SOME QUESTIONS WHICH MAY BE OF INTEREST TO DAVID HEMMER

JOHN C. MURRAY

1. Overall Question

Hi David, I'll try to formalise some of the problems I'm stuck on which you may find interesting, along with motivational comments and indications of possible methods and difficulties. Some of these I've already mentioned to you in Lausanne.

Question 1. *Given a field F (let's assume algebraically closed for the moment) and integers $l \leq n$, is the centraliser algebra $FS_n^{S_l}$ cellular, in the sense of Graham and Lehrer? If so, is the cellular structure compatible with that of FS_n or some Schur algebra? Is the cellular structure 'global', in that it can be defined for $\mathbb{Z}S_n^{S_l}$, and then taken modulo p?*

Motivation: use the cellular structure to describe the simple modules and ultimately the blocks of $FS_n^{S_l}$. Perhaps prove a version of the Jantzen-Schaper formula along the way. This aims to mimic the approach of [S. Lyle, A. Mathas, Blocks of cyclotomic Hecke algebras, Adv. Math. **216** (2007) 854–878.]

The easiest case I can't answer is the 14-dimensional algebra $FS_4^{S_2}$, where I guess F is of characteristic 2 or 3. Actually this should not be too hard to settle, as $FS_4^{S_2}$ is almost commutative.

The rest of this note is probably not relevant to the above question, at least for $l < n-1$. I outline how to construct explicit cellular bases for $Z(RS_n) = RS_n^{S_n}$ and for $RS_n^{S_{n-1}}$, where R is any integral domain. I include some off-topic results on centraliser algebras. What is nice about these constructions is that they involve the 'right' partial order on the indexing set for irreducibles (partitions or marked partitions, under the dominance order). Moreover, they can be done over $\mathbb{Z}$, and hence are global.

2. Cellular algebras

Recall that a cellular R-algebra (for R an integral domain which is Noetherian(?)) consists of an R-algebra A together with a *cell datum* (Λ, M, C, i). Here Λ is a poset, and associated to each $\lambda \in \Lambda$ there is a finite set $M(\lambda)$. Moreover, A has an R-basis $C_{S,T}^{\lambda}$, where λ runs over Λ, and (S,T) runs over $M(\lambda) \times M(\lambda)$. In addition i is an R-algebra anti-automorphism of A such that $C_{S,T}^{\lambda} = C_{T,S}^{\lambda}$, for all λ, S, T. Finally if $\lambda \in \Lambda$ and $S, T \in M(\lambda)$ and $a \in A$, then

$$aC_{S,T}^{\lambda} = \sum_{U \in M(\lambda)} r_U C_{U,T}^{\lambda} + \epsilon, \tag{1}$$

where r_U are scalars that depend on a, S, U, but not on T, and ϵ belongs to the R-span of $\{C_{X,Y}^{\mu} \mid \mu \geq \lambda \text{ and } X, Y \in M(\mu)\}$.

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Let I_λ be the R-span of all $C^\mu_{X,Y}$, with $\mu \geq \lambda$, and let J_λ be the R-span of all $C^\mu_{X,Y}$, with $\mu > \lambda$. It follows from the axioms that both I_λ and J_λ are 2-sided ideals of A. Each quotient I_λ/J_λ is isomorphic to a direct sum of $|M(\lambda)|$ copies of an A-module C^λ, a so-called *cell-module* for A. The equation (1) can be used to define a symmetric bilinear form on C^λ; the quotient D^λ of C^λ by the kernel of this form is either 0 or a simple A-module. Moreover, the non-zero D^λ that arise in this way are distinct, and give a complete set of representatives for the isoclasses of simple A-modules. We present a proof that every commutative algebra over a (suitable) field is cellular. In this case every cell module C^λ is 1-dimensional, and coincides with I^λ/J^λ.

Lemma 2. *Let F be a field and let A be a commutative unital finite dimensional F-algebra such that $A/J(A)$ is split semi-simple. Then A is cellular.*

Proof. We take $i = \mathrm{Id}_A$. Then i is an anti-automorphism of A, as A is commutative. We may assume that A is an indecomposable algebra i.e. that 1 is the unique idempotent in A. In particular A has a unique simple module S, and this module is 1-dimensional. Let

$$A = A_0 > A_1 > A_2 > \dots A_n > 0 = A_{n+1}$$

be a composition series for the regular A-module. So $A_i/A_{i+1} \cong S$, for $i = 0, \dots, n$. Set $\Lambda := \{0, \dots, n\}$, a poset under the usual total order $\leq$ on the integers. For each $\lambda \in \Lambda$ set $M(\lambda) := \{1\}$ and choose a basis element C^λ_{11} for A_λ modulo $A_{\lambda+1}$. Then (Λ, M, C, i) is a cell datum for A. □

Comment: surely this does not require the hypothesis of splittability?

In particular both $Z(\mathbb{Z}S_n) = \mathbb{Z}S_n^{S_n}$ and $\mathbb{Z}S_n^{S_{n-1}}$ are cellular. G. Murphy supplied me with an explicit cellular basis for the former. First we need:

Lemma 3. *Let R be an integral domain, with field of fractions F and let O be a commutative R-order such that $A := O \otimes_R F$ is split semi-simple. Let $\{e_\mu \mid \mu \in \Lambda\}$ be the set of idempotents in A (Λ an indexing set). Suppose that there is a partial order $\trianglerighteq$ on Λ, and an R-basis $\{c_\lambda \mid \lambda \in \Lambda\}$ of O such that*

$$c_\lambda = \sum_{\mu \trianglerighteq \lambda} d_{\lambda\mu} e_\mu,$$

where $d_{\lambda\mu} \in F$, for each $\lambda \in \Lambda$. Set $M(\lambda) = \{1\}$ and $C^\lambda_{11} = c_\lambda$. Then $(\Lambda, C, M, \mathrm{Id}_O)$ is a cellular datum for O.

Proof. This follows from the fact that $I^\lambda := \sum_{\mu \trianglerighteq \lambda} Rc_\mu$ and $J^\lambda := \sum_{\mu \triangleright \lambda} Rc_\mu$ are ideals of O that are pure as R-sublattices of O. □

Suppose that p is a maximal ideal of R, and set $k = R/p$ and $A := O/pO$. Identify elements of R, O with their images in k, A. Then A is a cellular algebra over k. Also

$$c_\lambda^2 = d_{\lambda\lambda} c_\lambda \mod(J^\lambda).$$

So the blocks and simple modules of A are indexed by $\{\lambda \mid d_{\lambda\lambda} \notin p\}$.

3. Cellular structure of $Z(RS_n)$

As a warm-up exercise we describe an explicit cellular datum for $Z(RS_n)$. It is enough to consider the case $R = \mathbb{Z}$ and $F = \mathbb{Q}$. The irreducible characters of S_n are indexed by the set Λ of partitions of n. Moreover, Λ is a poset under the dominance order $\trianglerighteq$. Corresponding to $\lambda \in \Lambda$ (notation $\lambda \vdash n$), we have

- S_λ, the Young submodule of S_n;
- M^λ, the RS_n-permutation module on the right cosets of S_λ;
- S^λ, the Specht RS_n-submodule of M^λ;
- χ_λ, the character of S^λ, an irreducible F-character of S_n;
- e_λ, the central primitive idempotent of FS_n such that $\chi(e_\chi) = \chi(1)$;
- ω_λ, the central character $\chi(\cdot)/\chi(1)$ of FS_n;
- C_λ, the S_n-conjugacy class of permutations of cycle type λ.

We next recall the notion of a *relative trace map*. Suppose that G is a finite group, K is a subgroup of G, R is a commutative ring, and V is an RG-module. Let $V^K = \{v \in V \mid vk = v, \forall k \in K\}$ denote the space of K-invariants in V. Recall that the *relative trace map* $\mathrm{Tr}_K^G : V^K \to V^G$ is defined by

$$\mathrm{Tr}_K^G(x) = \sum_{t \in K\backslash G} xt, \quad \text{for all } x \in V^K.$$

Here $K\backslash G$ is a right transversal to K in G. Then Tr_K^G does not depend on the choice of transversal $K\backslash G$. We will apply this to the RG-module RX, where $X = G$ is a G-set under conjugation.

Now the element sum S_λ^+ belongs to the centraliser algebra $RS_n^{N_{S_n}(S_\lambda)}$. So we may define an element of $Z(RS_n)$ via:

$$T_\lambda := \mathrm{Tr}_{N_{S_n}(S_\lambda)}^{S_n}(S_\lambda^+).$$

[30 Sep]

Lemma 4. $T_\lambda = \sum\limits_{\mu \trianglelefteq \lambda} \frac{1_{S_\lambda}^{S_n}(\mu)}{[N_{S_n}(S_\lambda):S_\lambda]} C_\mu^+$.

Proof. Here $1_{S_\lambda}^{S_n}(\mu)$ is the value of the induced character on a permutation of cycle type μ. Note that $1_{S_\lambda}^{S_n}(\mu) = 0$, if $\mu \not\trianglelefteq \lambda$. This is a special case of a more general result: consider a finite group G and $H \leq G$. Set $T = \mathrm{Tr}_N^G(H^+)$, where $N = N_G(H)$. For C a conjugacy class of G, the non-negative integers α_C are defined via $T = \sum_C \alpha_C C^+$. Then $|C|\alpha_C = [G : N]|C \cap H|$. So for $c \in C$ we have

$$\alpha_C = \frac{[G : N]|C \cap H|}{|C|} = \frac{1_H^G(c)}{[N : H]}.$$

□

Incidently, this proves that $[N : H]$ divides the permutation character values $1_H^G(g)$, for all $g \in G$. This can be seen directly by defining the following equivalence relation $\sim$ on the right cosets $H\backslash G$:

$$Hx \sim Hy \quad \text{if} \quad Hy = Hnx, \quad \text{for some } n \in N.$$

Then show that if $Hx, Hy \in H\backslash G$ and $Hx \sim Hy$ and $g \in G$ is such that $Hxg = Hx$, then also $Hyg = Hy$.

Lemma 5. *$\{T_\lambda \mid \lambda \vdash n\}$ is a basis for $Z(RS_n)$.*

Proof. The class sums C_μ^+, $\mu \vdash n$ form a basis of $Z(RS_n)$. We claim that

$$T_\lambda = C_\lambda^+ + \sum_{\lambda \rhd \mu} z_{\lambda\mu} C_\mu^+, \quad \text{where } z_{\lambda\mu} \in R.$$

This follows from the fact that each permutation $\sigma \in C_\lambda$ belongs to exactly one S_n-conjugate of S_λ, namely the stabiliser of the orbits of σ on $\{1, \ldots, n\}$. Now each orbit of $\tau \in S_\lambda \backslash C_\lambda$ is a subset of some λ-orbit. We conclude from this that the cycle type of τ is dominated by λ. □

Recall the hooklength degree formula of Frame, Thrall and Robinson

$$\chi_\lambda(1) = \frac{n!}{h(\lambda)},$$

where $h(\lambda)$ is the product of all the hook-lengths in the Young diagram $[\lambda]$. Also if $m_i = m_i(\lambda)$ is the multiplicity of i as a part of λ, we have

$$S_\lambda \cong \prod_{i=1}^n S_i^{m_i}, \quad N_{S_n}(S_\lambda) \cong \prod_{i=1}^n S_i \wr S_{m_i}.$$

So $[N_{S_n}(S_\lambda) : S_\lambda] = m(\lambda)$ where $m(\lambda) = \prod_{i=1}^n m_i!$.

If $\lambda, \mu \vdash n$, then the Kostka numbers $K_{\mu\lambda}$ equals

- the number of semistandard μ-tableau of type λ,
- the F-dimension of $\mathrm{Hom}_{FS_n}(S^\mu, M^\lambda)$,
- the inner product of characters $\langle \chi_\mu, 1_{S_\lambda}^{S_n} \rangle$.

Note that $K_{\lambda\lambda} = 1$, and $K_{\mu\lambda} \neq 0$ implies that $\mu \trianglerighteq \lambda$.

Theorem 6. *Let $\lambda \vdash n$. Then*

$$T_\lambda = \sum_{\mu \trianglerighteq \lambda} \frac{h(\mu) K_{\mu,\lambda}}{m(\lambda)} e_\mu.$$

In particular $\{T_\lambda \mid \lambda \vdash n\}$ is a cellular basis for $Z(RS_n)$.

Proof. We have

$$T_\lambda = \sum_{\mu \vdash n} \omega_\mu(T_\lambda) e_\mu,$$

and as χ_μ has the same value on each conjugate of S_λ^+, we can compute

$$\omega_\mu(T_\lambda) = [S_n : N_{S_n}(S_\lambda)] \frac{\chi_\mu(S_\lambda^+)}{\chi_\mu(1)} = [S_n : N_{S_n}(S_\lambda)] \frac{|S_\lambda| \langle \chi_\mu, 1_{S_\lambda}^{S_n} \rangle}{\chi_\mu(1)}.$$

□

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Note: if G is a finite group, $\chi \in \mathrm{Irr}(G)$, $H \le G$ and $N = N_G(H)$, then it can be shown that the coefficient of e_χ in $\mathrm{Tr}_N^G(H^+)$ which equals

$$\frac{[G:N]\,|H|}{\chi(1)}\langle \chi, 1_H^G \rangle$$

is actually a non-negative integer. This is a special case of [Theorem D of I. M. Isaacs, G. R. Robinson, Linear constituents of certain character restrictions, *Proc. Amer. Math. Soc.* **126** (9) (1998) 2615–2617].

Example: $Z(RS_3)$ has the following basis

$$\begin{array}{lcllllll} T_{(3)} &=& C^+_{(3)} & +C^+_{(21)} & +C^+_{(1^3)} &=& 6e_{(3)}, & \\ T_{(21)} &=& & C^+_{(21)} & +3C^+_{(1^3)} &=& 6e_{(3)} & +3e_{(21)}, \\ T_{(1^3)} &=& & & C^+_{(1^3)} &=& e_{(3)} & +e_{(21)} \quad +e_{(1^3)}. \end{array}$$

Example: $Z(RS_4)$ has the following basis

$$\begin{array}{llllllllll} T_{(4)} &= C^+_{(4)} & +C^+_{(31)} & +C^+_{(2^2)} & +C^+_{(21^2)} & +C^+_{(1^4)} & = 24e_{(4)}, & & & \\ T_{(31)} &= & C^+_{(31)} & & +2C^+_{(21^2)} & +4C^+_{(1^4)} & = 24e_{(4)} & +8e_{(31)}, & & \\ T_{(2^2)} &= & & C^+_{(2^2)} & +C^+_{(21^2)} & +4C^+_{(1^4)} & = 12e_{(4)} & +4e_{(31)} & +6e_{(2^2)}, & \\ T_{(21^2)} &= & & & C^+_{(21^2)} & +6C^+_{(1^4)} & = 12e_{(4)} & +8e_{(31)} & +6e_{(2^2)} & +4e_{(21^2)}, \\ T_{(1^4)} &= & & & & C^+_{(1^4)} & = e_{(4)} & +e_{(31)} & +e_{(2^2)} & +e_{(21^2)} \quad +e_{(1^4)}, \end{array}$$

4. Characters of the centraliser algebra FG^H

Let G be a finite group and let $H \le G$. Let F be a field such that FG, FH and FG^H are split-semisimple. Choose

$$\chi \in \mathrm{Irr}(G), \quad \phi \in \mathrm{Irr}(H) \quad \text{such that} \quad \langle \chi, \phi^G \rangle \neq 0. \tag{2}$$

Define a function $\chi/\phi : FG \to F$ via

$$\chi/\phi(g) = \frac{1}{|H|}\sum_{h\in H} \chi(gh)\phi(h^{-1}) = \chi(ge_\phi)/\phi(1), \quad \text{for all } g \in G.$$

According to [J. Alperin, Centralizer rings and Hecke algebra, 2010, 2pp, note on website], χ/ϕ restricts to an irreducible character of FG^H, and each irreducible character of FG^H is the restriction of exactly one such function χ/ϕ.

Note that χ/ϕ is constant on the H-orbits on G, because

$$\chi(ge_\phi) = \chi(h^{-1}ge_\phi h) = \chi(h^{-1}ghe_\phi), \quad \text{for all } h \in H.$$

Moreover, $\chi/\phi(1) = \chi/\phi(e_\chi e_\phi) = \langle \chi, \phi^G \rangle$. Finally, if M_χ is an FG-module affording χ and M_ϕ is an FH-module affording ϕ, then $\mathrm{Hom}_{FH}(M_\phi, M_\chi)$ is an irreducible FG^H-module affording the character χ/ϕ.

We use the notation $e_{\chi/\phi} := e_\chi e_\phi$. So $\{e_{\chi/\phi} \mid \chi/\phi \in \mathrm{Irr}(FG^H)\}$ is a basis for $Z(FG^H)$.

Lemma 7. *Let $z \in Z(FG^H)$. Then*

$$z = \sum_{\chi/\phi} \frac{\chi/\phi(z)}{\langle \chi, \phi^G \rangle} e_{\chi/\phi}$$

Proposition 8. *Let $H, K \leq G$ be such that $\mathrm{Tr}_{N_H(K)}^H(K^+) \in Z(FG^H)$. Then $\mathrm{Tr}_{N_H(K)}^H(K^+) = \sum \tau_{\chi/\phi} e_{\chi/\phi}$, where the coefficient $\tau_{\chi/\phi}$ is zero unless*

$$\langle \chi_K, 1_K \rangle \neq 0 \quad \text{and} \quad \langle \phi_{K\cap H}, 1_{K\cap H} \rangle \neq 0.$$

Proof. Let $\chi/\phi \in \mathrm{Irr}(FG^H)$ be such that $\tau_{\chi/\phi} \neq 0$. So we may assume that $\chi(K^+ e_\phi) \neq 0$.

Now the right FG-module K^+FG has character

$$\sum_{\psi \in \mathrm{Irr}(G)} \langle \psi_K, 1_K \rangle \psi.$$

But χ does not vanish on K^+FG, by our assumption. We deduce from these facts that $\langle \chi_K, 1_K \rangle \neq 0$.

If we take T_l to be a left transversal to $K \cap H$ in K, then

$$K^+ = T_l^+ (K \cap H)^+.$$

This, and our assumption, implies that $(K \cap H)^+ e_\phi \neq 0$. Using the same argument as above this implies that $\langle \phi_{K\cap H}, 1_{K\cap H} \rangle \neq 0$. □

Lemma 9. $\mathrm{Irr}(FG^H)$ *determines a basis for $Z(FG^H)$, in the sense that*

$$e_{\chi/\phi} = \frac{\phi(1)\chi(1)}{|G|} \sum_{g\in G} \chi/\phi(g^{-1}) g.$$

Proof. We have

$$e_\chi = \frac{\chi(1)}{|G|} \sum_{g\in G} \chi(g^{-1}) g, \quad e_\phi = \frac{\phi(1)}{|H|} \sum_{h\in H} \phi(h^{-1}) h,$$

So the coefficient of $x \in G$ in $e_{\chi/\phi}$ is

$$\frac{\phi(1)\chi(1)}{|G||H|} \sum_{h\in H} \chi(x^{-1}h)\phi(h^{-1}) = \frac{\phi(1)\chi(1)}{|G|} \chi/\phi(x^{-1}).$$

□

Lemma 10. *Let C be a conjugacy class of H. Then*

$$\chi/\phi(C^+) = \langle \chi_H, \phi \rangle \omega_\phi(C^+),$$

where ω_ϕ is the central character of FH corresponding to ϕ.

Proof. This is a consequence of the fact that $\chi(he_\phi) = \langle \chi_H, \psi \rangle \phi(h)$, for all $h \in H$. In addition, $\omega_\phi(C^+) = \phi(C^+)/\phi(1)$. □

Let R be a commutative ring. Then $RG = RH \oplus R[G \backslash H]$, as $RH \times H$-modules. We let $s = s_H$ denote the projection $RG \to RH$ with respect to this decomposition. So

$$s(g) = \begin{cases} g, & \text{if } g \in H; \\ 0, & \text{if } g \in G\backslash H. \end{cases}$$

Remark 11. *Suppose that $x \in RG^H$. Then $s(x) \in Z(RH)$.*

Proof. We have $xh = hx$, for $h \in H$. So $s(xh) = s(hx)$. But $s(xh) = s(x)h$ and $s(hx) = hs(x)$. We deduce that $hs(x) = s(x)h$, whence $s(x) \in RG^H \cap RH = Z(RH)$.

Alternatively, write x as a linear combination of H-orbit sums. Then $s(x)$ is the truncation of this expression to the H-orbits in H. But each H-orbit in H is a conjugacy class of H. So $s(x)$ is a linear combination of H-conjugacy class sums. □

Corollary 12. $s(e_{\chi/\phi}) = \frac{\langle \chi_H, \phi \rangle}{[G:H]} \frac{\chi(1)}{\phi(1)} e_\phi$.

Proof. We have $s(e_{\chi/\phi}) = s(e_\chi)e_\phi$. But $\{e_\psi \mid \psi \in \mathrm{Irr}(H)\}$ is an F-basis for $Z(FH)$. So $s(e_{\chi/\phi}) = me_\phi$, for some scalar m. We compute m by comparing the coefficients of 1 in $e_{\chi/\phi}$ and e_ϕ. These are

$$\frac{\chi(1)\phi(1)\langle \chi_H, \phi \rangle}{|G|} \quad \text{and} \quad \frac{\phi(1)^2}{|H|}, \quad \text{respectively.}$$

Note that this result can also be deduced directly from Lemmas 9 and 10. □

Lemma 13. $\mathrm{Tr}_H^G(e_{\chi/\phi}) = \langle \chi_H, \phi \rangle \frac{\phi(1)}{|H|} \frac{|G|}{\chi(1)} e_\chi$.

Proof. We have

$$\mathrm{Tr}_H^G(e_{\chi/\phi}) = e_\chi \mathrm{Tr}_H^G(e_\phi) = \frac{\chi(\mathrm{Tr}_H^G(e_\phi))}{\chi(1)} e_\chi.$$

As χ is a class function, we have

$$\chi(\mathrm{Tr}_H^G(e_\phi)) = [G:H]\chi(e_\phi) = [G:H]\phi(1)\langle \chi_H, \phi \rangle.$$

The lemma follows. □

5. Cellular structure of $RS_n^{S_{n-1}}$

A *marked partition of* n is a pair (λ, i), where $\lambda \trianglerighteq n$ and i is a part of λ. We use the notation $(\lambda, i) \Vdash n$. Replacing a part of length i in λ by one of length $i-1$ produces a partition of $n-1$, which we will denote by λ^i. In this way the marked partitions of n are in bijection with the edges in the Young graph between the partitions of $n-1$ and of n, and (hence) the irreducible representations of $FS_n^{S_{n-1}}$. We use the notation

$$\chi_{\lambda,i} = \chi_\lambda / \chi_{\lambda^i}.$$

The relevant partial order on marked partitions is

$$(\mu, i) \trianglerighteq (\lambda, j) \quad \text{if} \quad \begin{array}{l} \mu \trianglerighteq \lambda, \quad \text{or} \\ \mu = \lambda, \quad \text{and} \quad \mu^i \trianglerighteq \lambda^j \quad (\text{i.e. } i > j). \end{array}$$

The marked partitions of n also index the S_{n-1}-orbits on S_n. Specifically, corresponding to (λ, i) we have the S_{n-1}-orbit

$$C_{\lambda,i} := \{\sigma \in C_\lambda \mid n \text{ belongs to a } \sigma\text{-orbit of length } i\}.$$

In particular $\{C_{\lambda,i}^+ \mid (\lambda, i) \Vdash n\}$ is the "standard" basis of $RS_n^{S_{n-1}}$.

Set $S_{\lambda,i}$ as a conjugate of the Young subgroup S_λ which contains a permutation in $C_{\lambda,i}$. Now define

$$T_{\lambda,i} := \mathrm{Tr}_{N_{S_{n-1}}(S_{\lambda,i})}^{S_{n-1}}(S_{\lambda,i}^+).$$

So $T_{\lambda,i} \in RS_n^{S_{n-1}}$ does not depend on the choice of S_{n-1}-conjugate of S_λ.

Lemma 14. *$\{T_{\lambda,i} \mid (\lambda,i) \Vdash n\}$ is a basis for $RS_n^{S_{n-1}}$.*

Proof. Modify the proof of Lemma 5 in the 'obvious' ways. $\square$

Recall that

$$\{e_{\lambda,i} \mid (\lambda,i) \Vdash n\} \quad \text{is a basis of } FS_n^{S_{n-1}}.$$

Theorem 15. *Let $(\lambda,i) \Vdash n$. Then*

$$T_{\lambda,i} = \sum \tau_{\mu,j;\lambda,i} e_{\mu,j}, \quad \textit{where } \tau_{\mu,j;\lambda,i} \neq 0 \implies (\mu,j) \trianglerighteq (\lambda,i).$$

In particular $\{T_{\lambda,i} \mid (\lambda,i) \Vdash n\}$ is a cellular basis of $RS_n^{S_{n-1}}$.

Proof. We apply Proposition 8 with $G = S_n, H = S_{n-1}$ and $K = S_{\lambda.i}$. Now K is S_n-conjugate to S_λ while $K \cap H = S_{\lambda^i}$. Thus

$$T_{\lambda,i} = \sum_{(\mu,j) \Vdash n} \tau_{\mu,j;\lambda,i} e_{\mu,j}, \quad \text{for } \tau_{\mu,j;\lambda,i} \in F,$$

where $K_{\mu\lambda} = \langle \chi_\mu, 1_{S_\lambda}^{S_n} \rangle \neq 0$ and $K_{\mu^j \lambda^i} = \langle \chi_{\mu^j}, 1_{S_{\lambda^i}}^{S_{n-1}} \rangle \neq 0$. In particular $\mu \trianglerighteq \lambda$ and $\mu^j \trianglerighteq \lambda^i$. $\square$

Example: for $FS_3^{S_2}$ we get

$$\begin{array}{rcl} T_{(3),3} &=& 6e_{(3),3}, \\ T_{(21),1} &=& 2e_{(3),3} + 2e_{(21),1}, \\ T_{(21),2} &=& 4e_{(3),3} + e_{(21),1} + 3e_{(21),2}, \\ T_{(1^3),1} &=& e_{(3),3} + e_{(21),1} + e_{(21),2} + e_{(1^3),1}. \end{array}$$

Note that from the definitions $T_{(3)} = T_{(3),3}$ and $T_{(2,1)} = T_{(2,1),1} + T_{(2,1),2}$ and $T_{(1^3)} = T_{(1^3),1}$. This also follows from $e_{(3)} = e_{(3),3}$ and $e_{(2,1)} = e_{(2,1),1} + e_{(2,1),2}$ and $e_{(1^3)} = e_{(1^3),1}$.

We give the details for the 7-dimensional algebra $FS_4^{S_3}$. If $P, Q, R, \ldots$ is a partition of $\{1, \ldots, n\}$, we use the notation $S(P;Q;R;\ldots)$ for the Young subgroup $\mathrm{Sym}(P) \times \mathrm{Sym}(Q) \times \mathrm{Sym}(R) \times \ldots$ of S_n. We describe the ordered partition $P, Q, R, \ldots$ by writing the elements in each part with brackets removed, and using commas to separate the parts. Then

$$\begin{array}{rclcl} T_{(4),4} &=& S_{1234}^+ &=& S_4^+, \\ T_{(31),1} &=& S_{123;4}^+ &=& C_{(31),1}^+ + C_{(21^2),1}^+ + C_{(1^4)}^+, \\ T_{(31),3} &=& S_{124;3}^+ + S_{134;2}^+ + S_{234;1}^+ &=& C_{(31),3}^+ + C_{(21^2);1}^+ + 2C_{(21^2);2}^+ + 3C_{(1^4)}^+, \\ T_{(2^2),2} &=& S_{12;34}^+ + S_{13;24}^+ + S_{23;14}^+ &=& C_{(2^2)}^+ + C_{(21^2);1}^+ + C_{(21^2);2}^+ + 3C_{(1^4)}^+, \\ T_{(21^2),1} &=& S_{12;3;4}^+ + S_{13;2;4}^+ + S_{23;1;4}^+ &=& C_{(21^2),1} + 3C_{(1^4)}^+, \\ T_{(21^2),2} &=& S_{14;2;3}^+ + S_{24;1;3}^+ + S_{34;1;2}^+ &=& C_{(21^2);2}^+ + 3C_{(1^4)}^+, \\ T_{(1^4),1} &=& S_{1;2;3;4}^+ &=& C_{1^4;1}^+. \end{array}$$

In terms of primitive idempotents, we have

$$\begin{array}{lcl}
T_{(4),4} & = & 24e_{(4)}, \\
T_{(31),1} & = & 6e_{(4)} + 6e_{(31),3}, \\
T_{(31),3} & = & 18e_{(4)} + 2e_{(31),3} + 8e_{(31),1}, \\
T_{(2^2),2} & = & 12e_{(4)} + 4e_{(31),3} + 4e_{(31),1} + 6e_{(2^2),2}, \\
T_{(21^2),1} & = & 6e_{(4)} + 6e_{(31),3} + 3e_{(31),1} + 3e_{(2^2),2} + 3e_{(21^2),1}, \\
T_{(21^2),2} & = & 6e_{(4)} + 2e_{(31),3} + 5e_{(31),1} + 3e_{(2^2),2} + e_{(21^2),1} + 4e_{(21^2),2}, \\
T_{(1^4),2} & = & e_{(4)} + e_{(31),3} + e_{(31),1} + e_{(2^2),2} + e_{(21^2),1} + e_{(21^2),2} + e_{(1^4),1}.
\end{array}$$

6. Character Tables

Let $H \leq G$. Then for $\alpha \in \mathrm{Irr}(G), \beta \in \mathrm{Irr}(H)$ with $\langle \alpha, \beta^G \rangle \neq 0$, we define the H-invariant map $\alpha/\beta : G \to \mathbb{C}$ as above

$$\alpha/\beta(g) := \frac{1}{|H|} \sum_{h \in H} \alpha(gh)\beta(h^{-1}).$$

These maps are analogues for $\mathbb{C}G^H$ of the irreducible characters of $\mathbb{C}G$. First, they determine the central primitive idempotents viz

$$e_{\alpha/\beta} = e_\alpha e_\beta = \frac{\alpha(1)\beta(1)}{|G|} \sum_{g \in G} \alpha/\beta(g^{-1})g.$$

Note that for C a H-class in G, $\alpha/\beta(c)$ is constant for $c \in C$. Now if $z \in Z(\mathbb{C}G^H)$, we have

$$z = \sum_{\alpha/\beta} \frac{\alpha/\beta(z)}{\langle \alpha, \beta^G \rangle} e_{\alpha/\beta}.$$

It follows that the *central characters* $\omega_{\alpha/\beta} : Z(\mathbb{C}G^H) \to \mathbb{C}$, with

$$\omega_{\alpha/\beta}(z) := \frac{\alpha/\beta(z)}{\langle \alpha, \beta^G \rangle},$$

give all algebra homomorphisms from $Z(\mathbb{C}G^H)$ onto $\mathbb{C}$ (or irreps, maximal ideals).

Note that $Z(\mathbb{C}G^H)$ is generated as $\mathbb{C}$-algebra by its unital subalgebras $Z(\mathbb{C}H)$ and $Z(\mathbb{C}G)$. In fact, as abstract $\mathbb{C}$-algebra we have

$$Z(\mathbb{C}G^H) = \frac{Z(\mathbb{C}H) \otimes_{\mathbb{C}} Z(\mathbb{C}G)}{\langle e_\gamma \otimes e_\delta \mid < \gamma^G, \delta >= 0 \rangle}.$$

Lemma 16. *Suppose that α/β is a character of $\mathbb{C}G^H$. Then if C is a conjugacy class of G we have*

$$\omega_{\alpha/\beta}(C^+) = \omega_\alpha(C^+),$$

while if C is a conjugacy class of H we have

$$\omega_{\alpha/\beta}(C^+) = \omega_\beta(C^+).$$

Proof. In the former case we have

$$C^+ = \sum_{\alpha \in \mathrm{Irr}(G)} \omega_\alpha(C^+) e_\alpha = \sum_{\alpha/\beta \in \mathrm{Irr}(\mathbb{C}G^H)} \omega_\alpha(C^+) e_{\alpha/\beta}.$$

In the latter case we have

$$C^+ = \sum_{\beta \in \mathrm{Irr}(H)} \omega_\beta(C^+) e_\beta = \sum_{\alpha/\beta \in \mathrm{Irr}(\mathbb{C}G^H)} \omega_\beta(C^+) e_{\alpha/\beta}.$$

□

Next, suppose that $Z(\mathbb{C}G^H)$ has dimension l. Then $Z(\mathbb{C}G^H) \cap \mathbb{Z}G^H$ has a $\mathbb{Z}$-basis $z_1, \ldots, z_l$. So there exist integers c_{ij}^k such that

$$z_i z_j = \sum_{k=1}^{l} c_{ij}^k z_k.$$

Applying $\omega_{\alpha/\beta}$ we get

$$\omega_{\alpha/\beta}(z_i)\omega_{\alpha/\beta}(z_j) = \sum_{k=1}^{l} c_{ij}^k \omega_{\alpha/\beta}(z_k).$$

It follows, by the usual argument, that the $\omega_{\alpha/\beta}(z_i)$ are algebraic integers.

The same argument shows that $\omega_{\alpha/\beta}(z)$ is an algebraic integer, for all z in the $\mathbb{Z}$-algebra generated by $Z(\mathbb{Z}H)$ and $Z(\mathbb{Z}G)$. Or just note that any such z is a $\mathbb{Z}$-linear combination of $z_1, \ldots, z_l$.

We write down the character tables of $\mathbb{Q}S_n^{S_l}$, for $n = 2, 3, 4, 5$ and $l = n, n-1$. In each case we have a table $[\omega_\alpha(\beta)]$ indexed by pairs of partitions of n, or pairs of marked partitions of n.

The table for $\mathbb{Q}S_2^{S_2} = \mathbb{Q}S_2^{S_1}$ is

	(2)	(1^2)
(2)	1	1
(1^2)	-1	1

The table for $\mathbb{Q}S_3^{S_3}$ is

	(3)	$(2,1)$	(1^3)
(3)	2	3	1
$(2,1)$	-1	0	1
(1^3)	2	-3	1

The table for $\mathbb{Q}S_3^{S_2}$ is

	$(\dot{3})$	$(2,\dot{1})$	$(\dot{2},1)$	$(\dot{1}^3)$
$(\dot{3})$	2	1	2	1
$(2,\dot{1})$	-1	1	-1	1
$(2,\dot{1})$	-1	-1	1	1
$(\dot{1}^3)$	2	-1	-2	1

The table for $\mathbb{Q}S_4^{S_4}$ is

	(4)	$(3,1)$	(2^2)	$(2,1^2)$	(1^4)
(4)	6	8	3	6	1
$(3,1)$	−2	0	−1	2	1
(2^2)	0	−4	3	0	1
$(2,1^2)$	2	0	−1	−2	1
(1^4)	−6	8	3	−6	1

The table for $\mathbb{Q}S_4^{S_3}$ is

	$(\dot{4})$	$(3,\dot{1})$	$(\dot{3},1)$	$(\dot{2}^2)$	$(2,\dot{1}^2)$	$(\dot{2},1^2)$	$(\dot{1}^4)$
$(\dot{4})$	6	2	6	3	3	3	1
$(3,\dot{1})$	−2	2	−2	−1	3	−1	1
$(\dot{3},1)$	−2	−1	1	−1	0	2	1
$(\dot{2}^2)$	0	−1	−3	3	0	0	1
$(2,\dot{1}^2)$	2	−1	1	−1	0	−2	1
$(\dot{2},1^2)$	2	2	−2	−1	−3	1	1
$(\dot{1}^4)$	−6	2	6	3	−3	−3	1

The table for $\mathbb{Q}S_5^{S_5}$ is

	(5)	$(4,1)$	$(3,2)$	$(3,1^2)$	$(2^2,1)$	$(2,1^3)$	(1^5)
(5)	24	30	20	20	15	10	1
$(4,1)$	−6	0	−5	5	0	5	1
$(3,2)$	0	−6	4	−4	3	2	1
$(3,1^2)$	4	0	0	0	−5	0	1
$(2^2,1)$	0	6	−4	−4	3	−2	1
$(2,1^3)$	−6	0	5	5	0	−5	1
(1^5)	24	−30	−20	20	15	−10	1

The table for $\mathbb{Q}S_5^{S_4}$ is

	$(\dot{5})$	$(4,\dot{1})$	$(\dot{4},1)$	$(3,\dot{2})$	$(\dot{3},2)$	$(3,\dot{1}^2)$	$(\dot{3},1^2)$	$(2^2,\dot{1})$	$(\dot{2}^2,1)$	$(2,\dot{1}^3)$	$(\dot{2},1^3)$	$(\dot{1}^5)$
$(\dot{5})$	24	6	24	8	12	8	12	3	12	6	4	1
$(4,\dot{1})$	−6	6	−6	−2	−3	8	−3	3	−3	6	−1	1
$(\dot{4},1)$	−6	−2	2	−2	−3	0	5	−1	1	2	3	1
$(3,\dot{2})$	0	−2	−4	4	0	0	−4	−1	4	2	0	1
$(\dot{3},2)$	0	0	−6	−2	6	−4	0	3	0	0	2	1
$(3,\dot{1}^2)$	4	−2	2	−2	2	0	0	−1	−4	2	−2	1
$(\dot{3},1^2)$	4	2	−2	2	−2	0	0	−1	−4	−2	2	1
$(2^2,\dot{1})$	0	0	6	2	−6	−4	0	3	0	0	−2	1
$(\dot{2}^2,1)$	0	2	4	−4	−2	0	−4	−1	4	−2	0	1
$(2,\dot{1}^3)$	−6	2	−2	2	3	0	5	−1	1	−2	−3	1
$(\dot{2},1^3)$	−6	−6	6	2	3	8	−3	3	−3	−6	1	1
$(\dot{1}^5)$	24	−6	−24	−8	−12	8	12	3	12	−6	−4	1

Note: we get the columns for $C(3,\dot{2})$ and $C(\dot{3},2)$ using the facts that

$$C(3,2)^+ = C(3,\dot{2})^+ + C(\dot{3},2)^+, \quad C(3,\dot{2})^+ = C(3,1)^+L_5 - C(\dot{4},1)^+.$$

Orthogonality Relations

Suppose that α/β and λ/μ are irreducible characters of $\mathbb{C}G^H$. Then $e_{\alpha/\beta}e_{\lambda/\mu} = [\alpha = \lambda][\beta = \mu]e_{\alpha/\beta}$. By computing the coefficient of 1 in $e_{\alpha/\beta}e_{\gamma/\delta}$, we see that

$$\sum_{C\in \mathcal{C}l_H(G)} \alpha/\beta(C^+)\frac{\lambda/\mu(C^{o+})}{|C|} = \sum_{g\in G} \alpha/\beta(g)\lambda/\mu(g^{-1}) = [\alpha = \lambda][\beta = \mu]\frac{|G|\langle \alpha_H, \beta\rangle}{\alpha(1)\beta(1)}.$$

It follows from this that, choosing a suitable ordering on the α/β, and the H-orbits C on G, we have the matrix identity

$$\left[\omega_{\alpha/\beta}(C^+)\right]\left[\frac{\omega_{\alpha/\beta}(C^{o+})}{|C|}\right]^t = \operatorname{diag}\left(\frac{|G|}{\alpha(1)\beta(1)\langle \alpha_H, \beta\rangle}\right).$$

Set $m = \dim(\mathbb{C}G^H)$. Then in particular the $l \times m$-matrix $[\omega_{\alpha/\beta}(C^+)]$ has a right inverse.

Lemma 17. *The restriction of the standard symmetric bilinear form on functions $\mathbb{C}G \to \mathbb{C}$*

$$\langle \mathcal{S}, \mathcal{T}\rangle := \frac{1}{|G|}\sum_{g\in G} \mathcal{S}(g)\mathcal{T}(g^{-1})$$

to H-invariant functions $\mathbb{C}G \to \mathbb{C}$ satisfies

$$\langle \alpha/\beta, \gamma/\delta\rangle = [\alpha = \gamma][\beta = \delta]\frac{\langle \alpha, \beta^G\rangle}{\alpha(1)\beta(1)}.$$

Corollary 18. *Suppose that $\mathcal{S}$ is a $\mathbb{C}$-linear combination of $\operatorname{Irr}(\mathbb{C}G^H)$. Then*

$$\mathcal{S} = \sum_{\alpha/\beta} \frac{\alpha(1)\beta(1)}{\langle \alpha, \beta^G\rangle}\langle \mathcal{S}, \alpha/\beta\rangle\, \alpha/\beta.$$

7. Induction?

Why does 'ordinary' induction from the class functions on a subgroup H of a finite group G to the class functions on G sned characters to characters? This is related to the values of characters of G on the idempotents in $\mathbb{C}G$. We also need the relative trace map.

Lemma 19. *Let e be an idempotent in $\mathbb{C}G$. Then $\chi(e) \in \mathbb{N}_0$, for each character χ of G.*

Proof. We may assume that $\chi \in \mathbb{C}(G)$. Then $e_\chi\mathbb{C}Ge_\chi$ is isomorphic to a full matrix algebra over $\mathbb{C}$. Now $ee_\chi = e_\chi e$ is an idempotent in this algebra. So $ee_\chi = 0$ or ee_χ is conjugate $e_{11}+\cdots+e_{tt}$, for some t between 1 and $\chi(1)$. We deduce that $\chi(e) = 0$ or $ee_\chi \neq 0$ and

$$\chi(e) = \chi(ee_\chi) = t.$$

□

Corollary 20. *Suppose that $H \leq G$ and $\chi \in \operatorname{Irr}(G)$. Then $\chi(H^+)$ is a non-negative integer divisible by $|H|$.*

Proof. The element $H^+/|H|$ is an idempotent in $\mathbb{C}G$. □

Suppose that ϕ is class function on H. Extend ϕ^o to a function on G which is zero on $G\backslash H$. The induced class function on G is

$$\phi^G(g) := \frac{1}{|H|}\sum_{x\in G}\phi^o(xgx^{-1}), \quad \text{for all } g \in G.$$

Thus for C a conjugacy class of G, and $c \in C$ we have

$$\phi^G(c) = \frac{|C_G(c)|}{|H|}\phi((C\cap H)^+).$$

Compare the next Lemma to Lemma 13.

Lemma 21. *Let $H \leq G$ and $\phi \in \mathrm{Irr}(H)$. Then*

$$\chi(e_\phi) = \phi(1)\langle\chi, \phi^G\rangle \quad \textit{and} \quad \mathrm{Tr}_H^G(e_\phi) = \frac{\phi(1)}{|H|}\sum_{g\in G}\phi^G(g^{-1})g.$$

Proof. We have $\mathrm{Tr}_H^G(e_\phi) = \sum_{\chi\in\mathrm{Irr}(G)}\zeta_{\phi,\chi}e_\chi$, where $\zeta_{\phi,\chi} \in \mathbb{C}$. Applying χ, we get

$$\zeta_{\phi,\chi} = [G:H]\chi(e_\phi)/\chi(1) = \frac{\phi(1)}{|H|}\frac{|G|}{\chi(1)}\langle\chi, \phi^G\rangle.$$

Now write $\mathrm{Tr}_H^G(e_\phi) = \sum_{g\in G}\mu_{\phi,g}g$, where $\mu_{\phi,g} \in \mathbb{C}$. Then

$$\mu_{\phi,g} = \frac{\phi(1)}{|H|}\sum_{\chi\in\mathrm{Irr}(G)}\langle\chi, \phi^G\rangle\chi(g^{-1}) = \frac{\phi(1)}{|H|}\phi^G(g^{-1}).$$

The lemma follows from this. □

Now suppose that H, K are subgroups of a finite group G. We want to define induction from class functions on K to H-class functions on G. Or K-class functions on G to H-class functions on G.

What is the formula? We start with an approximation. If ϕ is a class function on K, we define the following *induced* H-invariant function on G by

$$\begin{aligned}\phi^{G^H}(g) &:= \frac{[G:K]}{|N_H(K)|}\sum_{x\in H}\phi^o(xgx^{-1}), \quad \text{for all } g \in G.\\ &= \frac{[G:K]|C_H(g)|}{|N_H(K)|}\phi((C^o\cap H)^+), \quad C \text{ is the } H\text{-orbit containing } g.\end{aligned}$$

Note that ϕ^{G^H} is constant on C. So

$$\phi^{G^H}(C^+) = [G:K][H:N_H(K)]\,\phi\left((C\cap K)^+\right).$$

Lemma 22. *Let $H, K \leq G$, and let $\phi \in \mathrm{Irr}(K)$. Then*

$$\mathrm{Tr}_{N_H(K)}^H(e_\phi) = \frac{\phi(1)}{|G|}\sum_{g\in G}\phi^{G^H}(g^{-1})g.$$

Proof. This follows from

$$\begin{aligned}\mathrm{Tr}^H_{N_H(K)}(e_\phi) &= |N_H(K)|\frac{\phi(1)}{|K|}\sum_{k\in K}\sum_{h\in H}\phi(k^{-1})(hkh^{-1})\\ &= \frac{|N_H(K)|\phi(1)}{|K|}\sum_{C\in \mathrm{Cl}_H(G)}|C_H(c)|\phi((C^o\cap K)^+)C^+\\ &= \frac{\phi(1)}{|G|}\sum_{g\in G}\phi^{G^H}(g^{-1})g.\end{aligned}$$

□

Lemma 23 (Frobenius Reciprocity). *For $\alpha/\beta\in\mathrm{Irr}(\mathbb{C}G^H)$ and $\phi\in\mathrm{Irr}(K)$ we have*

$$\langle\alpha/\beta,\phi^{G^H}\rangle_G \;=\; [H:N_H(K)]\,\langle\alpha/\beta\downarrow_K,\phi\rangle_K.$$

Proof. The inner product $\langle\alpha/\beta,\phi^{G^H}\rangle_G$ evaluates as

$$\frac{1}{|G|}\sum_{g\in G}\phi^{G^H}(g)\alpha/\beta(g^{-1}) = \frac{1}{|K|\,|N_H(K)|\,|H|}\sum_{g\in G,h_1,h_2\in H}\alpha(g^{-1}h_1)\beta(h_1^{-1})\phi^o(h_2gh_2^{-1}).$$

Now suppose that $k\in K$. Then the $(h_2,g)\in H\times G$ such that $h_2gh_2^{-1}=k$ are precisely the $|H|$-pairs $(h_2,h_2^{-1}kh_2)$, for $h_2\in H$. So the sum above is

$$\frac{1}{|K||N_H(K)||H|}\sum_{k\in K,h_1,h_2\in H}\alpha(h_2k^{-1}h_2^{-1}h_1)\beta(h_1^{-1})\phi(k) = \frac{[H:N_H(K)]}{|K|}\sum_{k\in K}\alpha/\beta(k^{-1})\phi(k).$$

□

Note that $\alpha/\beta\downarrow_K$ is a $\mathbb{C}$-valued function on K that is invariant on the intersections with K of the H-orbits on G. However, it is generally not a class function on K. So a priori, there is no reason why $\langle\alpha/\beta\downarrow_K,\phi\rangle$ is an integer (or even an element of $\mathbb{Q}_0$).

Here are two special cases.

Lemma 24. *If $K=G$ and $\chi\in\mathrm{Irr}(G)$, then*

$$\chi^{G^H} \;=\; \sum_{\substack{\phi\in\mathrm{Irr}(H)\\ \langle\chi,\phi^G\rangle\neq 0}}\phi(1)\chi/\phi.$$

If instead $K=H$ and $\phi\in\mathrm{Irr}(H)$, then

$$\phi^{G^H} \;=\; \sum_{\substack{\chi\in\mathrm{Irr}(G)\\ \langle\chi,\phi^G\rangle\neq 0}}\chi(1)\chi/\phi.$$

Proof. Let C be a H-orbit in G. Assume the first hypothesis. Then $\chi^{G^H}(C^+)=\chi(C^+)$. Also $\phi(1)\chi/\phi(C^+)=\chi(C^+e_\phi)$ and $e_\chi=\sum_\phi e_\chi e_\phi$. So

$$\sum_{\substack{\phi\in\mathrm{Irr}(H)\\ \langle\chi,\phi^G\rangle\neq 0}}\phi(1)\chi/\phi(C^+) = \sum_\phi\chi(C^+e_\phi)=\chi(C^+).$$

For the second hypothesis, we have $\phi^{G^H}(C^+) = [G : H]\phi(C^+)$, if C is a conjugacy class of H, and $\phi^{G^H}(C^+) = 0$, if $C \subseteq G\backslash H$. Now

$$\sum_{\substack{\chi \in \mathrm{Irr}(G) \\ \langle \chi, \phi^G \rangle \neq 0}} \chi(1)\chi/\phi(C^+) = \rho_G(C^+ e_\phi)/\phi(1) = \frac{|G|}{\phi(1)}\{\text{coeff. of } 1 \text{ in } C^+ e_\phi\}.$$

Suppose first that C is a conjugacy class of H. Then

$$\rho_G(C^+ e_\phi) = [G : H]\phi(C^+).$$

Suppose then that $C \subseteq G\backslash H$. Then

$$\frac{|G|}{\phi(1)}\rho_G(C^+ e_\phi) = 0.$$

□

We want to define induction from class functions on S_λ to S_{n-1}-class functions on S_n? Or S_λ-class functions on G to S_{n-1}-class functions on S_n?

What is the formula? We start with an approximation. If ϕ is a class function on S_λ, we define the following *induced* S_{n-1}-invariant function on S_n

$$\phi^{S_n^{S_{n-1}}}(g) := \frac{[S_n : S_\lambda]}{|N_{S_{n-1}}(S_\lambda)|} \sum_{x \in S_{n-1}} \phi^o(xgx^{-1}), \quad \text{for all } g \in S_n.$$

Now suppose that C is an S_{n-1}-orbit on S_n. Then $\phi^{S_n^{S_{n-1}}}$ is constant on C. It follows easily that

$$\phi^{S_n^{S_{n-1}}}(C^+) = [S_n : S_\lambda][S_{n-1} : N_{S_{n-1}}(S_\lambda)]\, \phi\left((C \cap S_\lambda)^+\right).$$

Example: The character table for $\mathbb{Q}S_3^{S_2}$ is

	$(\dot{3})$	$(2,\dot{1})$	$(\dot{2},1)$	$(\dot{1}^3)$
$(\dot{3})$	2	1	2	1
$(2,\dot{1})$	-1	1	-1	1
$(2,\dot{1})$	-1	-1	1	1
$(\dot{1}^3)$	2	-1	-2	1

If $\mu^1, \mu^2, \ldots$ is a finite sequence of partitions, there is a corresponding irreducible character of S_λ, where $\lambda_i = |\mu^i|$, for $i = 1, 2, \ldots$. We consider the induced function of each such character, with one partition μ^i marked as $\overline{\mu^i}$ to indicate the marked partition λ, i. This gives the following table of induced S_2-class functions:

	$(\dot{3})$	$(2,\dot{1})$	$(\dot{2},1)$	$(\dot{1}^3)$
$\overline{(3)}$	2	1	2	1
$\overline{(21)}$	−2	0	0	2
$\overline{(1^3)}$	2	−1	−2	1
$(2),\overline{(1)}$	0	3	0	3
$(1^2),\overline{(1)}$	0	−3	0	3
$\overline{(2)}(1)$	0	0	6	6
$\overline{(1^2)}(1)$	0	0	−6	6
$(1),(1),\overline{(1)}$	0	0	0	6

The transition matrix from the induced functions (in the order listed above) and the irreducible characters of $\mathbb{Q}S_3^{S_2}$ (in the order of the previous table) is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 2 & 1 & 3 & 0 \\ 0 & 3 & 1 & 2 \\ 1 & 2 & 2 & 1 \end{bmatrix}$$

Example: The table for $\mathbb{Q}S_4^{S_3}$ is

	$(\dot{4})$	$(3,\dot{1})$	$(\dot{3},1)$	$(\dot{2}^2)$	$(2,\dot{1}^2)$	$(\dot{2},1^2)$	$(\dot{1}^4)$
$(\dot{4})$	6	2	6	3	3	3	1
$(3,\dot{1})$	−2	2	−2	−1	3	−1	1
$(\dot{3},1)$	−2	−1	1	−1	0	2	1
$(\dot{2}^2)$	0	−1	−3	3	0	0	1
$(2,\dot{1}^2)$	2	−1	1	−1	0	−2	1
$(\dot{2},1^2)$	2	2	−2	−1	−3	1	1
$(\dot{1}^4)$	−6	2	6	3	−3	−3	1

	$(\dot{4})$	$(3,\dot{1})$	$(\dot{3},1)$	$(\dot{2}^2)$	$(2,\dot{1}^2)$	$(\dot{2},1^2)$	$(\dot{1}^4)$
$\overline{(4)}$	6	2	6	3	3	3	1
$\overline{(31)}$	-6	0	0	-3	3	3	3
$\overline{(2^2)}$	0	-2	-6	6	0	0	2
$\overline{(21^2)}$	6	0	0	-3	-3	-3	3
$\overline{(1^4)}$	-6	2	6	3	-3	-3	1
$(3)\overline{(1)}$	0	8	0	0	12	0	4
$(21)\overline{(1)}$	0	-8	0	0	0	0	8
$(1^3)\overline{(1)}$	0	8	0	0	-12	0	4
$\overline{(3)}(1)$	0	0	24	0	12	24	12
$\overline{(21)}(1)$	0	0	-24	0	0	0	24
$\overline{(1^3)}(1)$	0	0	24	0	-12	-24	12
$(2)\overline{(2)}$	0	0	0	18	18	18	18
$(1^2)\overline{(2)}$	0	0	0	-18	18	-18	18
$(2)\overline{(1^2)}$	0	0	0	-18	-18	18	18
$(1^2)\overline{(1^2)}$	0	0	0	18	-18	-18	18
$(2)(1)\overline{(1)}$	0	0	0	0	36	0	36
$(1^2)(1)\overline{(1)}$	0	0	0	0	36	0	36
$\overline{(2)}(1)(1)$	0	0	0	0	0	36	36
$\overline{(1^2)}(1)(1)$	0	0	0	0	0	-36	36
$(1)(1)(1)\overline{(1)}$	0	0	0	0	0	0	24

The transition matrix from the induced functions (in the order listed above) and the irreducible characters of $\mathbb{Q}S_4^{S_3}$ (in the order of the previous table) is

$$\left[\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
1 & 3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 2 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 3 & 1 \\
3 & 1 & 8 & 0 & 0 & 0 & 0 \\
\hline
0 & 4 & 5 & 6 & 5 & 4 & 0 \\
0 & 0 & 0 & 0 & 8 & 1 & 3 \\
\hline
3 & 3 & 6 & 6 & 0 & 0 & 0 \\
0 & 6 & 3 & 0 & 9 & 0 & 0 \\
0 & 0 & 9 & 0 & 3 & 6 & 0 \\
0 & 0 & 0 & 6 & 6 & 3 & 3 \\
\hline
3 & 9 & 9 & 6 & 9 & 0 & 0 \\
0 & 0 & 9 & 6 & 9 & 9 & 3 \\
\hline
3 & 3 & 15 & 6 & 3 & 6 & 0 \\
0 & 6 & 3 & 6 & 15 & 3 & 3 \\
\hline
1 & 3 & 6 & 4 & 6 & 3 & 1
\end{array}\right]$$

8. Symmetric Functions

[24 Sep]

The ring of symmetric functions Λ in a countable number of variables $x_1, x_2, \ldots$, over $\mathbb{Q}$ has integral basis the monomial sf's $\{m_\lambda\}$, the complete sf's $\{h_\lambda\}$ and the elementary sf's $\{e_\lambda\}$. Another integral basis is the Schur sf's $\{s_\lambda\}$. The power sum sf's $\{p_\lambda\}$ only give a rational basis. Set Λ^n as the subspace spanned by $\{m_\lambda \mid \lambda \vdash n\}$.

The usual symmetric bilinear form on Λ satisfies $\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}$. Also $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$ and $\langle e_\lambda, f_\mu \rangle = \delta_{\lambda\mu}$, where f_λ are the forgotten sf's.

The frobenius map is $\mathrm{Fr} : L_n \to \Lambda^n$, where L_n is the $\mathbb{Q}$-vector space of class functions on $\mathbb{Q}_n$:

$$\mathrm{Fr}(f) = \sum_{\lambda \vdash n} f(\lambda) \frac{p_\lambda}{c_\lambda}$$

where c_λ is the order $\prod_i i^{m_i(\lambda)} m_i(\lambda)!$ of the centraliser in S_n of a permutation of cycle type λ. Actually this extends to a ring isomorphism $\Lambda \to \oplus_{n \geq 0} L_n$, where on the rhs, for $f \in L_n$ and $g \in L_m$ we have $fg = f \otimes g \uparrow^{S_{n+m}}$ is given by outer tensor product followed by induction from $S_n \times S_m$ to S_{n+m}.

Let C_λ be the *indicator function* for the conjugacy class of type λ and let χ_λ be the irreducible character of S_n corresponding to $\lambda \vdash n$. Then $\mathrm{Fr}(C_\lambda) = p_\lambda / c_\lambda$, $\mathrm{Fr}(1_{S_\lambda}^{S_n}) = h_\lambda$ and $\mathrm{Fr}(\chi_\lambda) = s_\lambda$.

Thus

$$h_\lambda = \sum_{\mu \vdash n} 1_{S_\lambda}^{S_n}(\mu) \frac{p_\mu}{c_\mu} \quad = \sum_{\mu \vdash n} \frac{|C_\mu \cap S_\lambda|}{|S_\lambda|} p_\mu,$$

$$s_\lambda = \sum_{\mu \vdash n} \chi_\lambda(\mu) \frac{p_\mu}{c_\mu}.$$

Note: $h_\lambda = \sum_{\mu \vdash n} \langle 1_{S_\lambda}^{S_n}, \chi_\mu \rangle s_\mu$.

Let $t_\lambda = \mathrm{Fr}(T_\lambda)$. Then we see that

$$t_\lambda = \sum_{\mu \vdash n} \frac{|C_\mu \cap S_\lambda|}{|N_{S_n}(S_\lambda)|} p_\mu = \frac{1}{[N_{S_n}(S_\lambda) : S_\lambda]} h_\lambda.$$

To get T_λ in terms of the primitive idempotents e_μ, we note that e_μ represents the class function $I_\mu : \rho \to \frac{\chi_\mu(1)}{|S_n|} \chi_\mu(\rho)$. Set $i_\mu = \mathrm{Fr}(I_\mu)$. Then

$$i_\mu = \mathrm{Fr}(e_\mu) = \frac{\chi_\mu(1)}{|S_n|} s_\mu.$$

Warning: do not confuse the idempotent $e_\mu \in Z(\mathbb{Q}S_n)$ and the elementary symmetric function usually denoted by the same symbol. Now we have

$$t_\lambda = \frac{1}{[N_{S_n}(S_\lambda) : S_\lambda]} h_\lambda = \sum_{\mu \vdash n} \frac{\langle 1_{S_\lambda}^{S_n}, \chi_\mu \rangle \left(|S_n|/\chi_\mu(1)\right)}{[N_{S_n}(S_\lambda) : S_\lambda]} i_\mu.$$

Now consider the bilinear form B on all functions $S_n \to \mathbb{Q}$ such that $B(u, v) = \frac{1}{|G|} \sum_{g \in G} u(g) v(g^{-1})$. This restricts to L_n and makes the characters an orthonormal basis. Moreover, $B(i_\alpha, i_\beta) = \frac{\delta_{\alpha\beta}}{c_\alpha}$, for all $\alpha, \beta \vdash n$. As Fr is an isometry, $B(p_\alpha, p_\beta) = c_\alpha \delta_{\alpha\beta}$.

Vladimir Dotsenko, of TCD, arXiv:0802.1340v1[math.RT] 10 Feb 2008:

Theorem 25. *Suppose that X is an S_n-set, with (permutation) character θ_X. As usual, the Frobenius character of θ_X is the symmetric function* $\mathrm{Fr}_X := \sum_{\lambda \vdash n} \chi_X(\lambda) \frac{p_\lambda}{z_\lambda}$. *Then*

$$\mathrm{Fr}_X = \sum_{\lambda \vdash n} |X/S_\lambda| m_\lambda.$$

Here m_λ is the monomial s.f. and X/S_λ is the set of orbits of S_λ acting on X. In particular, if X is the set of right cosets of $H \leq S_n$, then $|X/S_\lambda|$ is the number of H, S_λ-double cosets in S_n.

9. General remarks on Hecke Algebras

Let G be a group, with subgroup H, let F be a field whose char does not divide $|G|$. Set $e = H^+/|H|$. Identify F_H^G with H^+FG, a left ideal of FG. Then $\mathrm{End}_{FG}(F_H^G)$ is isomorphic to the non-unital subalgebra $eFGe$ of FG. The isomorphism sends $f \in \mathrm{End}_{FG}(H^+FG)$ to left multiplication by $f(e) = ef(e)e$. Moreover, $eFGe$ has a Schur basis $\{(HgH)^+/|H| \mid HgH \subseteq G\}$ in bijection with the H, H-double cosets in G.

Lemma 26. *F_H^G is multiplicity free iff $\mathrm{End}_{FG}(F_H^G)$ is commutative.*

Note that $g \to g^{-1}$, for $g \in G$ induces an anti-automorphism on FG. We denote the image of $x \in FG$ under this by x^o. Now ^{o} restricts to an anti-automorphism of $eFGe$. This maps the Schur basis element $(HgH)^+/|H|$ to the Schur basis element $(Hg^{-1}H)^+/|H|$. Now for $a, b \in G$, we have

$$(Hb^{-1}H)^+(Ha^{-1}H)^+ = \left((HaH)^+(HbH)^+\right)^o.$$

Lemma 27. *Suppose that each H, H-double coset in G contains a group element of order 1 or 2. Then $\mathrm{End}_{FG}(F_H^G)$ is commutative.*

Proof. For $t \in G$, with $t^2 = 1$, we have $(Ht^{-1}H)^+ = (HtH)^+$. So ^{o} is the identity on $eFGe$. Now typical Schur basis elements can be written $(HsH)^+/|H|$ and $(HtH)^+/|H|$, for $s, t \in G$ with $s^2 = t^2 = 1$. So

$$\frac{(HsH)^+}{|H|}\frac{(HtH)^+}{|H|} = \left(\frac{(HsH)^+}{|H|}\frac{(HtH)^+}{|H|}\right)^o = \frac{(HtH)^+}{|H|}\frac{(HsH)^+}{|H|}.$$

□

Alternative proof (Gow): Write $1_H^G = \sum_{\chi \in \mathrm{Irr}(G)} n_\chi \chi$. Then by Wedderburn's Theorem, the number of H, H-double cosets in G is

$$\sum_{\chi \in \mathrm{Irr}(G)} n_\chi^2.$$

It turns out that the number of self-dual H, H-double cosets in G is given by

$$\sum_{\chi \in \mathrm{Irr}(G)} n_\chi \nu(\chi),$$

where $\nu(\chi)$ is the FS-indicator of χ. Equating these two quantities, under the hypothesis that all H, H-double cosets are self-dual, we get $\sum_{\chi \in \mathrm{Irr}(G)} n_\chi^2 = \sum_{\chi \in \mathrm{Irr}(G)} n_\chi \nu(\chi)$. The result follows, as $\nu(\chi) \in \{0, \pm 1\}$, for all χ. In fact, we get that $\nu(\chi) = +1$, for all χ appearing with non-zero multiplicity in 1_H^G.