IRMO CUBIC POLYNOMIAL PROBLEM

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Problem: Find all polynomials $f(x) = x^3 + bx^2 + cx + d$, where b, c, d are real numbers, such that $f(x^2 - 2) = -f(-x)f(x)$.

Solution: Set $a := 2$. Let $\beta_1, \beta_2, \beta_3$ be the roots of f. Then the hypothesis implies that

$$\prod_{i=1}^{3}(x - \sqrt{a + \beta_i})(x + \sqrt{a + \beta_i}) = \prod_{i=1}^{3}(x - \beta_i)(x + \beta_i)$$

We consider the various possibilities.

Assume first that $\sqrt{a + \beta_i} = \pm\beta_i$, for all $i = 1, 2, 3$. Then each β_i is a root of $x^2 - x - 2$. So $\beta_i = -1$ or 2. It can be checked that this gives four possible polynomials $f(x)$:

$$(x + 1)^3, \quad (x + 1)^2(x - 2), \quad (x + 1)(x - 2)^2, \quad (x - 2)^3. \tag{1}$$

Assume next that $\sqrt{a + \beta_1} = \pm\beta_1$ but $\sqrt{a + \beta_2} \neq \pm\beta_2$. Then $\sqrt{a + \beta_2} = \pm\beta_3$ and so $\sqrt{a + \beta_3} = \pm\beta_2$. Now $\beta_1 = -1$ or 2, as before. Also $\beta_2 = \beta_3^2 - 2$ and $\beta_3 = \beta_2^2 - 2$. So β_2, β_3 are roots of

$$(x^2 - 2)^2 - x - 2 = (x^2 - x - 2)(x^2 + x - 1)$$

and hence the two complex conjugate roots $(-1 \pm \sqrt{5})/2$ of $x^2 + x - 1$. In this way we obtain two additional possibile polynomials $f(x)$:

$$(x + 1)(x^2 + x - 1), \quad (x - 2)(x^2 + x - 1). \tag{2}$$

Finally we consider the case that $\sqrt{a + \beta_i} \neq \pm\beta_i$, for $i = 1, 2, 3$. Then we may choose notation so that (with i considered mod 3):

$$\sqrt{a + \beta_{i+1}} = \pm\beta_i,$$

whence $\beta_{i+1} = \beta_i^2 - a$, for $i = 1, 2, 3$. Thus $f(x)$ has the three roots:

$$\beta_i, \quad \beta_i^2 - a, \quad (\beta_i^2 - a)^2 - a = \beta_i^4 - 2a\beta_i^2 + a^2 - a.$$

Now $-b$ is the sum of the roots of f. So β_i is a root of the quartic

$$g(x) := x^4 + (1 - 2a)x^2 + x + a^2 - 2a + b = 0.$$

As its roots are distinct, it follows that $f(x)$ divides $g(x)$. As $g(x)$ has zero x^3 term, its easy to see that the quotient g/f must be $x - b$. Thus

$$(x - b)(x^3 + bx^2 + cx + d) = x^4 + (1 - 2a)x^2 + x + a^2 - 2a + b.$$

Thus we get

$$c - b^2 = 1 - 2a, \quad d - bc = 1, \quad -bd = a^2 - 2a + b.$$

In particular $c = b^2 - 2a + 1$ and $d = bc + 1$.

Now we expand $f(x^2 - a) = -f(-x)f(x)$, using $f(x) = x^3 + bx^2 + cx + d$. The coefficient of x^2 gives:

$$b - 3a = 2c - b^2.$$

Date: 6 Apr 2010.

But $c = b^2 - 2a + 1$, from the previous paragraph. We deduce that $2(b^2 - 2a + 1) = b^2 + b - 3a$, whence $b^2 - b + (2 - a) = 0$. As $a = 2$, we conclude that $b = 0$ or 1. Thus we get two additional possible polynomials $f(x)$:

$$x^3 - 3x + 1, \quad x^3 + x^2 - 2x - 1. \tag{3}$$

The set of all possible polynomials $f(x)$ is contained in (1), (2) and (3).

Notes: Its slightly harder to classify all $f(x)$ such that $f(x^2 - a)/f(x)$ is a polynomial.

You can change $a = 2$ to any positive real number. For example $a = 29/16$ gives 'unusual' cubic solutions for $f(x)$:

$$x^3 + \frac{3}{4}x^2 - \frac{33}{16}x - \frac{35}{64}, \quad x^3 + \frac{1}{4}x^2 - \frac{41}{16}x - \frac{23}{64}.$$

The latter is irreducible over $\mathbb{Q}$, while the former has rational roots

$$5/4 \quad , -1/4, \quad , -7/4.$$

These form a '3-cycle' under the $x \to (x^2 - 29/16)$ operation:

$$5/4 = (-7/4)^2 - 29/16, \quad -1/4 = (5/4)^2 - 29/16, \quad -7/4 = (-1/4)^2 - 29/16.$$

It can be shown that all rational 3-cycles are got from pairs s, t of coprime integers as:

$$\frac{s^3 - s^2t + t^3}{2st(s-t)}, \quad \frac{s^3 - 3s^2t + 2st^2 - t^3}{2st(s-t)}, \quad \frac{-s^3 + s^2t - 2st^2 + t^3}{2st(s-t)}.$$

Using the idea of cycles, an alternative problem is to find all rational numbers β such that

$$\beta = \sqrt{29/16 + \sqrt{29/16 + \sqrt{29/16 + \beta}}}.$$

The 'obvious' approach to such a question is to consider roots of the associated octic polynomial:

$$\begin{array}{rlrl} x^8 - & (29/2) \; x^7 & +(11 \times 17 \times 29/2^6) & x^6 \\ & -(5 \times 29^2 \times 31/2^8) \; x^5 & +(3 \times 7 \times 29 \times 22291/2^{15}) & x^4 \\ & -(3 \times 29^2 \times 59 \times 307/2^{17}) \; x^3 & +(3^3 \times 29^2 \times 23173/2^{22}) & x^2 \\ & -(5 \times 7 \times 13 \times 23 \times 29^3/2^{25}) \; x & -(13 \times 29 \times 397 \times 48371/2^{32}) & . \end{array}$$

Its probably impossible to guess the rational roots of this! However, its not too difficult to see that $(x - 29/16)^2 - x - 29/16$ is a quadratic factor.The trick is to look for cubic factors, which will have roots

$$\beta, \quad \beta^2 - 29/16, \quad (\beta^2 - 29/16)^2 - 29/16.$$

Finally, the problem is based on a rather unusual polynomial identity:

$$(((x^2 - a)^2 - a)^2 - x - a = (x^2 - x - a)\, f(x, b)\, f(x, 1 - b),$$

where x, b are commuting indeterminates, and $a = b^2 - b + 2$ and

$$f(x, b) = x^3 + bx^2 - (b^2 - 2b + 3)x - (b^3 - 2b^2 + 3b - 1).$$

The discriminant of $f(x, b)$, as polynomial in x, is equal to

$$\Delta(f(x)) = (4b^2 - 6b + 9)^2 = ((2b - \frac{3}{2})^2 + 27/4)^2.$$

The extreme situation that $\Delta(f(x)) = (27/4)^2$ is minimal occurs when $b = 3/4$ and hence $a = 29/16$.

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