TWO RESULTS ON SKEW DIAGRAMS

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1. Partitions

We are interested in hooks in skew diagrams because we would like to determine the blocks of centralizer algebras of the form $RS_n^{S_\ell}$, where R is a commutative ring, $\ell < n$ and S_n is the symmetric group of degree n. When $R = \mathbb{Q}$, the irreducible representations of $RS_n^{S_\ell}$ are indexed by skew-diagrams $[\alpha]/[\beta]$, for $\alpha \vdash n$ and $\beta \vdash \ell$. The contents of the nodes in such diagrams gives the possible simultaneous eigenvalues of certain Young-Jucys-Murphy operators on the representation spaces. Taking these integers modulo a prime p gives information on the corresponding p-modular representation. See [2] for further details.

Our note is independent of this project. The definitions and combinatorial and the proofs are almost completely self-contained and elementary.

A partition is a sequence of non-negative integers $\alpha = (\alpha_1 \geq \alpha_2 \geq \dots)$ which is non-increasing and eventually 0. The terms α_i are called the *parts* of α. We say that α is a partition of n, and write $\alpha \vdash n$, if $\sum \alpha_i = n$. The *Young diagram* of α is the following collection of *nodes* in $\mathbb{Z}^2$:

$$[\alpha] := \{(i, j) \mid i \geq 1, 1 \leq j \leq \alpha_i\}.$$

In the anglophone tradition, the X-axis is oriented left-to-right, but the Y-axis is oriented top-to-bottom.

The *content* of a node (i, j) is the integer $j - i$, and for $e > 0$, the *e-residue* of (i, j) is the image of $j - i$ in $\mathbb{Z}/e\mathbb{Z}$. We use $\mathrm{res}_e(\alpha)$ to denote the multi-set of e-residues of the nodes of $[\alpha]$.

As α is non-increasing, $\alpha_i - i = \alpha_j - j$ if and only if $i = j$, for all $i, j \in \mathbb{Z}$. So we can recover α from its *content set* $\{\alpha_j - j \mid j \geq 1\}$. The *partition sequence* $P = P(\alpha) = \{p_i\}_{i\in\mathbb{Z}}$ of α is the indicator function of the complement of this set in $\mathbb{Z}$. Thus

$$p_i = \begin{cases} 0, & \text{if } \alpha_j - j = i, \text{ for some } j \geq 1. \\ 1, & \text{if } \alpha_j - j \neq i, \text{ for all } j \geq 1. \end{cases}$$

Note that P describes the shape of the *extended boundary* of $[\alpha]$ ($[\alpha]$ extended down the Y axis and across the X-axis); a 0 corresponds to a vertical step and a 1 corresponds to a horizontal step. This is elegantly described in [1].

A *removable* hook of α is a $(1, 0)$-pair of entries in P viz $p_u = 1$ and $p_v = 0$ where $u < v$. The distance $h = v - u$ is the length of the hook. The hook is removed by exchanging the

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$(1,0)$ entries for $(0,1)$ entries at the same positions. Likewise, an *addable* hook of α is a $(0,1)$-pair of entries in P viz $p_w = 0$ and $p_x = 1$ where $w < x$. The distance $h = x - w$ is the length of the hook. The hook is added by exchanging the $(0,1)$ entries for $(1,0)$ entries at the same positions.

Suppose that β is a partition. We say that β *is contained in* α and write $\beta \subseteq \alpha$, if $\beta_i \leq \alpha_i$ for all i. If $\beta \subseteq \alpha$, we write α/β, and refer to difference in diagrams $[\alpha]/[\beta]$ as a *skew-diagram*. We say that α/β is *connected* if $\beta_i < \alpha_{i+1}$ or $\alpha_{i+1} = 0$, for all $i \geq 1$. Each skew-diagram $[\alpha]/[\beta]$ decomposes into a disjoint union of connected *components*.

Now $[\alpha]/[\beta]$ contains a number of nodes. These nodes have contents and e-residues, for $e > 0$. We use $\text{res}_e(\alpha/\beta)$ to denote the multi-set of the latter.

It is convenient to describe a connected skew-diagram $[\alpha]/[\beta]$ using a pair of finite $0,1$ sequences (p,q) related to α and β. For convenience, we apply the same translation to the partition sequences of α and β so that the first difference between the sequences is at position 0 and the last difference is at position n. Thus we have two $0,1$-sequences $p = (p_0, \ldots, p_n)$ and $q = (q_0, \ldots, q_n)$ of equal length n subject to the rules:

(i) $\sum_{j=0}^{i}(p_j - q_j) > 0$, for each $i = 0, 1, \ldots, n-1$.
(ii) $\sum_{j=0}^{n}(p_j - q_j) = 0$.

These conditions imply that $p_0 = q_n = 1$ and $p_n = q_0 = 0$. Here p describes the *outer boundary* of $[\alpha]/[\beta]$ and q describes the *inner boundary*. The sum $\sum_{j=1}^{i}(p_j - q_j)$ is the *diagonal distance* between the inner and outer boundary points on the $(i+1)$-th diagonal. As the diagram is connected, this distance must be positive, for $i < n$, but vanishes at the end point on the $n+1$-th diagonal.

We say that (p,q) is *expanding* at i if $p_i = 1$ and $q_i = 0$; then $[\alpha]/[\beta]$ contains one more node on the $(i+1)$-th diagonal than the i-th diagonal. Likewise, we say that (p,q) is *contracting* at i if $p_i = 0$ and $q_i = 1$; then $[\alpha]/[\beta]$ contains one less node on the $(i+1)$-th diagonal than the i-th diagonal. Finally, (p,q) is *parallel* at i if $p_i = q_i$; then $[\alpha]/[\beta]$ contains an equal number of nodes on the i-th and $(i+1)$-th diagonals.

We call n the *hook length* of α/β - it is the length of the largest hook that can be removed from α to give a partition still containing β.

Example: (p,q) describes the diagram of a proper partition if q has the form $q = (0, \ldots, 0, 1, \ldots, 1)$. This is a partition of $\sum_{i=1}^{n}(n+1-j)(p_j - q_j)$.

Example: (p,q) describes a rim-hook or ribbon skew-diagram (connected, but with no 2×2-subdiagrams) if $p_i = q_i$, for $i = 1, \ldots, n-1$. This ensures that the diagonal distance is always 1.

Nodes and hooks can be removed from both the outer and inner boundaries of a skew-diagram (p,q) to give a smaller skew-diagram. For this reason, we give the following definitions.

A *removable inner* hook is a $(0,1)$-pair of entries in q viz $q_r = 0$ and $q_s = 1$ with $0 \leq r < s \leq n$. The distance $s - r$ is the length of the hook. The hook is *removed* by exchanging the $(0,1)$-pair for a $(1,0)$-pair at the same positions.

Likewise, a *removable outer* hook is a $(1,0)$-pair of entries in p viz $p_t = 1$ and $p_u = 0$ with $t < u$. The distance $u - t$ is the length of the hook. The hook is *removed* by exchanging the $(1,0)$-pair for a $(0,1)$-pair at the same positions.

Removing a hook of either type leaves a skew-diagram. However, it need not be connected. Note that a removable inner hook corresponds to an addable hook of β, and a removable outer hook corresponds to a removable hook of α.

The proof of the following lemma is straightforward, and omitted:

Lemma 1. *Suppose that a skew-diagram contains a removable (inner or outer) hook of length e. Then for each divisor d of e, it contains a removable hook of length d (of the same type).*

The following striking result is an easy consequence of the partition sequence formalism:

Lemma 2. *Suppose that a connected skew-diagram has hook length n. Then the diagram has a removable (inner or outer) hook of each of the lengths $1, \ldots, n$.*

Proof. Let $e \in \{1, \ldots, n\}$. Describe the skew-diagram by a pair (p, q) of finite $0,1$-sequences as above. We assume for the sake of contradiction that (p, q) contains no removable inner or outer hooks of length e. In view of the previous Lemma, (p, q) contains no removable hook of length a multiple of e.

Recall that the $(1,0)$ pair of entries $p_0 = 1, p_n = 0$ is a removable outer hook of length n (and likewise the $(0,1)$ pair of entries $q_0 = 0, q_n = 1$ is a removable inner hook of length n). It follows that e does not divide n. So $n = qe + r$, where $0 < r < e$ and $q \geq 1$.

Let a, b be the number of 0's respectively 1's in $p_0, \ldots, p_{r-1}$ and let c, d be the number of 0's respectively 1's in $p_{qe+1}, \ldots, p_n$. So $a + b = r = c + d$. Suppose that $0 \leq i < r$ and $p_i = 1$. Then $p_{qe+i} = 1$, as there are no removable outer hooks of length qe. So

$$b \leq d. \tag{1}$$

Now let e, f be the number of 0's respectively 1's in $q_0, \ldots, q_{r-1}$ and g, h be the number of 0's respectively 1's in $q_{qe+1}, \ldots, q_n$. So $e + f = r = g + h$. Suppose that $0 \leq i < r$ and $q_i = 0$. Then $q_{qe+i} = 0$, as there are no removable inner hooks of length qe. Thus $e \leq g$, and hence

$$h \leq f. \tag{2}$$

As the skew-diagram is connected,

$$b - f = \sum_{j=0}^{r-1} (p_j - q_j) > 0 \tag{3}$$

Likewise $\sum_{j=0}^{qe} (p_j - q_j) > 0$, but $\sum_{j=0}^{n} (p_j - q_j) = 0$. It follows that

$$d - h = \sum_{j=qe+1}^{n} (p_j - q_j) < 0, \tag{4}$$

Now (1),(4),(2) and (3) give the impossible chain of inequalities:

$$b \le d < h \le f < b.$$

□

Lemma 3. *Suppose that α/β contains an equal number of nodes of each e-residue, for some $e > 0$. Then there exists a partition γ with $\beta \subseteq \gamma \subset \alpha$ and α/γ contains exactly one node of each e-residue.*

Proof. As we are not assuming that α/β is connected, we describe α/β using the pair (p, q) of infinite partitions sequences of α and β. Recall that a removable outer hook H of α/β is a removable hook of α, and hence consists of a $(1, 0)$ pair of entries $p_s = 1, p_t = 0$, with $s < t$. The contents of the nodes in H form the interval of integers $[s+1, s+2, \dots, t-1, t]$. This means that removing H results in a partition α' with $\beta \subseteq \alpha' \subseteq \alpha$ such that α/α' consists of nodes with contents $s+1, \dots, t$. From now on we identify H and $[s+1, \dots, t]$.

Each connected component D of α/β has a hook length h and a corresponding removable outer hook $[s+1, s+2, \dots, t]$ with $h = t - s$. Then the pair of finite $0, 1$-sequences $p_D := (p_s, \dots, p_t), q_D := (q_s, \dots, q_t)$ form the partition sequence description of D. We write $D = [[s, \dots, t]]$.

Choose a connected component $D_1 = [[s_1, \dots, t_1]]$ of α/β, and $c_1 \in \{s_1+1, \dots, t_1\}$ such that D_1 is expanding at $c_1 - 1$. Set r_1 as the e-residue of c_1 and r_2 as the e-residue of $t_1 + 1$. Then D_1 has a removable outer hook $[c_1, \dots, t_1]$, whose e-residues form the set $\{r_1, r_1 + 1, \dots, r_2 - 1\}$.

As α/β has the same number of nodes of e-residue $r_2 - 1$ and r_2, there must be a component $D_2 = [[s_2, \dots, t_2]]$ and $c_2 \in \{s_2 + 1, \dots, t_2\}$ such that $c_2 \equiv r_2 \mod e$ and D_2 is expanding at $c_2 - 1$. Set r_3 as the e-residue of $t_2 + 1$. Then D_2 contains a removable outer hook $[c_2, \dots, t_2]$, whose e-residues form the set $\{r_2, r_2 + 1, \dots, r_3 - 1\}$.

Continuing in this fashion, we get outer hooks of α/β:

$$[c_1, \dots, t_1], [c_2, \dots, t_2], \dots, [c_m, \dots, t_m], \tag{5}$$

where $t_i + 1 \equiv c_{i+1} \mod e$, for all $i = 1, \dots, m - 1$, and $t_m + 1 \equiv c_j \mod e$, for some $j \ge 1$. Without loss of generality, we can and do assume that $t_m + 1 \equiv c_1 \mod e$. Then the e-residue of the integers $\bigcup_{i=1}^{m}\{c_i, \dots, t_i\}$ are uniformly distributed across $\mathbb{Z}/e\mathbb{Z}$.

Choose a sequence of the form (5) of smallest total length $\sum_{i=1}^{m}(t_i - c_i + 1) = qe$, for $q \ge 1$. We claim first that there does not exist $i \ne j$ with $c_j \in \{c_i, c_i + 1, \dots, t_i\}$. Suppose otherwise. Then $t_i = t_j$. Then we can replace (5) by the strictly smaller sequence

$$[c_{i+1}, \dots, t_{i+1}], \dots, [c_j, \dots, t_j],$$

where the indices are taken modulo m. This contradicts the minimality of the length, and proves the claim. Notice that this implies that the contents of the nodes in (5) are distinct.

We claim now that $q = 1$. Suppose otherwise. Then there exists $i \in \{1, \dots, m\}$ and $c \in \{c_i, c_i + 1, \dots, s_i\}$ such that $c \neq c_1$ but $c \equiv c_1 \bmod e$. Then replace (5) by

$$\begin{array}{ll} [c_1, \dots, t_1],\ [c_2, \dots, t_2], \dots, [c_i, \dots, c-1], & \text{if } p_{c-1} = 0. \\ [c, \dots, t_i],\ [c_{i+1}, \dots, t_{i+1}], \dots, [c_m, \dots, t_m], & \text{if } p_{c-1} = 1. \end{array}$$

This is a contradiction, as each of these sequences has smaller length than (5). This proves our claim. Finally, removing the e nodes in a minimal set of hooks (5), we get a partition γ with the required properties. □

The hypothesis that there are equal numbers of nodes of each e-residue is necessary, even when $[\alpha]/[\beta]$ contains nodes of each e-residue. For example, no such γ exists when $\beta = (0, 0, \dots)$ and $\alpha = (3, 1, 0, \dots)$ and $e = 3$.

Recall the terminology of e-blocks and e-cores of partitions c.f. [3]. Note that the hypothesis implies that α and β have the same e-core. Represent α on an abacus with e-runners. Then the e-abacus of β can be obtained by moving beads up and down runners on this abacus. The number of bead moves can be arbitrarily large compared to the number of nodes in $[\alpha]/[\beta]$. This can be seen by taking $e = 2, m > 2$ and $\alpha = [m+1, m-1, \dots, 3, 2, 1]$ and $\beta = [m, m-1, \dots, 3, 2]$.

Set $w_e(\alpha)$ as the e-weight of α - this is the largest number of hooks of length e that can be sucessively removed from α. We can reformulate the last Lemma as

Lemma 4. *Suppose that $\beta \subseteq \alpha$ are partitions with the same e-core κ. Then for each w with $w_e(\beta) \leq w \leq w_e(\alpha)$ there exists a partition γ with e-core κ and e-weight w such that $\beta \subseteq \gamma \subseteq \alpha$.*

References

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