GENERATING FUNCTIONS ASSOCIATED WITH 2-RESIDUES

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1. Mathoverflow Question

Express the following power series in 2-variables x, y as an infinite product, or a short sum of infinite products:

$$\frac{P(xy)^2}{(1-x)} \sum_{k=-\infty}^{\infty} (2k+1) x^{k^2} y^{k^2+k}.$$

Does it have any special properties e.g. automorphic form?

Motivation

The *2-residue* of an integer node (x, y) in the plane is $y - x$ mod 2. So the 2-residues alternate as $0, 1$ in a checkerboard pattern. Let λ be a partition. Its *Young diagram* $[\lambda]$ consists of a set of nodes in the plane. A node in the plane is an *addable node* of λ if it can be adjoined to $[\lambda]$ to give a partition (of $|\lambda| + 1$).

Now define, for $i = 0, 1$:

(i) $c_i(\lambda)$ is the number of nodes in $[\lambda]$ with 2-residue i.
(ii) $a_i(\lambda)$ is the number of addable nodes of λ with 2-residue i.

Then my power series is the generating function of

$$\sum_{\lambda} a_0(\lambda) x^{c_0(\lambda)} y^{c_1(\lambda)}$$

Here λ ranges over all partitions. I'll leave it as an exercise to work out the *complementary* generating function $\sum_\lambda a_1(\lambda) x^{c_0(\lambda)} y^{c_1(\lambda)}$ from the first.

The coefficient of a given monomial $x^a y^b$ is the dimension of a certain algebra, naturally associated to the symmetric group S_{a+b}, defined in characteristic 2.

Other Information

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Using the Jacobi Triple Product identity I can factorize the generating function for the 2-residues of partitions as

$$\begin{aligned}\textstyle\sum_\lambda x^{c_0(\lambda)} y^{c_1(\lambda)} &= P(xy)^2 \sum_{k=-\infty}^{\infty} x^{k^2} y^{k^2+k} \\ &= \prod_{i=1}^{\infty} \tfrac{(1+x^{2i-1}y^{2i})(1+x^{2i-1}y^{2i-2})(1+x^i y^i)}{(1-x^i y^i)}.\end{aligned}$$

Experts on the modular representation of the symmetric group will understand the significance of the left hand side and the first equality.

The generating function I'm interested in can be got using partial differentiation from this.

Also if we set $x = y$ in the original, standard results give:

$$\begin{aligned}\textstyle\sum_\lambda a_0(\lambda) x^{|\lambda|} &= \tfrac{1}{2(1-x)} \tfrac{P(x)^4 + P(x^2)^2}{P(x)^3} \\ \textstyle\sum_\lambda a_1(\lambda) x^{|\lambda|} &= \tfrac{1}{2(1-x)} \tfrac{P(x)^4 - P(x^2)^2}{P(x)^3}\end{aligned}$$

2. Background

The aim of this note is to provide the representation theory and combinatorial background for Question 8 below. Many results are well-known. Others have been stated without proof. We apologise to the reader for omitting most references and proofs at this stage!

The partition generating function $P(x) = \sum_{n \geq 0} p_n x^n$, where p_n is the number of partitions of n. It can be expressed as the infinite product

$$P(x) = \prod_{i=1}^{\infty} (1 - x^i)^{-1}.$$

We identify a partition $\lambda = [\lambda_1, \ldots, \lambda_l]$ with its Ferrer's diagram in $\mathbb{R}^2$ (oriented in the Anglo-American way i.e. the x-axis from left-to-right, the y-axis from top-to-bottom). The *nodes* of λ are the integer points (i, j) with $1 \leq j \leq l$ and $1 \leq i \leq \lambda_j$. The *content* of (i, j) is the integer $c(i, j) := j - i$. If e is a non-zero integer, the *e-residue* of (i, j) is $r(i, j) := j - i$ modulo e (suppressing e in the notation).

The *rim* $\mathrm{R}(\lambda)$ of λ is the piecewise linear curve going from ∞ on the y-axis up to l, then to (λ_l, l) and successively from (λ_j, j) to $(\lambda_j, j-1)$ to $(\lambda_{j-1}, j-1)$, for $j > 1$, then to λ_1 on the x-axis, and continuing along the x-axis to ∞.

If (i, j) is a point in $\mathrm{R}(\lambda)$ with integer coordinates, then either the vertical line segment $[(i, j), (i, j+1)]$ or the horizontal segment $[(i, j), (i+1, j)]$ is contained in $\mathrm{R}(\lambda)$. In either case, we label the segment by $c(i, j) = j - i$. This establishes a bijection between $\mathrm{R}(\lambda)$ and

$\mathbb{Z}$. The labels of the vertical segments form the infinite sequence

$$\beta_1 := \lambda_1 - 1, \ldots, \beta_l := \lambda_l - l, \beta_{l+1} := -l-1, \beta_{l+2} := -l-2, \ldots.$$

The β_i are an instance of a set of *β-numbers* for λ.

Example: $\lambda = [3,2,1,1]$ has β-numbers $[2,0,-2,-3,-5,-6,-7,\ldots]$

Consider an *abacus* with e-runners, labelled from left to right by $0,1,\ldots,e-1$ modulo e. The positions on the runner i are labelled from top to bottom by the integers congruent to i mod e. Position $en+i$ occurs immediately to the left of position $en+i+1$, for $i \neq e-1$. Placing beads at the positions labelled by its β-numbers gives a e-abacus representation of a partition λ. A partition is an *e-core* if there are no spaces above beads in its e-abacus representation. Note that

Lemma 1. *For $n \geq 0$, set $C_e(n)$ as the number of e-cores of n. Then*

$$\sum_{n=0}^{\infty} C_e(n)x^n \;=\; P(x)P(x^e)^{-e} = \prod_{i=1}^{\infty} \frac{(1-x^{ie})^e}{(1-x^i)}.$$

Lemma 2. *The 2-cores are the triangular partitions $[n, n-1, \ldots, 1]$, for $n \geq 0$, of the triangular numbers $t_n := n(n+1)/2$.*

Now if $k \in \mathbb{Z}$, note that

$$2k^2 + k = \begin{cases} t_{2k}, & \text{if } k \geq 0. \\ t_{-2k-1}, & \text{if } k < 0. \end{cases}$$

So there is a 1-1 correspondence between integers and 2-cores:

$$T_k := \begin{cases} [\;2k, \quad 2k-1, \ldots \quad, 1], & \text{if } k \geq 0. \\ [-2k-1, \quad -2k-2 \ldots \quad, 1], & \text{if } k < 0. \end{cases}$$

Note that the difference in the number of beads on runner 1 and runner 0 of T_k is always $2k$.

Consider the e-abacus representation of λ. Pushing all beads as far up the runners as possible, we get the e-abacus representation of an e-core. This is said to be the *e-core of* λ. The number of bead moves required to produce this e-core is called the *e-weight* of λ.

3. 2-residues

Set $c_i = c_i(\lambda)$ as the number of nodes in λ that have 2-residue i, for $i = 0, 1$. We can describe the 2-weight and 2-core of λ in terms (c_0, c_1).

Lemma 3. *λ has 2-weight $w = w(\lambda) = c_0 - (c_0 - c_1)^2$ and 2-core T_k, where $k = c_1 - c_0$.*

More generally, we have

Lemma 4. *Let λ be a partition and let $e > 0$. Suppose that λ has c_i nodes of residue i modulo e, for $i = 0, \ldots, e-1$. Then λ has e-weight*

$$w = c_0 - \frac{1}{2} \sum_{i \in \mathbb{Z}_e} (c_i - c_{i+1})^2.$$

An *e-block* of a positive integer n consists of the set of all partitions of n that have a fixed e-tuple $(c_0, \ldots, c_{e-1})$ of e-residue numbers. Write $P(x)^e = \sum_{w=1}^{\infty} n_w x^w$. It is known that n_w is equal to the number of partitions in an e-block of weight w.

Example: if $e = 2$ its easy to see that $n_0 = 1, n_1 = 2, n_2 = 5, n_3 = 10$.

Corollary 5. *Let $e = 2$. Then*

$$\sum_{\lambda} x^{c_0(\lambda)} y^{c_1(\lambda)} = P(xy)^2 \sum_{k=-\infty}^{\infty} x^{k^2} y^{k^2+k}.$$

Proof. The coefficient of $x^{c_0} y^{c_1}$ on the lhs is the number n_w of partitions (of $c_0 + c_1$) in a 2-block of weight $w = c_0 - (c_0 - c_1)^2$. On the rhs, $x^{c_0} y^{c_1}$ occurs only in the monomial $x^{k^2} y^{k^2+k} x^w y^w$ with $k = c_1 - c_0$ and w as before. The weight of this monomial is n_w also. □

Let $a_i = a_i(\lambda)$ be the number of addable nodes of λ that have e-residue i, and let $r_i = r_i(\lambda)$ be the number of removable nodes of λ that have e-residue i.

Lemma 6. *For $e = 2$ and any partition λ we have*

$$(a_0 - r_0) = 2(c_1 - c_0) + 1, \quad (a_1 - r_1) = -2(c_1 - c_0).$$

Proof. Use the 2-abacus representation of λ, noting that the 2-core T_k, where $k = (c_1 - c_0)$ has $2k$ more beads on runner 1 than runner 0. □

Corollary 7. *For $e = 2$ we have*

$$\sum_{\lambda} a_0(\lambda) x^{c_0(\lambda)} y^{c_1(\lambda)} = P(xy)^2 (1-x)^{-1} \sum_{k=-\infty}^{\infty} (2k+1) x^{k^2} y^{k^2+k}.$$

$$\sum_{\lambda} a_1(\lambda) x^{c_0(\lambda)} y^{c_1(\lambda)} = P(xy)^2 (1-y)^{-1} \sum_{k=-\infty}^{\infty} -2k x^{k^2} y^{k^2+k}.$$

Question 8. *Express the right hand sides of each of the two power series in the above Corollary as infinite products.*

4. Dropping down a dimension

Dropping down a dimension, we obtain the generating functions for addable nodes of 2-residue 0 and 1.

Proposition 9.

$$\sum_\lambda a_0(\lambda)x^{|\lambda|} = \frac{1}{2(1-x)}\left[P(x) + \frac{P(x^2)^2}{P(x)^3}\right],$$
$$\sum_\lambda a_1(\lambda)x^{|\lambda|} = \frac{1}{2(1-x)}\left[P(x) - \frac{P(x^2)^2}{P(x)^3}\right].$$

Proof. The generating function $\sum_\lambda a(\lambda)x^{|\lambda|}$ for addable nodes is known to be $\frac{1}{(1-x)}P(x)$. So we only need to prove the identity for $a_0(\lambda)$. Setting $y = x$ in Corollary 7, we have

$$\sum_\lambda a_0(\lambda)x^{|\lambda|} = P(x^2)^2(1-x)^{-1}\sum_{k=-\infty}^{\infty}(2k+1)x^{2k^2+k}. \tag{1}$$

Let k be an indice that runs through all integers and define

$$m := \begin{cases} 2k, & \text{if } k \geq 0. \\ -2k-1, & \text{if } k < 0. \end{cases}$$

Then m runs through all non-negative integers and

$$2k^2 + k = \frac{m(m+1)}{2}. \tag{2}$$

It can be checked that $4k+1 = (-1)^m(2m+1)$. So

$$2k+1 = \frac{1}{2}\left((-1)^m(2m+1)+1\right). \tag{3}$$

Now (2) and (3) imply that

$$\sum_{k=-\infty}^{\infty}(2k+1)x^{2k^2+k} = \frac{1}{2}\left(\sum_{m=0}^{\infty}x^{m(m+1)/2} + \sum_{m=0}^{\infty}(-1)^m(2m+1)x^{m(m+1)/2}\right)$$
$$= \frac{1}{2}\left(P(x)P(x^2)^{-2} + P(x)^{-3}\right),$$

the last equality following from classical identities of Gauss and Jacobi. The first identity of the Proposition follows from this and (1). □

5. Connections to the Jacobi Triple Product identity

The famous Jacobi triple product identity asserts that:

$$\prod_{i=1}^{\infty}(1+st^i)(1+s^{-1}t^{i-1})(1-t^i) = \sum_{k=-\infty}^{\infty} s^k t^{k(k+1)/2}. \tag{4}$$

We use $T(s,t)$ to denote the left hand infinite product above. Applying the change of variables $s = x^{-1}$ and $t = x^2y^2$ this becomes:

$$\prod_{i=1}^{\infty}(1+x^{2i-1}y^{2i})(1+x^{2i-1}y^{2i-2})(1-x^{2i}y^{2i}) = \sum_{k=-\infty}^{\infty} x^{k^2}y^{k^2+k}. \tag{5}$$

Using Corollary 5 and (5), the generating function for the 2-residues of 2-blocks can be expressed as:

$$\begin{aligned}\sum_{\lambda} x^{r_0(\lambda)}y^{r_1(\lambda)} &= P^2(xy)\sum_{k=-\infty}^{\infty} x^{k^2}y^{k^2+k} \\ &= \prod_{i=1}^{\infty}(1+x^{2i-1}y^{2i})(1+x^{2i-1}y^{2i-2})\frac{(1+x^ry^r)}{(1-x^ry^r)}.\end{aligned}$$

We now *fail* to compute the partial derivative with respect to s of both sides of (4), and then multiply through by s. The right hand side is:

$$\sum_{k=-\infty}^{\infty} ks^kt^{k(k+1)/2} = \sum_{k=-\infty}^{\infty} kx^{k^2}y^{k^2+k},$$

on applying the substitution $s = x^{-1}, t = x^2y^2$, as before. The importance of the right hand side of this expression is clear from Corollary 7. Now we compute:

$$s\frac{\partial T}{\partial s} = \left(\sum_{i=1}^{\infty}\frac{st^i}{1+st^i} - \sum_{j=0}^{\infty}\frac{s^{-1}t^j}{1+s^{-1}t^j}\right)T$$

Now the expression in brackets on the right hand side equals

$$\sum_{i=1}^{\infty}\sum_{u=1}^{\infty}(-1)^{u-1}s^ut^{iu} + \sum_{j=0}^{\infty}\sum_{v=1}^{\infty}(-1)^v s^{-v}t^{jv}.$$

This evaluates to

$$\sum_{u=1}^{\infty}\frac{(-1)^{u-1}s^ut^u}{1-t^u} + \sum_{v=1}^{\infty}\frac{(-1)^v s^{-v}}{1-t^v} = \sum_{u=-\infty, u\neq 0}^{\infty}\frac{(-1)^u s^{-u}}{1-t^u}.$$

Substituting $s = x^{-1}, t = x^2y^2$, this becomes

$$\sum_{u=1}^{\infty} \frac{(-1)^{u-1}x^u y^{2u}}{1 - x^{2u}y^{2u}} + \sum_{v=1}^{\infty} \frac{(-1)^v x^v}{1 - x^{2v}y^{2v}}.$$

6. Odd Partitions

Let $\mathcal{O}$ be the partitions with all parts odd. The generating function for odd partitions can be given in various ways:

$$\begin{aligned} O(x) = \sum_{\lambda \in \mathcal{O}} x^{|\lambda|} &= \prod_{n=1}^{\infty} (1 - x^{2n-1})^{-1} \\ &= D(x) = \prod_{n=1}^{\infty} (1 + x^n) \\ &= \sum_{k=0}^{\infty} x^k \prod_{i=1}^{k} (1 - x^{2i})^{-1}. \end{aligned}$$

The generating function for partitions of n with all parts odd and distinct is

$$OD(x) = \prod_{n=1}^{\infty} (1 + x^{2n-1}) = \frac{P(x)P(x^4)}{P(x^2)^2}.$$

Now Olsson has noted that a p-block of weight w has coefficient of x^w in $P(x)^{p-1}$ irreducible modules. Also by Brauer, the number of irreducible S_n-modules is equal to the number of p-regular classes of S_n. Thus

$$O(x) = P(x^2) \sum_{k=-\infty}^{\infty} x^{2k^2+k}.$$

Now all odd partitions λ of n belong to the principal 2-block of n, as $c_0(\lambda) = \lfloor (n+1)/2 \rfloor$ and $c_1(\lambda) = \lfloor (n-1)/2 \rfloor$. It follows from this that

$$O_0(x) := \sum_{\lambda \in \mathcal{O}, |\lambda| \in 2\mathbb{Z}} x^{c_0(\lambda)} y^{c_1(\lambda)} = P(xy) \sum_{k=-\infty}^{\infty} x^{4k^2+k} y^{4k^2+k}$$

$$O_1(x) := \sum_{\lambda \in \mathcal{O}, |\lambda| \notin 2\mathbb{Z}} x^{c_0(\lambda)} y^{c_1(\lambda)} = P(xy) \sum_{k=-\infty}^{\infty} x^{4k^2-3k+1} y^{4k^2-3k}$$

Now Hardy & Wright give the following special cases of Jacobi's identity:

$$\prod_{n=1}^{\infty}(1+x^{8n-5})(1+x^{8n-3})(1-x^{8n}) = \sum_{k=-\infty}^{\infty} x^{4k^2+k}$$
$$\prod_{n=1}^{\infty}(1+x^{8n-7})(1+x^{8n-1})(1-x^{8n}) = \sum_{k=-\infty}^{\infty} x^{4k^2-3k}$$

These and the previous paragraph give

$$O_0(x) = \prod_{n=1}^{\infty}\frac{(1+x^{8n-5})(1+x^{8n-3})(1-x^{8n})}{(1-x^n)}$$
$$O_1(x) = x\prod_{n=1}^{\infty}\frac{(1+x^{8n-7})(1+x^{8n-1})(1-x^{8n})}{(1-x^n)}$$

We can interpret these functions as follows. Recall that a regular McMahon diagram is a pair (λ, R), where λ is a partition, and R is a subset of the part-lengths of λ - called marked parts. Then there is a bijection between the odd partitions of an even integer $2n$ and the regular McMahon partitions of n which have no parts divisible by 8, where only parts congruent to 3 or 5 modulo 8 can be marked. Likewise, there is a bijection between the odd partitions of an odd integer $2n+1$ and the regular McMahon partitions of n which have no parts divisible by 8, where only parts congruent to 1 or 7 modulo 8 can be marked.

7. Derivative of $P(x)$

We end this note by computing the derivative of $P(x)$. This may or may not be relevant.

For $n > 0$ set $\sigma(n)$ as the sum of all positive divisors of n.

Lemma 10. $P'(x) = P(x)\sum_{n=0}^{\infty}\sigma(n+1)x^n$.

Proof. From Euler's formula for $P(x)$ we get

$$P'(x) = \sum_{i=1}^{\infty}(P(x)(1-x^i))ix^{i-1}(1-x^i)^{-2} = P(x)\sum_{i=1}^{\infty} i\sum_{j=1}^{\infty}x^{ij-1}.$$

For a given n, the integer i contributes i to the coefficient of x^n if $ij-1 = n$, and 0 otherwise. So the coefficient of x^n in $P'(n)$ is $\sigma(n+1)$. $\square$

Note that $P'(x)/P(x)$ is the derivative of $\log P(x)$. Given a power series $C(x) = \sum_{n=0}^{\infty} c_n x^n$, MacDonald has a formula for $\log C(x)$.

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