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p-Length and Character Heights in Blocks of p-Solvable Groups

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Abstract

We give the lower bounds 2^ℓ and $2^{\ell-2}+2^{\ell-4}+\dots$ for the number of irreducible characters of height zero and positive height, respectively, in a p-block of a p-solvable groups, where ℓ is the p-length of an associated p-solvable group. We also prove some results on extensions of linear characters in p-subgroups of p-solvable groups.

1 Introduction

Suppose that G is a finite p-solvable group, where p is a prime integer. According to [2] if B is a p-block of G then there is an irreducible character θ of $O_{p'}(G)$ such that all irreducible characters in B lie over θ. Both θ and its inertia group G_θ in G are determined by B up to G-conjugacy. For example, the irreducible characters in the principal p-block of G are the inflations of the irreducible characters of $G/O_{p'}(G)$. So they lie over the trivial character of $O_{p'}(G)$.

Recently, the second author proved that if G has p-length ℓ, then G has at least 2^ℓ irreducible characters of degree coprime to p which take values in the cyclotomic field $\mathbb{Q}_p$, obtained by adjoining a primitive p-th root of unity to $\mathbb{Q}$ (see [16] for the case $p = 2$ and [17] for the general bound). The characters constructed in the proof of this result are of a special nature: they are p'-degree irreducible constituents of the principal projective indecomposable character Φ_{1_G} of G (see Theorem 2.2 of [1]). In particular they belong to the principal p-block of G. Such characters play an important role in the work of I. M. Isaacs and G. Navarro [12].

In this note, we generalize the results of [16] and [17] to all p-blocks which are weakly regular with respect to $O_{p'}(G)$. Recall that a p-block is real if it contains the complex conjugates of its irreducible characters. It is known that every real 2-block has a real irreducible character of height zero. We will show:

Theorem 1. *Let B be a real 2-block of a finite solvable group G that is weakly regular with respect to $O_{2'}(G)$. Suppose that all irreducible characters in B lie over $\theta \in \mathrm{Irr}(O_{2'}(G))$ and that G_θ has 2-length ℓ. Then B has at least 2^ℓ real-valued 2-rational irreducible characters of height zero.*

If χ is a real-valued irreducible character of odd degree of a solvable group, then χ is rational-valued by a result of R. Gow [6]. So Theorem 1 implies the main result of [16].

When p is an odd prime, we obtain the same lower bound 2^ℓ for the number of height-zero characters in any p-block of G which is weakly regular with respect to $O_{p'}(G)$, although we have less control on the field of values of such characters: they take values inside the cyclotomic field $\mathbb{Q}_n$, where $n = p|G|_{p'}$ (see Theorem 5 below). We also give lower bounds for the number of positive height characters in such blocks (see Theorems 4 and 6 below). The latter is an easy consequence of Theorem 5 and the main result of [4].

In [7] R. Gow used an ingenious argument to show that the p'-degree irreducible characters of Φ_{1_G} correspond to families of linear characters of a Sylow p-subgroup of G. In the last section we show that Gow's approach is compatible with the normal structure of G. Making use of these arguments we are able to give a proof of the main result in [17] which does not use Isaacs' π-theory (see Theorem 13 below).

We are also able to give straighforward proofs for some results on extensions of linear characters in p-subgroups of G. Some of these results already appear in [12]. We thank G. Navarro for pointing out how to prove the next result using a theorem of S. Gagola. Our original statement required G to be p-solvable.

Theorem 2. *Suppose that S is a normal subgroup of a finite group G such that S has no proper quotient that is a p-group. Let P be a Sylow p-subgroup of G and let μ be a linear character of $P \cap S$. Then μ has an extension to P if and only if μ is P-invariant.*

Proof. Theorem A of [3], which does not require $S = O^p(S)$, gives

$$\frac{P' \cap S}{[P \cap S, P]} \cong \frac{(PS)' \cap S}{[PS, S]}.$$

The left hand side is a p-group while the right hand side is a subquotient of S/S'. As $S = O^p(S)$, we deduce that $P' \cap S = [P \cap S, P]$.

Now the 'only if' part of the conclusion is obvious. So suppose that μ is a P-invariant linear character of $P \cap S$. Equivalently μ is a linear character of $(P \cap S)/[P \cap S, P]$. Now by the previous paragraph

$$\frac{P \cap S}{[P \cap S, P]} \cong \frac{P'(P \cap S)}{P'}.$$

So we can inflate μ to a character of $P'(P \cap S)/P'$. But $P'(P \cap S)/P'$ is a subgroup of the abelian group P/P'. So μ extends to a linear character of P. The 'if' part of the Theorem follows. ∎

We prove a π-generalization of this theorem for an arbitrary set of primes π, and π-solvable groups in Theorem 10 below. This makes it natural to ask whether Gagola's result may hold for an arbitrary set π of primes and groups containing a Hall π-subgroup, or at least that Theorem 10 is true for any arbitrary group containing a Hall π-subgroup.

2 Characters heights

In this section we prove Theorem 1 and some related results, including a version for odd primes. We start by recalling some well known facts and fixing some notation which we shall use throughout the section. As usual $\mathrm{Irr}(G)$ is the set of complex irreducible characters of G, and if θ is an irreducible character of a subgroup of G then $\mathrm{Irr}(G \mid \theta)$ is the set of irreducible characters of G whose restrictions contains θ. Now $\mathrm{Irr}(G)$ is partitioned by the p-blocks of G c.f. [15, 3.6.4]. We use $\mathrm{Irr}(B)$ to denote the irreducible characters contained in a p-block B. Recall that a defect group of B is a p-subgroup of G, uniquely determined up to G-conjugacy.

Unless otherwise stated G is a finite p-solvable group, B is a p-block of G and N is the largest normal p'-subgroup $O_{p'}(G)$ of G. As already mentioned, there is $\theta \in \mathrm{Irr}(N)$ such that all irreducible characters of B lie over θ. By the Fong-Reynolds Theorem [15, 5.5.10] there exists a unique p-block β of the inertia group G_θ such that the irreducible characters of β are precisely the Clifford correspondents of the irreducible characters of B with respect to θ. In particular induction of characters defines a height-preserving bijection between $\mathrm{Irr}(B)$ and $\mathrm{Irr}(\beta)$.

Now $\mathrm{Irr}(\beta) \subseteq \mathrm{Irr}(G_\theta \mid \theta_1)$ for some irreducible character θ_1 of $O_{p'}(G_\theta)$. Moreover each defect group of β is a defect group of B. Suppose that a defect group of β is a Sylow p-subgroup of G_θ. Then B is said to be *weakly regular* with respect to N. In that case θ_1 is fixed by some Sylow p-subgroup of G_θ and $\mathrm{Irr}(\beta) = \mathrm{Irr}(G_\theta \mid \theta_1)$, by Theorem (1E) of [2].

Fields of values of characters can be controlled by the action of suitable Galois groups on characters. For this reason, it is convenient for us to consider a p-group Q acting on the irreducible characters of each characteristic subgroup of G_θ in such a way that Q preserves the determinantal orders of characters (as Galois action does). Assume also that θ is Q-invariant. Furthermore, suppose that Q contains a normal subgroup D which is a Sylow p-subgroup of G_θ, such that the action of D on characters is given by conjugation in G_θ.

We consider the following normal series for G

$$G = L_0 \trianglerighteq M_1 \trianglerighteq L_1 \trianglerighteq \ldots \trianglerighteq M_\ell \trianglerighteq L_\ell \trianglerighteq N, \tag{1}$$

where $M_i/N = O^{p'}(L_{i-1}/N)$ and $L_i/N = O^p(M_i/N)$ for $i \geq 1$. Note that since N has p'-order $O^p(M_i/N) = O^p(M_i)/N$ and $L_i = O^p(M_i)$. In particular $O^p(L_i) = L_i$, for $i \geq 1$. By definition, the *p-length* of G/N is the smallest integer ℓ such that $M_{l+1} \leq N$. Now $L_\ell/M_{\ell+1}$ and $M_{\ell+1}$ have p'-order and $N \leq L_\ell \trianglelefteq G$. So $L_\ell = N$. Again because N has p'-order, the p-length of G/N coincides with the p-length of G.

Proposition 3. *Let B be a p-block of G that is weakly regular with respect to N. Suppose that all irreducible characters in B lie over $\theta \in \mathrm{Irr}(N)$ and that G_θ has p-length ℓ. Then B has at least 2^ℓ height-zero Q-invariant irreducible characters, where Q is as above.*

Proof. By the discussion above we can assume that B has a defect group D which is a Sylow p-subgroup of G, that $\mathrm{Irr}(B) = \mathrm{Irr}(G \mid \theta)$ and that θ is D-invariant.

Let $i \in \{0, \dots, \ell\}$. We use backwards induction on i to prove that G has at least $2^{\ell-i}$-orbits on $\mathrm{Irr}(L_i \mid \theta)$ containing a Q-invariant character of p'-degree. The base case $i = \ell$ is trivial as $L_\ell = N$ has p'-order and θ is Q-invariant. So suppose that $1 \leq i \leq \ell$.

Suppose that $\chi \in \mathrm{Irr}(L_i \mid \theta)$ has p'-degree and is Q-invariant. Then χ is invariant in M_i because $M_i/L_i \leq DL_i/L_i$. Moreover $o(\chi)$ is coprime to p as $L_i = O^p(L_i)$. As $\gcd(\chi(1)o(\chi), [M_i : L_i]) = 1$, Gallagher's Theorem (6.28 in [10]) implies that there exists a unique extension ψ of χ to M_i such that $o(\psi) = o(\chi)$. Our assumptions on Q imply that ψ is Q-invariant.

Let ψ have inertia group I_i in L_{i-1}. Then

$$\sum_{\alpha \in \mathrm{Irr}(L_{i-1})} \langle \alpha, \psi^{L_{i-1}} \rangle^2 = \frac{\psi^{L_{i-1}}(1)}{[L_{i-1} : I_i]\psi(1)} = [I_i : M_i]. \tag{2}$$

As $p \nmid [I_i : M_i]$ we may choose a Q-invariant $\chi_1 \in \mathrm{Irr}(L_{i-1} \mid \psi)$.

Observe that the p-group Q acts on the linear characters of the p-group M_i/L_i. These characters form a non-trivial p-group, with a Q-invariant identity element (the trivial character). So we can choose a non-trivial Q-invariant linear character $\delta_i \in \mathrm{Irr}(M_i/L_i)$. Now $\delta_i\psi$ is also Q-invariant. So the arguments above show that there exists a Q-invariant $\chi_2 \in \mathrm{Irr}(L_{i-1} \mid \delta_i\psi)$,

We claim that χ_1 and χ_2 are not conjugate in G. For the irreducible constituents of $(\chi_1)_{M_i}$ have p'-determinantal order $o(\psi) = o(\chi)$. On the other hand, $\det(\delta_i\psi) = \delta_i^{\psi(1)} \det\psi$. So the irreducible constituents of $(\chi_2)_{M_i}$ have determinantal order $o(\delta_i)o(\chi)$, which is divisible by p. The claim now follows.

Now by induction we may assume that G has at least $2^{\ell-i}$-orbits on $\mathrm{Irr}(L_i \mid \theta)$ containing a Q-invariant character of p'-degree. The work above implies that G has at least $2^{\ell-(i-1)}$-orbits on $\mathrm{Irr}(L_{i-1} \mid \theta)$ containing a Q-invariant character of p'-degree. This completes the inductive step and the result follows. ∎

We use Proposition 3 to prove Theorem 1. Set $\mathcal{G} := \mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q})$ where $\mathbb{Q}_{|G|}$ is the field obtained by adjoining a primitive $|G|$-th root of unity to $\mathbb{Q}$. The characters of subgroups of G take values inside $\mathbb{Q}_{|G|}$. Now $\mathcal{G}$ acts naturally on $\mathrm{Irr}(K)$ for each $K \leq G$ via

$$\theta^\tau(x) = \theta(x)^\tau$$

for all $\tau \in \mathcal{G}$, $\theta \in \mathrm{Irr}(K)$ and $x \in K$. Write $|G| = p_1^{a_1} \cdots p_t^{a_t}$, where $p_1, \ldots, p_t$ are the distinct prime divisors of $|G|$. Then

$$\mathcal{G} \cong \mathrm{Gal}(\mathbb{Q}_{p_1^{a_1}}/\mathbb{Q}) \times \cdots \times \mathrm{Gal}(\mathbb{Q}_{p_t^{a_t}}/\mathbb{Q}).$$

Recall that $\mathrm{Gal}(\mathbb{Q}_{2^a}/\mathbb{Q})$ is isomorphic to $C_2 \times C_{2^{a-2}}$ and $\mathrm{Gal}(\mathbb{Q}_{p^a}/\mathbb{Q})$ is cyclic of order $(p-1)p^{a-1}$ if p is an odd prime.

Suppose that $p = 2$ and the 2-block B in the statement of Proposition 3 is real. Then by Theorem 5.1 of [5] B contains a real irreducible character φ. With the notation of Theorem 1, it follows from Clifford Theory that $\overline{\theta} = \theta^t$ for some $t \in G$ (because φ lies over both θ and $\overline{\theta}$). It is easy to see that t normalizes G_θ and $t^2 \in G_\theta$. Replacing t by a generator of the Sylow 2-subgroup of $\langle t \rangle$, we may assume that t has order a power of 2. For each characteristic subgroup K of G_θ define a map σ on $\mathrm{Irr}(K)$ by

$$\eta^\sigma = \overline{\eta^t}, \quad \text{for } \eta \in \mathrm{Irr}(K).$$

Then σ fixes θ and σ^2 is induced by the conjugation action of $t^2 \in G_\theta$.

Let E be a Sylow 2-subgroup of the group $G_\theta\langle t \rangle$ that contains t and set $D := E \cap G_\theta$. Then $D\langle\sigma\rangle$ is a group acting on each $\mathrm{Irr}(K)$ and we can consider the external direct product

$$Q := D\langle\sigma\rangle \times \mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q}_{|G|_{2'}}).$$

Then Q is a 2-group, as $\mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q}_{|G|_{2'}}) \cong \mathrm{Gal}(\mathbb{Q}_{|G|_2}/\mathbb{Q})$ is a 2-group. Since $\mathcal{G}$ is abelian and Galois action commutes with the conjugation action of G on the characters of its normal subgroups, it is clear that Q acts on the set of characters of each characteristic subgroup of G_θ.

Proof of Theorem 1. By Proposition 3 the block β of G_θ contains 2^ℓ irreducible characters of height zero which are Q-invariant. Now let $\psi \in \mathrm{Irr}(\beta)$ be Q-invariant. As $\psi^\sigma = \psi$ we have

$$\psi^G = (\overline{\psi^t})^G = \overline{\psi^G}.$$

Also, for each $\tau \in \mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q}_{|G|_{2'}})$ we have

$$(\psi^G)^\tau = (\psi^\tau)^G = \psi^G.$$

It follows that each Q-invariant irreducible character of β induces a real 2-rational character of G, since the 2-rational characters of G are precisely those fixed by $\mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q}_{|G|_{2'}})$. Thus, the 2-block B of G contains 2^ℓ real-valued 2-rational irreducible characters of height zero. ∎

For real irreducible characters of positive height in the principal 2-block we have the following weaker estimate, which we obtain using a different type of argument.

Theorem 4. *Let G be a finite solvable group that has 2-length $\ell \geq 1$. Then the principal 2-block of G contains at least $\ell - 1$ real 2-rational irreducible characters of even degree.*

Proof. We may assume that $\ell > 1$. Consider the normal series for G:

$$G = U_0 \trianglerighteq V_1 \trianglerighteq U_1 \trianglerighteq \ldots \trianglerighteq V_\ell \trianglerighteq U_\ell \trianglerighteq V_{\ell+1} = 1,$$

where $U_i = O_{2'}(G/V_{i+1})$ and $V_i/U_i = O_2(G/U_i)$ for $0 \leq i \leq \ell$. In particular $U_\ell = O_{2'}(G)$. We use induction on the 2-length ℓ.

As noted in the introduction $\mathrm{Irr}(G/U_\ell)$ is the set of irreducible characters in the principal 2-block of G. As $G/U_{\ell-1}$ has 2-length $\ell-1$, we may assume by induction that the principal 2-block of $G/U_{\ell-1}$ has at least $\ell-2$ real 2-rational irreducible characters of even degree. The inflations of these characters to G belong to the principal 2-block.

The group G/V_ℓ has a non-trivial normal odd order subgroup $U_{\ell-1}/V_\ell$ and hence by Lemma (3A) of [2], G/V_ℓ has a 2-block B of non-maximal defect. Corollary 5.9 of [8] implies that B can be chosen to be real.

Now B has a real 2-rational irreducible character χ, according to Theorem 5.1 of [5]. Moreover χ is not a character of $G/U_{\ell-1}$ and $\chi(1)$ is even. However the inflation of χ to G belongs to the principal 2-block. This brings to $\ell - 1$ our lower bound for the number of real 2-rational irreducible characters of even degree in the principal 2-block. ∎

Suppose now that the prime p in the statement of Proposition 3 is odd. Take P to be the unique Sylow p-subgroup of $\mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q}_{|G|_{p'}})$ and set $Q = D \times P$, where D is a Sylow p-subgroup of G_θ. Then we can repeat the arguments used to prove Theorem 1 to get:

Theorem 5. *Let B be a p-block of a finite p-solvable group G that is weakly regular with respect to N. Suppose that all irreducible characters in B lie over $\theta \in \mathrm{Irr}(N)$ and that G_θ has p-length ℓ. Then B has at least 2^ℓ height-zero irreducible characters with values in $\mathbb{Q}_n$ where $n = p|G|_{p'}$.*

If we drop the 'real' requirement in the statement of Theorem 4, we get a much stronger estimate which holds for all primes p.

Theorem 6. *Let B be a p-block of a finite p-solvable group G that is weakly regular with respect to N. Suppose that all irreducible characters in B lie over $\theta \in \mathrm{Irr}(N)$ and that G_θ has p-length $\ell \geq 2$. Then B has at least $2^{\ell-2} + 2^{\ell-4} + \dots$ irreducible characters of positive height.*

Proof. As above we can assume that $\mathrm{Irr}(B) = \mathrm{Irr}(G \mid \theta)$ and that θ is D-invariant, where D is a Sylow p-subgroup of G. Set $\beta_i := (4^i - 1)/3$ for $i \geq 0$. Then our lower bound is

$$\frac{2^\ell + ((-1)^\ell - 3)/2}{3} = \begin{cases} \beta_{\ell/2}, & \text{if } \ell \text{ is even,} \\ 2\beta_{(\ell-1)/2}, & \text{if } \ell \text{ is odd.} \end{cases}$$

Suppose first that ℓ is even. Consider the normal series for G given by (1). It is convenient to set $J_i := L_{\ell-2i}$, for $0 \leq i \leq \ell/2$. So J_i is p-solvable of even p-length $2i$. We prove by induction on i that $\mathrm{Irr}(J_i \mid \theta)$ has β_{i+1} irreducible characters in distinct G-orbits, of which β_{i+1-j} have height at least j, for $j = 0, 1, \dots, i$. The base case is $i = 0$ and $J_0 = L_\ell = N$. Then the conclusion is trivial as $\beta_1 = 1$.

Now let $i \geq 1$. Then by our inductive hypothesis $\mathrm{Irr}(J_{i-1} \mid \theta)$ has β_i irreducible characters in distinct G-orbits, of which β_{i-j} have height at least j, for $j = 0, 1, \dots, i-1$. The group J_i/J_{i-1} is p-solvable of p-length 2. In particular it has nonabelian Sylow p-subgroups, by Theorem A of [9]. Let $\varphi \in \mathrm{Irr}(J_{i-1})$. Then there exists $\chi \in \mathrm{Irr}(J_i \mid \varphi)$ such that $p \mid (\chi(1)/\varphi(1))$, by Theorem A of [4]. We deduce from this that $\mathrm{Irr}(J_i \mid \theta)$ has β_i irreducible characters in distinct G-orbits, of which β_{i+1-j} have height at least j, for $j = 1, \dots, i$. Examining the proof of Proposition 3 (applied with $Q = 1$) we see that $\mathrm{Irr}(J_i \mid \theta)$ also has 2^{2i} height-zero irreducible characters in distinct G-orbits. This gives a total of $\beta_{i+1} = \beta_i + 4^i$ irreducible characters in distinct G-orbits, and the inductive step follows.

The case that ℓ is odd is similar. We omit the proof. ∎

We note that it was already noted in Theorem D of [4] that the block B in Theorem 6 has an irreducible character of height at least $(\ell-1)/2$.

Let $\mathrm{Irr}_0(B)$ denote the set of height-zero irreducible characters in B. Then we have the following estimate:

Theorem 7. *Let B be a p-block of a finite p-solvable group G that is weakly regular with respect to N. Suppose that all irreducible characters in B lie over $\theta \in \mathrm{Irr}(N)$ and that G_θ has p-length ℓ. If q is the smallest prime divisor of $|G_\theta|$ not equal to p then*

$$\sum_{\chi \in \mathrm{Irr}_0(B)} \chi(1)^2 \geq \theta(1)^2 [G : G_\theta]^2 (q^2+1)^\ell$$

Proof. We use the notation in the proof of Proposition 3. We may assume as before that B has a defect group $D \in \mathrm{Syl}_p(G)$, that θ is D-invariant and that $\mathrm{Irr}(B) = \mathrm{Irr}(G \mid \theta)$. Recall that given $i = 1, \ldots, \ell$ each Q-invariant $\chi \in \mathrm{Irr}(L_i \mid \theta)$ has a Q-invariant extension ψ to M_i. Also δ_i is a Q-invariant linear character of M_i/L_i. Then we produced Q-invariant $\chi_1 \in \mathrm{Irr}(L_{i-1} \mid \psi)$ and $\chi_2 \in \mathrm{Irr}(L_{i-1} \mid \delta_i \psi)$ which are not conjugate in G.

We claim that $\delta_i \psi$ is not invariant in L_{i-1}. For suppose otherwise. Then $(\chi_2)_{M_i} = e\delta_i\psi$ for some positive integer $e \mid [L_{i-1} : M_i]$ and hence

$$\det((\chi_2)_{M_i}) = \delta_i^{e\psi(1)} \det(\psi)^e.$$

As $e\psi(1)$ is coprime to p and $o(\delta_i)$ is a power of p, it follows that $p \mid o((\chi_2)_{M_i})$ and hence that $p \mid o(\chi_2)$. This contradicts the fact that $L_{i-1} = O^p(L_{i-1})$. This proves our claim.

The previous paragraph implies that $\chi_2(1) \geq q\chi(1)$ and hence $\chi_1(1)^2 + \chi_2(1)^2 \geq (q^2+1)\chi(1)^2$. Noting that this process is repeated ℓ times between N and G, we see that

$$\sum_{\chi \in \mathrm{Irr}_0(B)} \chi(1)^2 \geq \theta(1)^2 (q^2+1)^\ell.$$

The factor $[G : G_\theta]^2$ in the statement arises from the fact that induction from G_θ to G multiplies each character degree by $[G : G_\theta]$. ∎

Examination of the previous proof shows that B contains a Q-invariant irreducible character χ of height zero such that the total number of p'-prime factors in $\chi(1)$ is at least ℓ, counting multiplicities.

With a bit of work, we can get relative versions of the above results. By this we mean that we can replace G and N in the hypotheses of Proposition 3 or Theorems 1 or 5 by any finite group G and any normal subgroup N of G such that G/N is p-solvable. In the conclusion ℓ is the p-length of G_θ/N. We merely sketch a proof.

Let $M \trianglelefteq L$ be groups and let (F, R, k) be a splitting p-modular system for L and its subgroups and let b_L be a p-block of L that covers a p-block b_M of M. Suppose that b_L is weakly regular with respect to M. Recall that b_M has a central character $\omega : Z(kM) \to k$ and b_L has a primitive idempotent $e \in Z(kL)$ which has a unique lift to an idempotent $\hat{e} \in Z(RL)$. The *Brauer map* $\mathrm{Br}_M^L : kL \to kM$ is the linear map induced by

$$\mathrm{Br}_M^L(x) = \begin{cases} x, & \text{if } x \in L, \\ 0, & \text{if } x \in L \backslash M. \end{cases}$$

It is a result of M. Murai [13] that

$$\omega(\mathrm{Br}_M^L(e)) \neq 0_k.$$

Moreover, let $\theta \in \mathrm{Irr}(b_M)$ and set $(\theta^L)_{b_L} := \sum_{\chi \in \mathrm{Irr}(b_L)} \langle \theta^L, \chi \rangle \chi$. Then it is straightforward to show that (in R)

$$\frac{(\theta^L)_{b_L}(1)}{\theta(1)} = [L : M]\, \omega_\theta(\mathrm{Br}_M^L(\hat{e})). \tag{3}$$

Now let $N \leq M \leq L \leq G$ be a normal chain of groups such that $p \nmid [L : M]$ and let Q be a p-group that acts on the irreducible characters of each normal subgroup of G. Suppose that B covers the p-block b_M of M and that $\theta \in \mathrm{Irr}(b_M)$ is Q-invariant. Then we can use (3) to show that there exists a p-block b_L of L that is covered by B and $\psi \in \mathrm{Irr}(b_L \mid \theta)$ such that ψ is Q-invariant. Use this in place of (2) in order to prove the analogue of Proposition 3 in this situation.

3 Linear characters of Hall π-subgroups

We continue to assume that G is a finite p-solvable group. The main result in [17] states that if G has p-length ℓ, then G has at least 2^ℓ irreducible characters of p'-degree having values inside $\mathbb{Q}_p$. The proof of this theorem is constructive and the characters so obtained actually belong to Isaacs' canonical set $B_p(G) \subseteq \mathrm{Irr}(G)$ (we refer the reader to [11] for the definition and basic properties of the B_π-characters of a π-separable group, where π is any set of primes).

In their deep work [12] M. Isaacs and G. Navarro construct a bijection between the $B_p(G)$-characters of p'-degree and the $N_G(P)$-orbits of linear characters of $P \in \mathrm{Syl}_p(G)$. This enables them to prove a strong form for the Alperin-McKay conjecture for p-solvable groups. By the results in Section 4 of [12], this association of characters is well behaved with respect to the normal structure of G, which implies some results on extension of linear characters in P.

We observe that by Theorem 2.2 of [1], the B_p-characters of p'-degree of G are precisely the irreducible constituents of the principal projective indecomposable character Φ_1 of p'-degree. In particular, the characters we obtained above in Theorems 1 and 5 will not belong to $B_p(G)$ if our block B is not the principal p-block of G.

Note that as G is p-solvable, Φ_1 coincides with the permutation character of G acting on the cosets of a Hall p'-subgroup of G (see for instance Problem (2.8) of [14]). In [7], R. Gow studied the p'-degree characters in Φ_1 using elementary methods, and our next goal is to show that Gow's treatment is compatible with the normal structure of G and give some consequences of this fact.

All the above can be stated and proved with exactly the same amount of work for π-separable groups G, considering now an arbitary set of primes π instead of a single prime p. We chose this more general framework, and we start by recalling Gow's approach in [7].

Let λ be a linear character of a Hall π-subgroup H of G. If M is a π-complement of G, then $(\lambda^G)_M$ is the regular character of M. Thus

$$\text{there is a unique } \chi \in \mathrm{Irr}(G) \text{ with } \langle \chi, \lambda^G \rangle \neq 0 \text{ and } \langle \chi, (1_M)^G \rangle \neq 0.$$

Clearly $\langle \chi, \lambda^G \rangle = \langle \chi, (1_M)^G \rangle = 1$. By [7], χ is monomial with Schur index 1 over $\mathbb{Q}$ and $\chi(1)$ is a π'-number. More significantly, if σ is any linear character of H with $\langle \chi, \sigma^G \rangle \neq 0$, then $\sigma = \lambda^n$, for some $n \in N_G(H)$.

We call χ the *Gow character* of G corresponding to λ, and denote it by $\Psi_G(\lambda)$. Let $\mathrm{Irr}_\Psi(G)$ be the set of Gow characters of G, and for $L \leq G$ and $\theta \in \mathrm{Irr}(L)$, let $\mathrm{Irr}_\Psi(G \mid \theta)$ be the set of Gow characters of G lying over θ.

We sketch Gow's proof for the reader's convenience. Work by induction on $|G|$. Then we may assume that χ is faithful. In particular $O_{\pi'}(G) = 1$.

Set $U := O_\pi(G)$ and $\mu := \lambda_U$. If $U = G$ then $\chi = \mu$ and the result holds. Otherwise Lemma 1.2.3 of [9] can be used to show that G_μ is a proper subgroup of G. Clearly $H \leq G_\mu$. Moreover $\varphi := \Psi_{G_\mu}(\lambda)$ is the Clifford correspondent of χ with respect to μ. Then χ inherits the properties stated from φ, using the inductive hypothesis.

Clearly there is a pair (W, τ) with $H \leq W \leq G_\mu$ and $\tau \in \mathrm{Lin}(W)$ such that $\tau_H = \lambda$ and $\tau^{G_\mu} = \varphi$, whence $\tau^G = \chi$. Examination of Gow's proof shows that $o(\tau)$ is a π-number. The pair (W, τ) is determined up to G-conjugacy in the following strong sense:

Lemma 8. *Suppose that $W_1 \leq G$ and $\tau_1 \in \mathrm{Lin}(W_1)$ is such that $\tau_1^G = \chi$. Then there exists $g \in G$ such that $W_1^g = W$ and $\tau_1^g = \tau$.*

Proof. We prove this by induction on $|G|$. Then with the notation used above, it is clear that we may assume that $G_\mu < G$. Note that $\langle \chi_H, (\tau_1)_H \rangle \neq 0$. So by Gow's result $(\tau_1)_H$ is conjugate in $N_G(H)$ to λ. Conjugating W_1 and τ_1 by an element of G, if necessary, we may assume that $(\tau_1)_H = \lambda$. Then W_1 fixes $\lambda_{H\cap U}$ and hence also μ. As $W_1 \leq G_\mu$, and $|G_\mu| < |G|$ the claim follows from our inductive hypothesis. ∎

As we have already claimed, Gow characters interact well with the normal subgroups of G. Our next result states this precisely.

Lemma 9. *Let $S \trianglelefteq G$, let $\lambda \in \mathrm{Lin}(H)$ and let $\gamma \in \mathrm{Lin}(H \cap S)$. Set $\chi = \Psi_G(\lambda)$ and $\theta = \Psi_S(\tau)$. Then*

(i) Let $\{\lambda_i\}_{i=1}^r$ be the set of $N_G(H \cap S)$-conjugates of $\lambda_{H\cap S}$. Then

$$\chi_S = e \sum_{i=1}^{r} \Psi_S(\lambda_i) \quad \text{for some integer } e > 0.$$

(ii) Suppose that G/S is a π'-group. Then $e = 1$. Moreover $\Psi_G(\gamma)$ is the unique Gow character in $\mathrm{Irr}(G \mid \theta)$.

(iii) Suppose that G/S is a π-group. Then $\chi_S \in \mathrm{Irr}(S)$. Moreover the Gow characters in $\mathrm{Irr}(G \mid \theta)$ are the extensions of θ to G.

Proof. First note that $H \cap S$ is a Hall π-subgroup of S and $M \cap S$ is a Hall π'-subgroup of S. As $\langle \chi_M, 1_M \rangle = 1$ and $(1_M)_{M\cap S} = 1_{M\cap S}$, there is an irreducible constituent ψ of χ_S such that $\langle \psi_{M\cap S}, 1_{M\cap S} \rangle \neq 0$. But then

$\langle (\psi^g)_{M\cap S}, 1_{M\cap S}\rangle \neq 0$, for all $g \in G$, as $M \cap S$ and $(M\cap S)^g$ are Hall π'-subgroups of S. Replacing ψ by a G-conjugate, if necessary, we may assume that $\langle \psi_{H\cap S}, \lambda_{H\cap S}\rangle \neq 0$. So $\psi = \Psi_S(\lambda_{H\cap S})$.

Now $G = SN_G(H\cap S)$, by the Frattini argument. So each G-conjugate of ψ has the form ψ^n, for some $n \in N_G(H\cap S)$. Then $\psi^n = \Psi_S(\lambda^n_{H\cap S})$. This proves (i).

Assume the hypothesis of (ii). Then $G = MS$ and H is a Hall π-subgroup of S. So

$$\langle (\psi^G)_M, 1_M\rangle = \langle (\psi_{M\cap S})^M, 1_M\rangle = \langle \psi_{M\cap S}, 1_{M\cap S}\rangle = 1.$$

This proves the conclusions in (ii).

Assume the hypothesis of (iii). Then $G = SH$ and M is a Hall π'-subgroup of G. As $1 = \langle \chi_M, 1_M\rangle = \langle (\chi_S)_M, 1_M\rangle$, we see that

$$\sum_{\nu\in \mathrm{Irr}_\Psi(S)} \langle \chi_S, \nu\rangle = 1.$$

It follows from this and (i) that $\chi_S = \psi$. Now suppose that $\mathrm{Irr}_\Psi(G \mid \theta)$ is non-empty. Then the previous paragraph implies that θ extends to G. Conversely let φ be an extension of θ to G. By Gow's theorem there exists a subgroup X of S containing $H\cap S$ and $\delta \in \mathrm{Lin}(X)$ such that $\theta = \delta^S$. Let W be the inertia group of δ in $N_G(X)$. Now $\langle \varphi_X, \delta\rangle = \langle \theta_X, \delta\rangle = 1$. So there exists a unique $\tau \in \mathrm{Irr}(W)$ such that $\langle \varphi_W, \tau\rangle = 1 = \langle \tau_X, \delta\rangle$. As $X \trianglelefteq W$ and δ is invariant in W, we see that $\tau_X = \delta$. In particular $\tau \in \mathrm{Lin}(W)$. As $\tau_{S\cap W}$ is an extension of δ to $S\cap W$ and δ^S is irreducible we must have $S \cap W = X$. Now θ is G-invariant. So $G = SW$, using Lemma 8 and the Frattini argument. We have $(\tau^G)_S = (\tau_{S\cap W})^S = \delta^S = \theta$. It follows that $\varphi = \tau^G$. But $\langle \varphi_M, 1_M\rangle = \langle \theta_M, 1_M\rangle = 1$. We conclude that $\varphi = \Psi_G(\tau_H)$. ∎

The following observation, which is independent of the previous lemma, can be proved using part (iii) of Lemma 9. Of course, this implies Theorem 2 in the Introduction. In particular this gives a character-theoretic proof which is different to the methods in [3].

Theorem 10. *Let G be a finite π-separable group and suppose that $S \trianglelefteq G$ satisfies $S = O^\pi(S)$. Let H be a Hall π-subgroup of G and let μ be a linear character of $H \cap S$. Then μ has an extension to H if and only if μ is H-invariant.*

Proof. Assume that μ is H-invariant. Since $H \cap S$ is a Hall π-subgroup of S, we may consider $\theta := \Psi_S(\mu)$, an irreducible character of S. We claim that θ is H-invariant. In fact, for any $h \in H$, θ^h is the Gow character of S associated with $\mu^h = \mu$, so $\theta^h = \theta$ by uniqueness of Gow characters.

Now as S has no proper normal subgroups of index a π-number $o(\theta)$ is a π'-number. Then $\gcd(\theta(1)o(\theta), [HS:S]) = 1$. So by Gallagher's Theorem θ extends to $\chi \in \mathrm{Irr}(HS)$. Since $\langle \theta_{H\cap S}, \mu \rangle = 1$, Lemma 4.1 of [11] implies that there is a unique $\lambda \in \mathrm{Irr}(H \mid \mu)$ such that λ is a constituent of χ_H. Also, by the same result

$$\frac{\chi(1)}{\lambda(1)} = \frac{\theta(1)}{\mu(1)}$$

and it follows that λ extends μ, as wanted. ∎

Our next result should be compared with Theorem 4.4 of [12].

Proposition 11. *Suppose that $S \trianglelefteq G$, where G is π-separable. Let μ be a linear character of a Hall π-subgroup K of S and let θ be the Gow character of K associated with μ. Then μ has an extension to some Hall π-subgroup of G containing K if and only if there exists a Gow character χ of G such that θ is a constituent of χ_S. Also, χ and the extension of μ can be chosen so that χ is the Gow character associated to the extension of μ.*

Proof. Suppose that there exists a Gow character χ of G such that θ is a constituent of χ_S. Let λ be a linear character of a Hall π-subgroup H of G such that $\chi = \Psi_S(\lambda)$. By Lemma 9(i), we know that

$$\chi_S = e \sum_{i=1}^{r} \Psi_S(\lambda_i)$$

where $\lambda_1, \ldots, \lambda_r$ are the $N_G(H \cap S)$-conjugates of $\lambda_{H\cap S}$. Choose notation so that $\theta = \Psi(\lambda_1)$. First we claim that it is no loss to assume that $\lambda_{H\cap S} = \lambda_1$. Write $\lambda_1 = (\lambda_{H\cap S})^y$, where $y \in N_G(H \cap S)$. The Hall π-subgroup H^y of G has a linear character λ^y. By the induction formula, $(\lambda^y)^G = \lambda^G$ and thus χ is the Gow character associated to λ^y, by uniqueness of the Gow character. Note that $H^y \cap S = H \cap S$ and $(\lambda^y)_{H\cap S} = (\lambda_{H\cap S})^y = \lambda_1$, so after conjugating H and λ by an element of $N_G(H \cap S)$ if necessary, we can assume that λ restricts to λ_1.

Now let $s \in S$ be such that $H^s \cap S = K$ and let $\lambda_1^s \in \mathrm{Lin}(K)$. Then $\theta = \Psi(\lambda_1^s)$, and thus there exists $n \in N_S(K)$ such that $\mu = \lambda_1^{sn}$, by Gow's theorem. Now $H^{sn} \cap S = K$ and $\lambda^{sn} \in \mathrm{Irr}(H^{sn})$ restricts to μ. Also, χ is the Gow character associated with λ^{sn}, as wanted.

The reverse implication follows from Lemma 9(i). ∎

Observe that it follows from Lemma 9(i) that if χ is a Gow character of π-separable group G and $S \lhd\lhd G$, then all irreducible constituents of χ_S are Gow characters of S. The following is now easy to prove by standard arguments, and we omit the proof.

Corollary 12. *Proposition 11 holds under the weaker hypothesis that* $S \lhd\lhd G$.

Finally, we combine our results on Gow characters with the type of arguments in the previous section to prove a result that easily implies the main theorem in [17].

Theorem 13. *Let G be a finite p-solvable group and let S be a normal subgroup of G such that $S = O^p(S)$. Suppose that P is a Sylow p-subgroup of G and θ is an irreducible Gow character of S which is P-invariant. If G/S has p-length ℓ then G has at least 2^ℓ irreducible Gow characters which lie over θ.*

Proof. Consider the normal series for G modulo S

$$G = L_0 \trianglerighteq M_1 \trianglerighteq L_1 \trianglerighteq \ldots \trianglerighteq M_\ell \trianglerighteq L_\ell \trianglerighteq S$$

with $M_i/S = O^{p'}(L_{i-1}/S)$ and $L_i/S = O^p(M_i/S)$ for $i \geq 1$. Now $\ell + 1$ is the smallest index such that $M_{\ell+1} \leq S$. In particular L_ℓ/S is a p'-group. We claim that $O^p(L_i) = L_i$, for $i \geq 1$. To see this, notice that $S/S \cap O^p(L_i)$ is isomorphic to a subgroup of the p-group $M_i/O^p(L_i)$. As $O^p(S) = S$, it follows that $S \leq O^p(L_i)$. But then $M_i/O^p(L_i)$ is a p-quotient of M_i/S. So $L_i \leq O^p(L_i)$, which proves our claim.

Set $\theta := \Psi_S(\mu)$. Then θ is P-invariant. We claim that there are at least $2^{\ell-i}$ irreducible Gow characters of L_i lying over θ for $i = 0, \ldots \ell$. The proof is by backwards induction on i. For $i = \ell$, take $\theta_\ell := \Psi_{L_\ell}(\mu)$. Then θ_ℓ is the unique Gow character in $\mathrm{Irr}(L_\ell \mid \theta)$, by Lemma 9(ii). In particular θ_ℓ is P-invariant.

Let $i \leq \ell$ and suppose that $\theta_i \in \mathrm{Irr}_\Psi(L_i \,|\, \theta)$ is P-invariant. Then $o(\theta_i)$ is a p'-number as $L_i{=}O^p(L_i)$. As $\gcd(\theta_i(1)o(\theta_i), [M_i : L_i]) = 1$, it follows from Gallagher's extension theorem that θ_i has a unique extension φ_i to M_i with $o(\varphi_i) = o(\theta_i)$. This uniqueness implies that φ_i is P-invariant.

Now M_i/L_i is a p-group. So Lemma 9(iii) implies that $\varphi_i = \Psi_{M_i}(\mu_i)$ for some linear character μ_i of $P_i = P \cap M_i$. Let τ_i be a P-invariant linear character of the p-group M_i/L_i. Then $\varphi_i\tau_i = \Psi_{M_i}(\mu_i(\tau_i)_{P_i})$. Notice that $o(\varphi_i)$ is a p'-number while $o(\varphi_i\tau_i)$ is divisible by p. It follows that φ_i and $\varphi_i\tau_i$ are not G-conjugate. Since $\Psi_{M_i}(\mu_i^n) = \Psi_{M_i}(\mu_i)^n$ for all $n \in N_G(P_i)$, and the same holds for $\mu_i(\tau_i)_{P_i}$, we have that μ_i and $\mu_i(\tau_i)_{P_i}$ are not conjugate in $N_G(P_i)$.

Set $\chi_1 := \Psi_{L_{i-1}}(\mu_i)$ and $\chi_2 := \Psi_{L_{i-1}}(\mu_i\tau_i)$. Lemma 9(ii) implies that χ_1 and χ_2 are the unique Gow characters in $\mathrm{Irr}(L_{i-1} \mid \varphi_i)$ and $\mathrm{Irr}(L_{i-1} \mid \varphi_i\tau_i)$, respectively. In particular both χ_1 and χ_2 are P-invariant. As μ_i and $\mu_i(\tau_i)_P$ are not conjugate in $N_G(P \cap L_{i-1})$, Lemma 9(i) implies that χ_1 and χ_2 are not conjugate in G.

It follows from the previous three paragraphs that if L_i has $2^{\ell-i}$ irreducible Gow characters lying over θ, then L_{i-1} has $2^{\ell-(i-1)}$ irreducible Gow characters lying over θ. This is the inductive step and our claim follows. ∎

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