	1	0	ε	ε_j	1
χ_3	1	1	1	ε	ε_j	-1
χ_4	1	0	1	$-\varepsilon$	$-\varepsilon_j$	1
$\chi^{(i)}$	0	1	0	2ε	$*$	$*$
$\chi^{(d-3)}$	0	1	0	-2ε	$*$	$*$

(v) *B has 3 irreducible modules M_1, M_2, M_3 and is Morita equivalent to the principal block of $\mathrm{PSL}(2,q)$, with $q \equiv 1$ mod 4.*

	M_1	M_2	M_3	s_1	s_j	s
χ_1	1	0	0	ε	ε_j	1
χ_2	1	1	1	ε	ε_j	1
χ_3	1	1	0	ε	ε_j	-1
χ_4	1	0	1	ε	ε_j	-1
$\chi^{(i)}$	2	1	1	2ε	$*$	$*$
$\chi^{(d-3)}$	2	1	1	-2ε	$*$	$*$

(vi) *B has 3 irreducible modules M_1, M_2, M_3 and is Morita equivalent to the principal block of $\mathrm{PSL}(2,q)$, with $q \equiv 3$ mod 4.*

	M_1	M_2	M_3	s_1	s_j	s
χ_1	1	0	0	$-\varepsilon$	$-\varepsilon_j$	-1
χ_2	1	1	1	ε	ε_j	1
χ_3	0	1	0	ε	ε_j	-1
χ_4	0	0	1	ε	ε_j	-1
$\chi^{(i)}$	0	1	1	2ε	$*$	$*$
$\chi^{(d-3)}$	0	1	1	-2ε	$*$	$*$

Proof. Brauer showed in [3] that $d^{(s_1)}_{\chi,\theta} = \pm 1$ or ± 2, depending on whether χ has height 0 or 1. Moreover, $d^{(s_1)}_{\chi,\theta}$ is constant on each of the families of 2-conjugate characters $F_1, \ldots, F_{d-3}$. If B is of type (bb) or (ab), and $x = t, st$, then $d^{(x)}_{\chi,\theta} = \pm 1$ or 0, depending on whether χ has height 0 or 1. The columns for s_1, t, st, s_j, s can then be recovered, up to the signs $\varepsilon, \varepsilon', \varepsilon_j$, and possible rearranging of the χ_i, using orthogonality with the columns of the decomposition matrix of B. $\square$

Corollary 6.2. *Suppose that B has a dihedral defect group of order at least 16 and that all height 1 irreducible characters in B are real-valued. Then at least one of these characters has FS-indicator $+1$.*

Proof. The multiplicity of an irreducible B-module M in $k\mathfrak{I}B$ is given by $\sum_{\chi\in\mathrm{Irr}(B)}\epsilon(\chi)d_{\chi,M}$. In particular this sum is non-negative. The hypothesis on the defect group implies that B has at least 3 irreducible characters of height 1. We assume that all irreducible characters in B of height 1 have FS-indicator -1, and derive a contradiction.

Suppose that B is nilpotent. So $M = M_1$. The contribution of the height zero characters to the above sum is at most $+4$. On the other hand, the height 1 characters contribute an integer ≤ -6, whence the sum is negative.

Suppose that B is not nilpotent. Take $M = M_2$. The contribution of the height zero characters to the above sum is at most $+2$. On the other hand, the height 1 characters contribute an integer ≤ -3, whence the sum is negative. □

We will use the following rather technical result:

Lemma 6.3. *Let f be a $\mathbb{C}$-valued function on $\mathrm{Irr}(B)$ that is constant on each of $F_1,\dots,F_{d-3}$. Then*

$$\sum_{\chi\in\mathrm{Irr}(B)} f(\chi)d^{(s^2)}_{\chi,\theta} = \sum_{i=1}^{4} f(\chi_i)d^{(s^2)}_{\chi_i,\theta} + f(\chi^{(0)})d^{(s^2)}_{\chi^{(0)},\theta}.$$

Proof. Let $1 \leq r \leq d-3$. Then $\zeta+\zeta^{-1}$ takes on 2^{r-1} different values as ζ ranges over the primitive 2^{r+1}-th roots of unity. A careful reading of Section 6 of [3] shows that for each ζ there are exactly two irreducible characters $\chi \in F_r$ with $d^{(s^2)}_{\chi,\theta} = \zeta+\zeta^{-1}$. Now $f(\chi)$ is constant on F_r. So the net contribution of the two characters associated with each of ζ and $-\zeta$ to $\sum_{\chi\in\mathrm{Irr}(B)} f(\chi)d^{(s^2)}_{\chi,\theta}$ is 0. The lemma follows from this. □

7. Type (a): totally split extended defect groups

In this section $E = D\times\langle e\rangle$ i.e. $e \in \mathrm{C}(D)$ and $e^2 = 1_G$. This includes the principal 2-block case, when $e = 1_G$. By Theorem 5.7, all height 1 irreducible characters in B are real-valued. Using Lemma 5.5 and Theorem 5.7, we can show that if B is nilpotent then all its irreducible characters are real valued. This also follows from Lemma 2.2 of [22].

Lemma 7.1. *There is an indecomposable B-module M_D such that M_D has vertex D and $M_D \oplus M_D$ is the sum of all components of $k\mathfrak{I}B$ that have vertex D. Moreover $\nu(\dim(M_D)) = \nu[G : D]$.*

Proof. Let b be the 2-block of $\mathrm{N}(D)$ that is the Brauer correspondent of B. Then b is real and has defect couple (D,E). Now $\mathrm{N}(D)$ acts trivially on $Z(D)$. So by Lemma 3.4, there is a self-dual irreducible b-module I, such that the sum of all components of $k\mathfrak{I}_{\mathrm{N}(D)}b$ that have vertex D is isomorphic to $I \oplus I$. Set M_D as the Green correspondent of I, with respect to $(G, D, \mathrm{N}(D))$. Then the sum of all components of $k\mathfrak{I}B$ that have vertex D is isomorphic to $M_D \oplus M_D$.

As I is irreducible and projective as $\mathrm{N}(D)/D$-module, $\nu(\dim(I)) = \nu[\mathrm{N}(D) : D]$. Now $I{\uparrow}^G = M_D \oplus W$, where every component of W has a vertex that is a proper subgroup of D. In particular $\nu(\dim(W)) > \nu[G : D]$. We conclude that $\nu(\dim(M_D)) = \nu[G : D]$. □

Lemma 7.2. *There are indecomposable B-modules M_{X_2}, M_{Y_2} such that $M_{X_2} \oplus M_{Y_2}$ is the sum of all components of $k\mathfrak{I}B$ that have vertex X_2 or Y_2. Here M_V has vertex B-subpair (V, b_V), for $V = X_2, Y_2$.*

Proof. Recall that X_2 is a Klein-four subgroup of D. Set $N := \mathrm{N}(X_2)$ and $C := \mathrm{C}(X_2)$. Then b_{X_2} is real, with defect couple $(X_2, X_2 \times \langle e \rangle)$. As b_{X_2} is nilpotent, Theorem 1.7 of [22] implies that $k\mathfrak{I}_C b_{X_2} \cong R^4$, where R is the unique irreducible b_{X_2}-module. Let $\mathrm{N}(R)$ denote the common inertia group of b_{X_2} and R in N.

Set $b := b_{X_2}^N$. Then b is real, with defect couple $(X_3, X_3 \times \langle e \rangle)$. By Lemma 7.1, there is an indecomposable b-module M_{X_3} such that $M_{X_3} \oplus M_{X_3}$ is the sum of all components of $k\mathfrak{I}_N b$ that have vertex X_3. We note that $k\mathfrak{I}_C b$ is the sum of all components of $k\mathfrak{I}_N b$ that have vertex X_2 or X_3.

Suppose first that $\mathrm{N}(R)/C \cong \mathbb{Z}_2$. Then R has a unique extension to $\mathrm{N}(R)$ and b is nilpotent. Let I be the unique irreducible b-module. Then R occurs once in the semisimple module $I{\downarrow}_C$. As $k\mathfrak{I}_C b_{X_2} \cong R^4$, it follows that $[k\mathfrak{I}_C : I] = 4$. Now $\nu(\dim(M_{X_3})) = \nu[N : X_3] = \nu(\dim(I))$. So $[M_{X_3} : I]$ is odd. The only possibility is that $M_{X_3} \cong I$. We may now write $k\mathfrak{I}_C b \cong I \oplus I \oplus W$, where $[W : I] = 2$, and every component of W has vertex X_2. As I does not itself have vertex X_2, it follows that W is indecomposable.

Suppose then that $\mathrm{N}(R)/C \cong \mathfrak{S}_3$. Then two irreducible $\mathrm{N}(R)$-modules lie over R. So b has two irreducible modules I_1 and I_2. We may assume that I_1 has vertex X_3 and $I_1{\downarrow}_C \cong R$, while I_2 has vertex X_2 and $I_2{\downarrow}_C \cong R \oplus R$. By considering the Clifford theory of the ordinary characters of b, we see that b is Morita equivalence to the principal 2-block of $\mathrm{PGL}(2,3) \cong \mathfrak{S}_4$. Arguing as in the previous paragraph, $M_{X_3} \cong I_1$ and $k\mathfrak{I}_C b \cong I_1 \oplus I_1 \oplus W$, where W is indecomposable with vertex X_2. With more work, we can even show that $W \cong I_2$.

Regardless of the structure of $\mathrm{N}(R)/C$, let M_{X_2} denote the Green correspondent of W. Then M_{X_2} is the unique component of $k\mathfrak{I}B$ that has vertex B-subpair (X_2, b_{X_2}). Analogous arguments show that $k\mathfrak{I}B$ has a unique component M_{Y_2} that has vertex B-subpair (Y_2, b_{Y_2}). But Lemma 5.1 shows that $(X_2, b_{X_2}) \not\sim (Y_2, b_{Y_2})$. So $M_{X_2} \ncong M_{Y_2}$. □

Theorem 7.3. *$k\mathfrak{I}B \cong M_D \oplus M_D \oplus M_{X_2} \oplus M_{Y_2}$. If B is nilpotent then all its irreducible characters have FS-indicator $+1$, M_D is irreducible and M_{X_2}, M_{Y_2} are each of composition length 2^{d-2}.*

Proof. Suppose first that B is nilpotent, with unique irreducible module M_1. Let θ be the Brauer character of M_1. From the discussion in Section 1, M_1 occurs with multiplicity $\sum \epsilon(\chi) d_{\chi,\theta}$ in $k\mathfrak{I}B$. By Part (i) of Lemma 6.1, this is at most $4 + 2\sum_{j=0}^{d-3} 2^j = |D|/2 + 2$. Now $[M_D : M_1] \geq 1$ and by consideration of vertices, $|D|/4$ divides both $[M_{X_2} : M_1]$ and $[M_{Y_2} : M_1]$. It then follows from Lemmas 7.1 and 7.2 that $[k\mathfrak{I}B : M_1] \geq |D|/2 + 2$. All statements of the theorem now follows for nilpotent B.

Now let B be of arbitary Morita equivalence type and let (V, b_V) be a vertex B-subpair of a component of $k\mathfrak{I}B$. Lemmas 3.1 and 5.1 imply that (V, b_V) is a vertex b_x-subpair of a component of $k\mathfrak{I}_{\mathrm{C}(x)}b_x$, where $x = s_1, t$ or st. Now b_{s_1} is real and nilpotent, with defect couple $(D, D \times \langle e\rangle)$. So by the previous paragraph, (V, b_V) is conjugate to $(D, b_D), (X_2, b_{X_2})$ or (Y_2, b_{Y_2}), if $x = s_1$. The block b_t is real with defect couple $(X_2, X_2 \times \langle e\rangle)$ and Sylow B-subpair (X_2, b_{X_2}). As X_2 is a Klein-four group, Theorem 1.7(i) of [22] implies that (V, b_V) is conjugate to (X_2, b_{X_2}), if $x = t$. In the same way, (V, b_V) is conjugate to (Y_2, b_{Y_2}), if $x = st$. The first statement of the Theorem now follows from Lemmas 7.1 and 7.2. □

Theorem 7.4. *Let $\chi \in \mathrm{Irr}(B)$. Then $\nu(\chi) = +1$, unless B is of type (vi) and $\chi = \chi_3$ or χ_4; in that case $\overline{\chi_3} = \chi_4$.*

Proof. As b_{s_1} is nilpotent, real, and has defect couple $(D, D \times \langle e\rangle)$, Theorem 7.3 shows that $\epsilon(\psi) = +1$, for all $\psi \in \mathrm{Irr}(b_{s_1})$. From our knowledge of $d^{(s_1)}_{\psi,\theta}$, and the positivity assertion in Lemma 1.4, we get

$$(2) \qquad \sum_{\psi \in \mathrm{Irr}(b_{s_1})} \epsilon(\psi) d^{(s_1)}_{\psi,\theta} = \pm(4 + 2(2^{d-3} - 1) - 2.2^{d-3}) = 2.$$

Let $d \geq 4$. Then b_{s^2} is real, with defect couple $(S, S \times \langle e\rangle)$. Theorem 1.6 of [22] shows that b_{s^2} has two real-valued irreducible characters, ψ_1 and ψ_2. Moreover, $\epsilon(\psi_i) = +1$, for $i = 1, 2$. It follows that

$$(3) \qquad \begin{aligned} \sum_{\psi \in \mathrm{Irr}(b_{s^2})} \epsilon(\psi) d^{(s^2)}_{\psi,\theta} &= \pm\epsilon(\psi_1) \pm \epsilon(\psi_2) \\ &= +2, \quad \text{by Lemma 1.3, as } s^2 = (se)^2. \end{aligned}$$

Suppose first that B has type (ii) or (v). Lemma 6.1 gives the form of the generalized decomposition matrix of B. Applying Lemma 1.4 to the column $d^{(s_1)}_{\chi\theta}$, equality (2) gives

$$2 = \sum_{\chi\in\mathrm{Irr}(B)} \epsilon(\chi) d^{(s_1)}_{\chi,\theta} = \varepsilon(\sum_{i=1}^{4} \epsilon_i + 2\sum_{j=0}^{d-4} \epsilon^{(j)} 2^j - 2\epsilon^{(d-3)}2^{d-3}).$$

In particular $\sum_{i=1}^4 \epsilon_i \equiv 0 \bmod 4$. It then follows from Corollary 4.3 that $\epsilon_i = 1$, for $i = 1, 2, 3, 4$. Then the above equation rearranges to

$$\sum_{j=0}^{d-4} \epsilon^{(j)} 2^j - \epsilon^{(d-3)} 2^{d-3} \; = \; \varepsilon - 2 \in \{-1, -3\}.$$

By Lemma 1.5 the two solutions: $\varepsilon = +1$ and $\epsilon^{(j)} = +1$, for $j \geq 0$, or $\varepsilon = -1$, $\epsilon^{(0)} = -1$ and $\epsilon^{(j)} = +1$ for $j > 0$, are the only ones. We claim that the latter solution does not occur. For, taking $f(\chi) = d^{(s_1)}_{\chi,\theta}$ in Lemma 6.3, we get $\sum_{\chi\in\mathrm{Irr}(B)} d^{(s_1)}_{\chi,\theta} d^{(s^2)}_{\chi,\theta} = 4\varepsilon\varepsilon_{d-2} + 2\varepsilon d^{(s^2)}_{\chi^{(0)},\theta}$. This sum is zero, by column orthogonality. So $d^{(s^2)}_{\chi^{(0)},\theta} = -2\varepsilon_{d-2}$. Now

$$2 = \sum_{\psi\in\mathrm{Irr}(b_{s^2})} \epsilon(\psi) d^{(s^2)}_{\psi,\theta} = \sum_{\chi\in\mathrm{Irr}(B)} \epsilon(\chi) d^{(s^2)}_{\chi,\theta} = \varepsilon_{d-2}(4 - 2\epsilon^{(0)}),$$

by (3), Lemma 1.4, and an application of Lemma 6.3, with $f(\chi) = \epsilon(\chi)$. Thus $\varepsilon_{d-2} = +1$ and $\epsilon^{(0)} = +1$, which proves the claim.

Suppose then that B has type (iii), (iv) or (vi). Then from the decomposition matrices given in Lemma 6.1, we have $\sum_{i=1}^4 \epsilon_i d^{(s_1)}_{\chi,\theta} = 0, \pm 2$. However, just as above, this sum is $\equiv 0$ modulo 4. So it must equal 0. If B is of type (iii) or (iv), we get $\epsilon_i = +1$, for all i; if B is of type (vi), we get $\epsilon_1 = \epsilon_2 = +1$ and $\epsilon_3 = \epsilon_4 = 0$.

From knowledge of $d^{(s_1)}_{\chi,\theta}$, Lemmas 1.4, (2) and the work above give

$$2\varepsilon(\sum_{j=0}^{d-4} \epsilon^{(j)} 2^j - \epsilon^{(d-3)} 2^{d-3}) = +2.$$

Thus $\epsilon^{(j)} = -\varepsilon = \mp 1$, for all $j = 0, \ldots, d-3$.

We claim that $\varepsilon = -1$ and $\epsilon^{(j)} = +1$, for all j. If $|D| \geq 16$, this is a consequence of Corollary 6.2. So from now on we assume that $|D| = 8$ and $\epsilon^{(0)} = -1$, and argue to a contradiction.

Recall that $k\mathfrak{I}B = M_D^2 \oplus M_{X_2} \oplus M_{Y_2}$. Moreover, $M_{X_2} \ncong M_{Y_2}$, as $(X_2, b_{X_2}) \not\sim (Y_2, b_{Y_2})$. So M_{X_2} and M_{Y_2} are self-dual. Also $[M_D : M_1]$ is odd. We claim that $M_D \cong M_1$. Otherwise M_D is reducible, whence by Lemma 1.6, M_D^2 has a composition factor that occurs with multiplicity ≥ 4. Using (1), and Lemma 6.1, $[k\mathfrak{I} : M_1] \leq 4$, $[k\mathfrak{I} : M_2] = 1$ and $[k\mathfrak{I} : M_3] \leq 2$. So $[M_D : M_1] = 2$. This contradiction proves our claim.

If B is of type (iii), then $[M_{X_2} + M_{Y_2} : M_1] = 2$ and $[M_{X_2} + M_{Y_2} : M_2] = 1$, using (1) and Lemma 6.1. We choose notation so that M_{X_2} is irreducible. So $M_{X_2} \cong M_2$. Now the $\mathcal{O}$-lift of M_{X_2} has character $\chi^{(0)}$. So $\chi^{(0)}$ appears with multiplicity 1 in the permutation character of $\mathcal{O}\mathfrak{I}$. But then $\epsilon(\chi^{(0)}) = +1$, contrary to hypothesis.

If B is of type (iv), then $[M_{X_2} + M_{Y_2} : M_1] = 2$, $[M_{X_2}+M_{Y_2} : M_2] = 1$ and $[M_{X_2} + M_{Y_2} : M_2] = 2$, using (1) and Lemma 6.1. Choose notation so that $[M_{X_2} : M_3] \neq 0$. Then χ_3 or χ_4 occurs in the character of the $\mathcal{O}$-lift of M_{X_2}. So $[M_{X_2} : M_1] \neq 0$. As M_{X_2} is reducible, Lemma 1.6 implies that $[M_{X_2} : M_1] = 2$. Then $[M_{Y_2} : M_1] = 0$, whence $[M_{Y_2} : M_3] = 0$. Thus $M_{Y_2} \cong M_2$, leading to the contradiction as in the previous paragraph.

Finally, if B is of type (vi), then $[M_{X_2} + M_{Y_2} : M_1] = 0$, $[M_{X_2} + M_{Y_2} : M_2] = 1$ and $[M_{X_2} + M_{Y_2} : M_2] = 1$, by (1) and Lemma 6.1. We may choose notation so that $M_{X_2} \cong M_2$ and $M_{Y_2} \cong M_3$. But $\overline{\chi_3} = \chi_4$. So $M_2^* \cong M_3$. We deduce that $M_{X_2} \cong M_{X_2}^* \cong M_{Y_2}$, which is a contradiction. □

8. Type (b) extended defect groups

In this section $E = D\langle e\rangle$, where $e^2 = s_1$ and $e \in \mathrm{C}(D)$. Table 1 shows that the involutions in $E \backslash D$ form a single E-class, constisting of $\{s_2 e, s_2^{-1} e\}$. Theorem 5.7 shows that all height 1 irreducible characters in B are real. We require the following subsidiary result:

Lemma 8.1. *Let b be a real nilpotent 2-block of a finite group H that has defect couple $(Q, Q \times \langle r\rangle)$, where Q is a Klein-four or dihedral group. Suppose that $\mathcal{C}l_H(r) \cap Qr = \{r\}$. Then $k\mathcal{C}l_H(r)b$ is irreducible.*

Proof. We prove the result for the case that $Q \cong \mathbb{Z}_2^2$. The case that Q is dihedral follows from similar arguments. Theorem 1.7 of [22] implies that $k\mathfrak{I}_H b \cong S^4$, where I is the unique irreducible b-module. So it is enough to show that $k\mathcal{C}l_H(r)b$ is indecomposable. Then by the Green correspondence theorem, we may assume that $H = \mathrm{N}_H(Q)$.

Let b_1 be a block of $C := \mathrm{C}_H(Q)$ that is covered by b. Then b_1 is nilpotent, real, and has defect couple $(Q, Q \times \langle r\rangle)$. So $k\mathfrak{I}_C b_1 \cong S_1^4$, where I_1 is the unique irreducible b_1-module. The hypothesis implies that Qr meets 4 distinct conjugacy classes of involutions of C. Using Theorem 3.2, we deduce that $k\mathcal{C}l_C(qr)b_1 = I_1$, for each $q \in Q$.

As $\mathrm{N}_H(b_1) = C$, we have $I = I_1{\uparrow}^H$ and hence $[I{\downarrow}_C : I_1] = 1$. We then conclude from the previous paragraph that $k\mathcal{C}l_H(r)b = I$. □

Let $X/Y/Z$ denote a module with successive Loewy factors X, Y, Z.

Theorem 8.2. *B is of type (i), (ii) or (v) and $k\mathfrak{I}B$ is indecomposable with vertex S. All 2^{d-3} irreducible characters in F_{d-3} have FS-indicator -1 and the remaining $2^{d-3}+3$ irreducible characters in B have FS-indicator $+1$. Moreover by the type of B we have:*

(i) $k\mathfrak{I}B \cong M_1/M_1$.
(ii) $k\mathfrak{I}B \cong M_1/M_2/M_1$.
(v) $k\mathfrak{I}B \cong M_1/(M_2 \oplus M_3)/M_1$.

Proof. Suppose that $(t, b_t) \not\sim (s_1, b_{s_1})$. Then [3] shows that $b_t = b_{X_2}^{\mathrm{C}(t)}$ and both b_{X_2} and b_t have defect group X_2. Lemma 5.5 implies that b_{X_2} is real, with defect couple $(X_2, X_2\langle e\rangle)$. Then by the results in Section 1, b_t is real, and also has defect couple $(X_2, X_2\langle e\rangle)$. As $X_2\langle e\rangle$ does not split over X_2, it follows from Theorem 3.2 that $k\mathfrak{I}_{\mathrm{C}(t)}b_t = 0$. Similarly $(st, b_{st}) \sim (s_1, b_{s_1})$ or $k\mathfrak{I}_{\mathrm{C}(st)}b_{st} = 0$.

Set $H := \mathrm{C}(s_1)$. Then the first paragraph implies that b_{s_1} is the unique 2-block of H such that $k\mathfrak{I}_H b_{s_1} \neq 0$ and $b_{s_1}^G = B$.

Table 1 shows that there is a single E-class of involutions in $E\backslash D$, consisting of $\{s_2e, s_2^{-1}e\}$. It then follows from Theorem 3.2 that $k\mathfrak{I}B = k\mathcal{C}l(s_2e)B$. Moreover, as $S = \mathrm{C}_D(s_2e)$, each component of $k\mathfrak{I}B$ has vertex S_i, for some $i > 0$, and at least one component has vertex S. As $\mathrm{N}(S_i) \leq H$, Green correspondence induces a bijection between the components of $k\mathfrak{I}B$ that have vertex S_i, and the components of $k\mathcal{C}l_H(s_2e)b_{s_1}$ that have vertex S_i.

Now b_{s_1} is nilpotent and real, and has defect couple $(D, D\langle e\rangle)$. Let I be the unique irreducible b_{s_1}-module and set $\overline{H} := H/\langle s_1\rangle$. Now b_{s_1} dominates a unique block $\overline{b}$ of $\overline{H}$, and $\overline{b}$ is nilpotent. Moreover, Lemma 1.7 implies that $\overline{b}$ is real, and has defect couple $(\overline{D}, \overline{D} \times \langle\overline{e}\rangle)$. Table 1 shows that $\overline{e}, \overline{s_2e}, \overline{te}$ and $\overline{ste}$ represent the distinct $\overline{E}$-classes of involutions in $\overline{E}\backslash\overline{D}$. Of these, only $\overline{s_2e}$ is the image of an involution in H. We conclude from Lemma 8.1 that $k\mathcal{C}l_{\overline{H}}(\overline{s_2e})\overline{b} \cong I$.

Now $[\mathrm{N}_H(\overline{s_2e}) : \mathrm{C}_H(s_2e)] = 2$, and $\mathrm{N}_H(\overline{s_2e})$ is the preimage of $\mathrm{C}_{\overline{H}}(\overline{s_2e})$ in H. Then as $k\mathcal{C}l_{\overline{H}}(\overline{s_2e})\overline{b} \cong I$, we get an exact sequence:

$$0 \to I \to k\mathfrak{I}_H b \to I \to 0, \qquad \text{as } kH\text{-modules.}$$

We cannot have $k\mathfrak{I}_H b \cong I \oplus I$, as I has no vertex contained in S. So $k\mathfrak{I}_H b_{s_1}$ is indecomposable, with vertex S. We conclude from Lemma 3.1 that $k\mathfrak{I}B$ is indecomposable, has vertex S, and is in Green correspondence with $k\mathfrak{I}_H b_{s_1}$.

We have $[k\mathfrak{I}_H b_{S_1} : I] = 2$. Let θ be the Brauer character of I. We temporarily apply the FS-indicator notation $\epsilon_i, \epsilon^{(j)}$ to $\mathrm{Irr}(b_{s_1})$. Then from (1) and the decomposition matrix of b_{s_1} we get:

$$\sum_{\psi\in\mathrm{Irr}(b_{s_1})}\epsilon(\psi)d_{\psi,\theta}=\sum_{i=1}^{4}\epsilon_i+2\sum_{j=0}^{d-3}\epsilon^{(j)}2^j=2.$$

The only solution is $\epsilon_i=1$, for $i=1,2,3,4$, $\epsilon^{(j)}=1$, for $j=0,\ldots,d-4$ and $\epsilon^{(d-3)}=-1$. In fact, all irreducible characters in $\overline{b}$ have FS-indicator 1, and these account for all characters in $\mathrm{Irr}(b_{s_1})$ that do not belong to the family F_{d-3}. We now use Lemma 6.1 to compute

$$\sum_{\psi\in\mathrm{Irr}(b_{s_1})}\epsilon(\psi)d^{(s_1)}_{\psi,\theta}=\pm(4+2(2^{d-3}-1)-(-1)2.2^{d-3})=\pm(2^{d-1}+2).$$

It then follows from Lemma 1.3, and the fact that $s_1=e^2$, that

$$\sum_{\chi\in\mathrm{Irr}(B)}\epsilon(\chi)d^{(s_1)}_{\chi,\theta}=2^{d-1}+2. \tag{4}$$

We claim that B is not of type (iii). For otherwise $\epsilon_i=+1$, for $i=1,2,3,4$, and we use the decomposition matrix of B to compute

$$\sum_{\chi\in\mathrm{Irr}(B)}\epsilon(\chi)d^{(s_1)}_{\chi\phi}=2\varepsilon\sum_{j=0}^{d-3}\epsilon^{(j)}2^j<2^{d-1}.$$

This contradicts (4), proving our claim. Similar contradictions rule out the Morita types (iv) and (vi).

Now suppose that B is of type (i), (ii) or (v). Then from the decomposition matrix of B, and (4) we get

$$\varepsilon(\sum_{i=1}^{4}\epsilon_i+2\sum_{j=0}^{d-4}\epsilon^{(j)}2^j-2\epsilon^{(d-3)}2^{d-3})=\sum_{\chi\in\mathrm{Irr}(B)}\epsilon(\chi)d^{(s_1)}_{\chi\phi}=2^{d-1}+2.$$

In particular $\sum\epsilon_i\equiv 0 \bmod 4$. As $\epsilon_i\geq 0$, we get $\epsilon_i=+1$, for $i=1,2,3,4$. Substituting these values into the equation above, we get

$$\sum_{j=0}^{d-4}\epsilon^{(j)}2^j-\epsilon^{(d-3)2^{d-3}}=\begin{cases}2^{d-2}-1, & \text{if } \varepsilon=+1;\\ -2^{d-2}-3, & \text{if } \varepsilon=-1.\end{cases}$$

As the left hand side has absolute value $<2^{d-2}$, we deduce that $\varepsilon=+1$. The resulting equation has the unique solution $\epsilon^{(j)}=+1$, for $j=0,\ldots,d-4$ and $\epsilon^{(d-3)}=-1$.

Let θ_1 be the Brauer character of M_1. Then (1) gives

$$[k\mathfrak{I}:M_1]=\sum_{\chi\in\mathrm{Irr}(B)}\epsilon(\chi)d_{\chi,\theta_1}=4+2(2^{d-3}-1)-2.2^{d-3}=2.$$

It follows from this that $k\mathfrak{I}B=M_1/M_1$, if B is nilpotent.

Now suppose that B has type (ii) or (v). Let θ_2 be the Brauer character of M_2. Using (1) and the decomposition matrix of B, we have:

$$[k\mathfrak{I}B : M_2] = \sum_{\chi\in\mathrm{Irr}(B)} \epsilon(\chi)d_{\chi,\theta_2} = 2 + (2^{d-3} - 1) - 2^{d-3} = 1.$$

In the same way $[k\mathfrak{I}B : M_3]{=}1$, if B is of Morita type (v). We deduce the structure of the successive Loewy layers of $k\mathfrak{I}B$ from these facts. □

9. Type (c): dihedral extended defect groups

In this section B is a real non-principal 2-block with a dihedral defect group D and a dihedral extended defect group $E = D\langle e\rangle$, where e has order 2 and $s = (te)^2$. In particular B is not the principal 2-block. We denote the projective cover of a module M by $P(M)$.

By Theorem 5.7, all height 1 irreducible characters in B are real-valued. Now s_1 is not the square of an element of $E\backslash D$. It then follows from Lemma 1.3 that

$$\sum_{\chi\in\mathrm{Irr}(B)} \epsilon(\chi)d^{(s_1)}_{\chi\theta} = 0. \tag{5}$$

Lemma 9.1. *B is not of type (ii), (iii) or (iv).*

Proof. Lemma 5.2 implies that B is not of type (ii) or (iii).

Suppose that B is of type (iv). An examination of the decomposition matrix of B shows that all three irreducible B-modules are self-dual. It then follows from Theorem 5.7 that all irreducible characters in B are real-valued. Corollary 4.3 shows that $\epsilon_i = +1$, for $i = 1, 2, 3, 4$. We can now use the decompostion matrix of B to compute

$$\sum_{\chi\in\mathrm{Irr}(B)} \epsilon(\chi)d^{(s_1)}_{\chi\theta} = 2\varepsilon(\sum_{j=0}^{d-4} \epsilon^{(j)}2^j - \epsilon^{(d-3)}2^{d-3}) \equiv 2 \mod 4.$$

This contradiction of (5) completes the proof. □

Theorem 9.2. *$k\mathfrak{I}B$ is indecomposable, has vertex $\mathrm{Z}(D)$, and is isomorphic to its Heller translate. Moreover, by the Morita type we have:*

- (i),(v) *Without loss of generality $\overline{\chi_3} = \chi_4$. All other irreducible characters in B have FS-indicator $+1$ and $P(k\mathfrak{I}B) \cong P(M_1)$;*
- (vi) *All irreducible characters in B have FS-indicator $+1$, and $P(k\mathfrak{I}B) \cong P(M_1) \oplus P(M_2) \oplus P(M_3)$.*

Proof. There is a single E-conjugacy class of involutions in $E\backslash D$, with representative e. Now $\mathrm{C}_D(e) = \mathrm{Z}(D) = \langle s_1 \rangle$ is cyclic of order 2. So by Theorem 3.2, all components of $k\mathfrak{I}B$ have vertex $\mathrm{Z}(D)$.

Suppose first that B is nilpotent. Lemmas 5.5 and 5.6 show that (t, b_t) and (st, b_{st}) are not real. Then by Theorem 5.7, B has two nonreal irreducible characters of height zero. We choose notation so that $\overline{\chi_3} = \chi_4$. Let θ_1 be the Brauer character of M_1 and let M be a component of $k\mathfrak{I}B$. Now $\dim(M) = [M : M_1]\dim(M_1)$, and $\nu\dim(M_1) = \nu[G : D]$. So $\nu\dim(M) = \nu[M : M_1] + \nu[G : D]$. As M has vertex $\mathrm{Z}(D)$, we have $\nu\dim(M) \geq \nu[D : \mathrm{Z}(D)] + \nu[G : D]$. We deduce that $2^{d-1} = [D : \mathrm{Z}(D)]$ divides $[M : M_1]$. On the other hand, $[k\mathfrak{I}B : M_1] = \sum_{\chi\in\mathrm{Irr}(B)} \nu(\chi)d_{\chi,\theta_1}$, which is $\leq 2^{d-1} + 2$, from the decomposition matrix of B. We conclude that $[M : M_1] = 2^{d-1}$, that $k\mathfrak{I}B = M$ is indecomposable, and that all irreducible characters in B, apart from χ_3, χ_4, have FS-indicator $+1$. Its easy to see that $k\mathfrak{I}B$ is the unique B-module that has vertex $\mathrm{Z}(D)$. So $k\mathfrak{I}B$ coincides with its Heller translate $\Omega(k\mathfrak{I}B)$. Now from the Cartan matrix of B, we have $[P(M_1) : M_1] = 2^d$. It follows that $P(k\mathfrak{I}B) \cong P(M_1)$.

Suppose that B is of type (v) or (vi). Then $(t, b_t) \sim (s_1, b_{s_1})$ and $(st, b_{st}) \sim (s_1, b_{s_1})$. So b_{s_1} is the unique 2-block of $H := \mathrm{C}(s_1)$ with $b_{s_1}^G = B$. The Green correspondence theorem establises a bijection between the components of $k\mathfrak{I}B$ and the components of $k\mathfrak{I}_H b_{s_1}$. Applying the previous paragraph to b_{s_1}, we see that $k\mathfrak{I}_H b_{s_1}$ is indecomposable and $k\mathfrak{I}_H b_{s_1} = \Omega(k\mathfrak{I}_H b_{s_1})$. So $k\mathfrak{I}B$ is indecomposable and $k\mathfrak{I}B = \Omega(k\mathfrak{I}B)$. We write $P(k\mathfrak{I}B) \cong P(M_1)^a \oplus P(M_2)^b \oplus P(M_3)^c$, where $a, b, c \geq 0$.

We now specialize to the case that B has Morita type (v). We see from the decomposition matrix of B, and (5) that

$$\sum_{i=1}^{4} \epsilon_i + 2\sum_{j=0}^{d-4} \epsilon^{(j)}2^j - 2\epsilon^{(d-3)}2^{d-3} = 0$$

The only solution is $\epsilon_1 = \epsilon_2 = 1$, $\overline{\chi_3} = \chi_4$, and $\epsilon^{(j)} = 1$, for $j = 0, \ldots, d-3$. As $\Omega(k\mathfrak{I}B) = k\mathfrak{I}B$, we can compute $[k\mathfrak{I}B : M_i]$ in terms of the unknown exponents a, b, c, and compare these with the values given by (1):

	M_1	M_2	M_3
$\sum \epsilon(\chi)d_{\chi,M_i}$	2^{d-1}	2^{d-2}	2^{d-2}
$P(\mathfrak{I}B)$	$2^{d-1}(a + \frac{(b+c)}{2})$	$2^{d-2}(a + \frac{(b+c)}{2}) + \frac{b}{2}$	$2^{d-2}(a + \frac{(b+c)}{2}) + \frac{c}{2}$

The only solution is $a = 1, b = c = 0$. So $P(k\mathfrak{I}B) \cong P(M_1)$.

Finally, we consider the case that B is of Morita type (vi). We see from the decomposition matrix of B, and (5) that

$$-\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4+2\sum_{j=0}^{d-4}\epsilon^{(j)}2^j-2\epsilon^{(d-3)}2^{d-3}=0$$

In this case the only solution is $\epsilon_i=\epsilon^{(j)}=+1$, for all i,j. Proceeding as in the previous paragraph, we get the following table involving a,b,c:

	M_1	M_2	M_3
$\sum\epsilon(\chi)d_{\chi,M_i}$	2	$2^{d-2}+1$	$2^{d-2}+1$
$P(\mathfrak{I}B)$	$a+\frac{(b+c)}{2}$	$2^{d-3}(b+c)+\frac{a+b}{2}$	$2^{d-3}(b+c)+\frac{a+c}{2}$

Thus $a=b=c=1$, whence $P(k\mathfrak{I}B)\cong P(M_1)\oplus P(M_2)\oplus P(M_3)$. $\square$

10. Type (d): semi-dihedral extended defect groups

In this section B is a real non-principal 2-block with a dihedral defect group D and a semi-dihedral extended defect group $E=D\langle e\rangle$, where $e^2=s_1$ and $s_1s=(te)^2$. By Theorem 5.7, all height 1 irreducible characters in B are real-valued. As E does not split over D, Theorem 3.2 implies that $k\mathfrak{I}B=0$. If M_i is an irreducible B-module, we use θ_i to denote its Brauer character. Then by (1) we have

$$\sum_{\chi\in\mathrm{Irr}(B)}\epsilon(\chi)d_{\chi,\theta_i}=0. \tag{6}$$

Theorem 10.1. *B is of Morita type (i) or (v). We may choose notation so that $\overline{\chi_3}=\chi_4$; the irreducible characters in F_{d-3} have FS-indicator -1; all other irreducible characters have FS-indicator $+1$.*

Proof. Lemma 5.2 implies that B is not of type (ii) or (iii).

Suppose that B is of Morita type (iv). Then from the decomposition matrix $\sum_{\chi\in\mathrm{Irr}(B)}\epsilon(\chi)d_{\chi,\theta_1}=\sum_{i=1}^4\epsilon_i>0$. This contradicts (6). So B is not of type (iv). A similar argument shows that B is not of type (vi).

Assume from now on that B is of Morita type (i) or (v). Taking $i=1$ in (6), the decomposition matrix of B shows that

$$\sum_{j=0}^{d-3}\epsilon^{(j)}2^j=-\sum_{i=1}^{4}\epsilon_i/2.$$

Now the left hand side is odd and $\epsilon_i\geq 0$, for $i=1,2,3,4$. So without loss of generality $\epsilon_1=\epsilon_2=1$ and $\overline{\chi_3}=\chi_4$. We then get the unique solution $\epsilon^{(j)}=+1$, for $j=0,\dots,d-4$ and $\epsilon^{(d-3)}=-1$. $\square$

11. Type (e) extended defect groups

In this section B is a real non-principal 2-block with extended defect group $E = D\langle e\rangle$, where $|D| \geq 16$, $e^2 = 1$, $s^e = s_1 s$ and $t^e = t$. By Theorem 5.7, no character in F_{d-3} is real, but the remaining characters of height 1 in $F_0, \dots, F_{d-4}$ are real.

Theorem 11.1. *B is of type (i), (ii) or (v). Let $\chi \in \mathrm{Irr}(B)$. Then χ is nonreal, if $\chi \in F_{d-3}$. Otherwise $\epsilon(\chi) = +1$. We may write $k\mathfrak{I}B = M_{X_{d-1}} \oplus M_{X_2}$, where M_{X_i} is an indecomposable module with vertex X_i, for $i = d-1, 2$. If B is nilpotent then $[M_{X_i} : M_1] = 2^{d-i}$.*

Proof. By Table 1, there are two E-conjugacy classes of involutions in $E\backslash D$, with representatives e and te. Now $\mathrm{C}_D(e) = X_{d-1}$ and $\mathrm{C}_D(te) = X_2$. So part (ii) of Theorem 3.2 implies that $k\mathcal{C}l_G(e)B$ has at least one component $M_{X_{d-1}}$ that has vertex X_{d-1}.

Set $b_2 := b_{X_2}^{\mathrm{N}(X_2)}$. Then b_2 is real and nilpotent, and has defect couple $(X_3, X_3 \times \langle e\rangle)$. As $X_3 \cong D_8$, Theorem 7.3 implies that $k\mathfrak{I}_{\mathrm{N}(X_2)}b_2$ has one component with vertex X_2. Set M_{X_2} as its Green correspondent with respect to $(G, X_2, \mathrm{N}(X_2))$. Then M_{X_2} is the unique component of $k\mathfrak{I}B$ that has vertex B-subpair (X_2, b_{X_2}).

Suppose that B is nilpotent. This holds for example if $G = \mathrm{C}(s_1)$. Then B is real and has defect couple (D, E). Let Φ be the unique principal indecomposable character of B. Then from (1) and the decomposition matrix of B, we have $[k\mathfrak{I} : M_1] = \epsilon(\Phi) \leq 2^{d-2} + 2$. On the other hand, $2 = [D : X_{d-1}]$ divides $[M_{X_{d-1}} : M_1]$, and $2^{d-2} = [D : X_2]$ divides $[M_{X_2} : M_1]$. We deduce that $k\mathfrak{I}B = M_{X_{d-1}} \oplus M_{X_2}$. Moreover, $\epsilon(\Phi) = 2^{d-2} + 2$. So $\epsilon(\chi) = +1$, if χ is an irreducible character in B that does not belong to F_{d-3}. The last statement of the Theorem also follows.

We now remove the nilpotency assumption on B. The last paragraph applies to b_{s_1}, allowing us to compute

$$\sum_{\psi \in \mathrm{Irr}(b_{s_1})} \epsilon(\psi) d_{\psi,\theta}^{(s_1)} = 2^{d-2} + 2. \tag{7}$$

Set $b_{d-1} := b_{X_{d-1}}^{\mathrm{N}(X_{d-1})}$. Then b_{d-1} is real and nilpotent, and has defect couple (D, E). The last paragraph shows that $k\mathfrak{I}_{\mathrm{N}(X_{d-1})}b_{d-1}$ has one component with vertex X_{d-1}. Set $M_{X_{d-1}}$ as its Green correspondent with respect to $(G, X_{d-1}, \mathrm{N}(X_{d-1}))$. Then $M_{X_{d-1}}$ is the unique component of $k\mathfrak{I}B$ that has vertex B-subpair $(X_{d-1}, b_{X_{d-1}})$.

Now let (V, b_V) be a vertex B-subpair of a component of $k\mathfrak{I}B$. Lemmas 3.1 and 5.1 imply that (V, b_V) is a vertex B-subpair of a component of $k\mathfrak{I}_{\mathrm{C}(x)}b_x$, where $x = s_1, t$ or st. Now b_{st} is real with vertex pair $(Y_2, Y_2\langle s_2e\rangle)$. As $Y_2\langle s_2e\rangle$ does not split over Y_2, Theorem 3.2 implies that $k\mathfrak{I}_{\mathrm{C}(st)}b_{st} = 0$. Our work above shows that every component of $k\mathfrak{I}_{\mathrm{C}(s_1)}b_{s_1}$ has vertex B-subpair (X_i, b_{X_i}), for $i = d-1, 2$. Theorem 1.7(i) of [22] shows that every component of $k\mathfrak{I}_{\mathrm{C}(t)}b_t$ has vertex B-subpair (X_2, b_{X_2}). We deduce from this that $k\mathfrak{I}B = M_{X_{d-1}} \oplus M_{X_2}$. This proves one statement of the Theorem.

Suppose for the sake of contradiction that B is of type (iii). Then all irreducible characters of height 0 are real, whence they have FS-indicator $+1$. From the decomposition matrix of B, we compute

$$\sum_{\chi\in\mathrm{Irr}(B)} \epsilon(\chi)d^{(s_1)}_{\chi\phi} = 2\sum_{j=0}^{d-4}\epsilon^{(j)}2^j < 2^{d-2}.$$

This contradicts Lemma 1.4 and (7). We can show that B is not of Morita type (iv) or (vi), using similar arguments.

Now suppose that B is of type (ii) or (v). Then we compute

$$\varepsilon(\sum_{i=1}^{4}\epsilon_i + 2\sum_{j=0}^{d-4}\epsilon^{(j)}2^j) = \sum_{\chi\in\mathrm{Irr}(B)}\epsilon(\chi)d^{(s_1)}_{\chi\theta} = 2^{d-2}+2,$$

using Lemma 1.4 and (7). Considering this equality modulo 4, and using the fact that $\epsilon_i \in \{0,1\}$, we see that $\epsilon_i = +1$, for $i = 1,2,3,4$. Substituting these values, we get two possibilities

$$\sum_{j=0}^{d-4}\epsilon^{(j)}2^j = \begin{cases} 2^{d-3}-1, & \text{if } \varepsilon = +1; \\ -2^{d-3}-3, & \text{if } \varepsilon = -1. \end{cases}$$

As the left hand side has absolute value $< 2^{d-3}$, we deduce that $\varepsilon = +1$. The resulting equation has the unique solution $\epsilon^{(j)} = +1$, for $j = 0, \ldots, d-4$. This completes the proof. □

12. Summary of results for blocks with dihedral defect groups

We summarize the results of Sections 7 through 11 in the following table. Recall that M_V is an indecomposable module that has vertex V.

Block Type	E type	ϵ_i $_{1\leq i\leq 4}$	$\epsilon^{(j)}$ $_{0\leq j\leq d-4}$	$\epsilon^{(d-3)}$	$k\Im B$
Nilpotent	(a)	$++++$	$+$	$+$	$M_1^2 \oplus M_{X_2} \oplus M_{Y_2}$
	(b)	$++++$	$+$	$-$	$M_S = M_1/M_1$
	(c)	$++00$	$+$	$+$	$M_{Z(D)}$
	(d)	$++00$	$+$	$-$	$-$
	(e)	$++++$	$+$	0	$M_{X_{d-1}} \oplus M_{X_2}$
$\mathrm{PGL}(2,q)$ $q \equiv 1(\mathrm{mod}\ 4)$	(a)	$++++$	$+$	$+$	$M_D^2 \oplus M_{X_2} \oplus M_{Y_2}$
	(b)	$++++$	$+$	$-$	$M_S = M_1/M_2/M_1$
	(e)	$++++$	$+$	0	$M_{X_{d-1}} \oplus M_{X_2}$
$\mathrm{PGL}(2,q)$ $q \equiv 3(\mathrm{mod}\ 4)$	(a)	$++++$	$+$	$+$	$M_D^2 \oplus M_{X_2} \oplus M_{Y_2}$
A_7	(a)	$++++$	$+$	$+$	$M_D^2 \oplus M_{X_2} \oplus M_{Y_2}$
$\mathrm{PSL}(2,q)$ $q \equiv 1(\mathrm{mod}\ 4)$	(a)	$++++$	$+$	$+$	$M_D^2 \oplus M_{X_2} \oplus M_{Y_2}$
	(b)	$++++$	$+$	$-$	$M_S = M_1/(M_2 \oplus M_3)/M_1$
	(c)	$++00$	$+$	$+$	$M_{Z(D)}$
	(d)	$++00$	$+$	$-$	$-$
	(e)	$++++$	$+$	0	$M_{X_{d-1}} \oplus M_{X_2}$
$\mathrm{PSL}(2,q)$ $q \equiv 3(\mathrm{mod}\ 4)$	(a)	$++00$	$+$	$+$	$M_D^2 \oplus M_{X_2} \oplus M_{Y_2}$
	(c)	$++++$	$+$	$+$	$M_{Z(D)}$

Table 2. Summary of results

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