ON A RESULT OF KIYOTA, OKUYAMA AND WADA

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Abstract. M. Kiyota, T. Okuyama and T. Wada recently proved that each 2-block of a symmetric group Σ_n contains a unique irreducible Brauer character of height 0. We present a more conceptual proof of this result.

1. Background on bilinear forms

According to the main result in [6], every 2-block of the symmetric group Σ_n has a unique irreducible Brauer character of height 0. This is not true for an arbitrary 2-block of a finite group. For example, let B be a real non-principal 2-block which is Morita equivalent to the group algebra of A_4 and which has a Klein-four defect group and a dihedral *extended defect group* (in the sense of [1]). Then one can show that B has three real irreducible Brauer characters of height 0. The non-principal 2-block of $((C_2 \times C_2) : C_9) : C_2$ is of this type.

In this note we place the results of [6] in a more general context using the approach to bilinear forms developed by R. Gow and W. Willems [2]. We use results and notation from [7] for representation theory, from [5] for symmetric groups, and from [8] for bilinear forms in characteristic 2.

Let G be a finite group and let (K, R, F) be a 2-*modular system* for G. So R is a complete discrete valuation ring with field of fractions K of characteristic 0, and residue field $R/J = F$ of characteristic 2. Assume that K contains a primitive $|G|$-th root of unity, and that F is perfect. Then K and F are splitting fields for each subgroup of G.

The anti-isomorphism $g \mapsto g^{-1}$ on G extends to an involutary F-algebra anti-automorphism $\sigma : FG \to FG$ called the *contragredient map.* Let V be a right FG-module. The linear dual $V^* := \mathrm{Hom}_F(V, F)$ is considered as a right FG-module via $(f.x)(v) := f(vx^\sigma)$, for $f \in V^*$, $x \in FG$ and all $v \in V$. The Frobenius automorphism $\lambda \mapsto \lambda^2$ of the field F induces an automorphism $(a_{ij}) \mapsto (a_{ij}^2)$ of the group $\mathrm{GL}_F(V)$. Composing the module map $G \to \mathrm{GL}_F(V)$ with this automorphism

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endows V with another FG-module structure. This module is called the *Frobenius twist* of V, and is denoted $V^{(2)}$.

Let $V^* \otimes V^*$ be the space of bilinear forms on V and let $\Lambda^2(V^*)$ be the subspace of *symplectic bilinear forms* on V; a bilinear form $b : V \times V \to F$ is symplectic if and only if $b(v, v) = 0$, for all $v \in V$. The quotient space $V^* \otimes V^*/\Lambda^2(V^*)$ is called the *symmetric square* of V^* and is denoted $S^2(V^*)$.

A *quadratic form* on V is a map $Q : V \to F$ such that $Q(\lambda v) = \lambda^2 Q(v)$ and $(u, v) \mapsto Q(u + v) - Q(u) - Q(v)$ is a bilinear form on V, for all $u, v \in V$ and $\lambda \in F$. Now if b is a bilinear form, its *diagonal* $\delta(b) : v \mapsto b(v, v)$ is a quadratic form. The assignment δ is linear with kernel $\Lambda^2(V^*)$. So there is a short exact sequence of vector spaces

$$(1) \qquad 0 \longrightarrow \Lambda^2(V^*) \longrightarrow V^* \otimes V^* \xrightarrow{\delta} S^2(V^*) \longrightarrow 0$$

We may identify $S^2(V^*)$ with the space of quadratic forms on V. If Q is a quadratic form, its *polarization* is the associated bilinear form $\rho(Q) : (u, v) \mapsto Q(u + v) - Q(u) - Q(v)$.

The dual $S^2(V)^*$ of the symmetric square $S^2(V)$ of V is the space of *symmetric bilinear forms* on V. As $\mathrm{char}(F) = 2$, each symplectic form is symmetric. If b is a symmetric bilinear form, $\delta(b)$ is additive and hence can be identified with a linear map $V^{(2)} \to F$. Thus there is a short exact sequence:

$$(2) \qquad 0 \longrightarrow \Lambda^2(V^*) \longrightarrow S^2(V)^* \xrightarrow{\delta} V^{(2)*} \longrightarrow 0$$

All of these F-spaces are FG-modules, and the maps are FG-module homomorphisms. It is a singular feature of the characteristic 2-theory that $S^2(V^*)$ and $S^2(V)^*$ need not be isomorphic as FG-modules.

Now let b be a bilinear form on V. We say that b is *G-invariant* if the associated map $v \mapsto b(v, \ )$ for $v \in V$, is an FG-module map $V \to V^*$. We say that b is *nondegenerate* if this map is an F-isomorphism. Taking G-fixed points in (2) we get a long exact sequence of the form

$$0 \longrightarrow \Lambda^2(V^*)^G \longrightarrow S^2(V)^{*G} \xrightarrow{\delta} V^{(2)*G} \longrightarrow H^1(G, \Lambda^2(V^*)) \longrightarrow \dots$$

In particular, if $V^{(2)*G} = 0$, then each G-invariant symmetric bilinear form on V is symplectic. Now the trivial FG-module equals its Frobenius twist. A simple argument then shows:

Lemma 1. *If $V \cong V^*$, and V has no trivial G-submodules, then each G-invariant symmetric bilinear form on V is symplectic.*

We will make use of Fong's Lemma:

Lemma 2. *Let V be an absolutely irreducible non-trivial FG-module. Then $V \cong V^*$ if and only if V affords a nondegenerate G-invariant symplectic bilinear form. In particular* $\dim(V)$ *is even.*

Let $t, h \in G$, with t an involution and h not an involution. Define quadratic forms Q_t and Q_h on FG by setting, for $u = \sum_{g\in G} u_g g \in FG$

$$\begin{aligned} Q_t(u) &= \sum_{\{g,tg\}\subseteq G} u_g u_{tg}, \\ Q_h(u) &= \sum_{g\in G} u_g u_{hg}. \end{aligned} \tag{3}$$

Then each G-invariant quadratic form on FG is a linear combination of Q_t's and Q_h's.

2. Real 2-blocks of defect zero

Assume that G has even order, and that B is a real 2-block of G which has a trivial defect group. Equivalently B is a simple F-algebra which is a σ-invariant $FG \times G$-direct summand of FG. Moreover, B has a unique irreducible K-character χ and a unique simple module S.

Let e_B be the identity element (or block idempotent) of B. Then

$$e_B = e_1 + e_2 + \cdots + e_d,$$

where $d = \dim_F(S)$ and the e_i are pairwise orthogonal primitive idempotents in FG. Each e_iFG is isomorphic to S. In particular S is a projective FG-module.

Let M be an RG-lattice whose character is χ. Then $M/J(R)M \cong S$, as FG-modules. Now M has a quadratic geometry because χ has Frobenius-Schur indicator $+1$. Thus S has a quadratic geometry.

By [2] there exists an involution t in G such that the restriction of the form Q_t of (3) to e_1FG is non-degenerate. It follows that e_1 can be chosen so that $e_1 = e_1^{t\sigma} = te_1^{\sigma}t$. We note that it can be shown that $\langle t\rangle$ is an extended defect group of B and S is a direct summand of $F_{C_G(t)}{\uparrow}^G$.

As $e_B = e_B^{t\sigma}$, we have $e_B = e_1 + e_2^{t\sigma} + \cdots + e_d^{t\sigma}$, and each $e_i^{t\sigma}$ is primitive in FG and $e_1 e_i^{t\sigma} = 0 = e_i^{t\sigma} e_1$, for $i > 1$.

Suppose next that V is a B-module, equipped with a (possibly degenerate) G-invariant symmetric bilinear form $\langle\, ,\, \rangle$. The G-invariance is equivalent to $\langle ux, v\rangle = \langle u, vx^{\sigma}\rangle$, for all $u, v \in V$ and $x \in FG$. Now $e_1e_i = 0$, for $i > 1$. So

$$\langle Ve_1, Ve_i^{\sigma}\rangle = 0, \quad \text{for } i > 1.$$

Following [6], we define a bilinear form b on the F-space Ve_1 by

$$b(ue_1, ve_1) := \langle ue_1, ve_1t\rangle, \quad \text{for all } ue_1, ve_1 \in Ve_1.$$

Then b is symmetric, as

$$b(ue_1, ve_1) = \langle ue_1 t, ve_1 \rangle = \langle ve_1, ue_1 t \rangle = b(ve_1, ue_1).$$

Now consider the radicals of the forms

$$\begin{aligned} \mathrm{rad}(V) \quad &:= \{u \in V \mid \langle u, v \rangle = 0, \forall v \in V\}, \\ \mathrm{rad}(Ve_1) \quad &:= \{ue_1 \in Ve_1 \mid b(ue_1, ve_1) = 0, \forall ve_1 \in Ve_1\}. \end{aligned}$$

We include a proof of Lemma 4.5 of [6] for the benefit of the reader:

Lemma 3. $\mathrm{rad}(Ve_1) = \mathrm{rad}(V)e_1$ *and* $Ve_1/\mathrm{rad}(Ve_1) \cong (V/\mathrm{rad}(V))e_1$.

Proof. Let $u \in \mathrm{rad}(V)$ and $ve_1 \in Ve_1$. Then

$$b(ue_1, ve_1) = \langle ue_1, ve_1 t \rangle = \langle u, ve_1 te_1^\sigma \rangle = 0.$$

So $\mathrm{rad}(Ve_1) \supseteq \mathrm{rad}(V)e_1$. Now let $ue_1 \in \mathrm{rad}(Ve_1)$ and $v \in V$. Writing $v = \sum_{i=1}^d ve_i^\sigma$, we have

$$\langle ue_1, v \rangle = \sum_{i=1}^d \langle ue_1, ve_i^\sigma \rangle = \langle ue_1, ve_1^\sigma \rangle = b(ue_1, vte_1) = 0.$$

So $\mathrm{rad}(Ve_1) \subseteq \mathrm{rad}(V)e_1$. The stated equality follows.

We have an F-vector space map $\phi : Ve_1 \to (V/\mathrm{rad}(V))e_1$ such that $\phi(ve_1) = ve_1 + \mathrm{rad}(V)$. Now $(v + \mathrm{rad}(V))e_1 = ve_1 + \mathrm{rad}(V)$ as $\mathrm{rad}(V)e_1 \subseteq \mathrm{rad}(V)$. So ϕ is onto. Moreover, $\ker(\phi) = \mathrm{rad}(V)e_1$. The stated isomorphism follows from this. □

3. Brauer characters of symmetric groups

Let n be a positive integer. Corresponding to each partition λ of n, there is a Young subgroup Σ_λ of Σ_n and a permutation $R\Sigma_n$-module $M^\lambda := \mathrm{Ind}_{\Sigma_\lambda}^{\Sigma_n}(R_{\Sigma_\lambda})$. This module has a Σ_n-invariant symmetric bilinear form with respect to which the permutation basis is orthonormal. The *Specht lattice* S^λ is a uniquely determined R-free $R\Sigma_n$-submodule of M^λ c.f. [5, 4.3]. Then $S^\lambda \otimes_R K$ is an irreducible $K\Sigma_n$-module and all irreducible $K\Sigma_n$-modules arise in this way.

Now S^λ is usually not a self-dual $R\Sigma_n$-module; the dual module $S_\lambda := S^{\lambda *}$ is naturally isomorphic to $S^{[1^n]} \otimes_R S_R^{\lambda^t}$ where λ^t is the transpose partition to λ. Note that $S^{[1^n]}$ is the 1-dimensional *sign module*.

Set $\overline{S^\lambda} := S^\lambda / JS^\lambda$. Then $\overline{S^\lambda}$ is a Specht module for $F\Sigma_n$. It inherits an Σ_n-invariant symmetric bilinear form $\langle\, ,\, \rangle$ from S^λ. This form is nonzero if and only if λ is 2-regular (i.e. if λ has different parts).

Suppose that λ is 2-regular. Then $D^\lambda := \overline{S^\lambda}/\mathrm{rad}(\overline{S^\lambda})$ is a simple $F\Sigma_n$-module, and all simple $F\Sigma_n$-modules arise uniquely in this way. The D^λ are evidently self-dual. Indeed, $\langle\, ,\, \rangle$ induces a nondegenerate form on D^λ, which by Fong's Lemma is symplectic if D^λ is non-trivial.

Note that $\overline{S^{[1^n]}}$ is the trivial $F\Sigma_n$-module, as $\mathrm{char}(F) = 2$. It follows that the dual of a Specht module in characteristic 2 is a Specht module:

$$\overline{S_\lambda} \cong \overline{S^{\lambda^t}}.$$

Let B be a 2-block of Σ_n. Then B is determined by an integer *weight* w such that $n-2w$ is a nonnegative triangular number $k(k+1)/2$. The partition $\delta := [k, k-1, \ldots, 2, 1]$ is called the 2-*core* of B. Each defect group of B is Σ_n-conjugate to a Sylow 2-subgroup of Σ_{2w}.

Recall that the 2-*core* of a partition λ is obtained by successively stripping removable domino shapes from λ. We attach to B all partitions of n which have 2-core δ.

Set $m := n-2w$ and identify $\Sigma_{2w}\times\Sigma_m$ with a Young subgroup of Σ_n. Now Σ_m has a 2-block B_δ of weight 0 and 2-core δ. This block is real and has a trivial defect group. Moreover, S_K^δ is the unique irreducible $K\Sigma_m$-module in B_δ and $D^\delta = \overline{S^\delta}$ is the unique simple B_δ-module. It is important to note that D^δ is a projective $F\Sigma_m$-module and every $F\Sigma_m$-module in B_δ is semi-simple.

Let e_δ be the block idempotent of B_δ. Following Section 2, choose an involution $t \in \Sigma_m$ and a primitive idempotent e_1 in $F\Sigma_m$ such that $e_1 = e_1 e_\delta$ and $e_1^{t\sigma} = e_1$. Note that $\dim_F(D^\delta e_1) = 1$.

Let μ be a 2-regular partition in B. Regard $V := \overline{S^\mu} e_\delta$ as an $F\Sigma_{2w}\times\Sigma_m$-module by restriction. Then Ve_1 is an $F\Sigma_{2w}$-module, as the elements of Σ_{2w} commute with e_1. Indeed

$$V \cong Ve_1 \otimes_F D^\delta \quad \text{as } F\Sigma_{2w}\times\Sigma_m\text{-modules.}$$

Now $\overline{S^\mu}$ and hence V affords a $\Sigma_{2w}\times\Sigma_m$-invariant symmetric bilinear form $\langle\,,\,\rangle$ such that $V/\mathrm{rad}(V) = D^\mu e_\delta$. It then follows from Lemma 3 that we may use the identity $e_1^{t\sigma} = e_1$ to construct a symmetric bilinear form b on Ve_1. Moreover, $Ve_1/\mathrm{rad}(Ve_1) \cong D^\mu e_1$. So the $F\Sigma_{2w}$-module $D^\mu e_1$ inherits a nondegenerate symmetric bilinear form b. Reviewing the construction of b from $\langle\,,\,\rangle$, we see that b is Σ_{2w}-invariant (as $t\in\Sigma_m$ commutes with all elements of Σ_{2w}, and $\langle\,,\,\rangle$ is Σ_n-invariant).

Lemma 4. *Suppose that $\mu \neq [k+2w, k-1, \ldots, 2, 1]$. Then $D^\mu e_1$ affords a non-degenerate Σ_{2w}-invariant symplectic bilinear form.*

Proof. In view of Lemma 1 and the discussion above, it is enough to show that $D^\mu e_1$ has no trivial $F\Sigma_{2w}$-submodules. Suppose otherwise. Then $F_{\Sigma_{2w}} \otimes_F D^\delta$ is a submodule of the restriction of D^μ to $\Sigma_{2w}\times\Sigma_m$. But D^μ is a submodule of $\overline{S_\mu}$. So D^δ is a submodule of $\mathrm{Hom}_{F\Sigma_{2w}}(F_{\Sigma_{2w}}, \overline{S_\mu})$ as $F\Sigma_m$-modules.

We have F-isomorphisms

$$\begin{aligned}\mathrm{Hom}_{F\Sigma_{2w}}(F_{\Sigma_{2w}}, \overline{S_\mu}) &\cong \mathrm{Hom}_{F\Sigma_n}(M^{[2w,1^m]}, \overline{S_\mu}), \quad \text{by Eckmann-Shapiro}\\ &\cong \mathrm{Hom}_{F\Sigma_n}(\overline{S^\mu}, M^{[2w,1^m]}), \quad \text{as } M^{[2w,1^m]} \text{ is self-dual.}\end{aligned}$$

As μ is 2-regular, it follows from [5, 13.13] that $\mathrm{Hom}_{F\Sigma_n}(\overline{S^\mu}, M^{[2w,1^m]})$ has a basis of semistandard homomorphisms. The argument of Theorem 4.5 of [4] now applies, and shows that

$$\mathrm{Hom}_{F\Sigma_{2w}}(F_{\Sigma_{2w}}, \overline{S_\mu}) \cong \overline{S^{\mu^t\backslash[1^{2w}]}} \quad \text{as } F\Sigma_m\text{-modules.}$$

Here $\mu^t\backslash[1^{2w}]$ is a skew-partition of m; it is empty if $\mu_1 < 2w$ (in which case $\mathrm{Hom}_{F\Sigma_{2w}}(F_{\Sigma_{2w}}, \overline{S_\mu}) = 0$). Otherwise its diagram is the set of nodes in the Young diagram of μ^t not in the top $2w$ rows of the first column. Now $\overline{S^{\mu^t\backslash[1^{2w}]}}$ has an $F\Sigma_m$-submodule isomorphic to D^δ if and only if $S_K^{\mu^t\backslash[1^{2w}]}$ has an $K\Sigma_m$-submodule isomorphic to S_K^δ, as $D^\delta = \overline{S^\delta}$, and using the projectivity of D^δ.

The multiplicity of S_K^δ in $S_K^{\mu^t\backslash[1^{2w}]}$ is the number of $\mu\backslash[2w]$-tableau of type $\delta^t = \delta$ which are strictly increasing along rows and nondecreasing down columns. Suppose for the sake of contradiction that such a tableau T exists.

We claim that $\mu_i \leq k-i+2$ for $i = 2, \ldots, k$, and $\mu_i = 0$ for $i > k+1$. This is true for $i = 2$, as the entries in the second row of T are different. Suppose that $i \geq 2$ and $\mu_{i-1} \leq k-i+3$. But $\mu_i < \mu_{i-1}$, as μ is 2-regular. So $\mu_i \leq k-i+2$, proving our claim.

On the other hand, $\mu_i \geq \delta_i = k-i+1$, for $i = 1, \ldots, k$, as μ has 2-core δ. It follows that $\mu\backslash\delta$ consists of the last $\mu_1 - k$ nodes in the first row of μ, and a subset of the nodes $(2,k), (3,k-1), \ldots, (k,2), (k+1,1)$. On the other hand, μ has 2-core δ. So $\mu\backslash\delta$ is a union of domino shapes. It follows that T does not exist if $\mu \neq [k+2w, k-1, \ldots, 2, 1]$. This contradiction completes the proof of the Lemma. □

Suppose that G is a finite group and that B is a 2-block of G with defect group $P \leq G$. Then it is known that $[G:P]_2$ divides the degree of every irreducible Brauer character in B. Recall that a Brauer character in B has *height zero* if the 2-part of its degree is $[G:P]_2$. We now prove the main result of [6].

Theorem 5. *Let B be a 2-block of Σ_n. Then B contains a unique irreducible Brauer character of height 0.*

Proof. Suppose as above that B has weight w and 2-core δ, and let θ be a height zero irreducible Brauer character in B. Then θ is the Brauer character of D^μ for some 2-regular partition μ of n belonging to B.

Let P be a vertex of D^μ. Then P is a defect group of B. We may assume that P is a Sylow 2-subgroup of Σ_{2w}. It is easy to show that $N_{\Sigma_n}(P) = P \times \Sigma_m$, a subgroup of $\Sigma_{2w} \times \Sigma_m$.

Let B_0 denote the principal 2-block of Σ_{2w}. Then $B_0 \otimes B_\delta$ is the Brauer correspondent of B with respect to $(\Sigma_n, P, \Sigma_{2w} \times \Sigma_m)$. So the Green correspondent of D^μ with respect to $(\Sigma_n, P, \Sigma_{2w} \times \Sigma_m)$ has the form $U^\mu \otimes D^\delta$, where U^μ is an indecomposable Σ_{2w}-direct summand of $D^\mu e_1$ which belongs to B_0. Moreover, U^μ is the unique component of $D^\mu e_1$ that has vertex P.

If $\mu = [k+2w, k-1, \ldots, 2, 1]$ it can be shown that U^μ is the trivial $F\Sigma_{2w}$-module. Suppose that $\mu \neq [k+2w, k-1, \ldots, 2, 1]$. Lemma 4 implies that $D^\mu e_1$ has a symplectic geometry. It then follows from the first proposition in [3] that U^μ has a symplectic geometry. In particular $\dim(U^\mu)$ is even.

Now the 2-part of $\dim(U^\mu \otimes D^\delta)$ divides $2|\Sigma_m|_2 = 2[\Sigma_n : P]_2$. A standard result on the Green correspondence implies that the 2-part of $\dim(D^\mu)$ divides $2[\Sigma_n : P]_2$. This contradicts the assumption that θ has height zero, and completes the proof. □

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