INTERESTING RESULTS ON GENERATING FUNCTIONS

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1. Partitions and diagrams

Notation: the q-series notation is

$$(a;q)_k := \prod_{n=0}^{k}(1-aq^n), \quad (aq)_\infty := \prod_{n=0}^{\infty}(1-aq^n).$$

We will use the parameter x instead of q.

Let $\mathcal{P}$ be the set of partitions, and let $p(n)$ be the number of partitions of n. For a partition λ, the sum of the parts is $|\lambda|$ and the number of parts is $l(\lambda)$. Note that the generating function for the number of partition of n is $P(x) = (x;x)_\infty^{-1}$ i.e.

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{k=0}^{\infty}(1-x^k)^{-1}.$$

A *MacMahon diagram* is a pair $([\lambda], N)$ where λ is a partition and N is a set of 'marked' end-of-row nodes in $[\lambda]$ such that no marked node occurs above an unmarked node. The generating function for the number of MacMahon diagrams is $P(x)^2$.

A *standard MacMahon diagram* is a MacMahon diagram $([\lambda], N)$ where only removable nodes can belong to N. So the generating function for the number of standard MacMahon diagrams is:

$$\prod_{n=0}^{\infty}\frac{(1+x^n)}{(1-x^n)} = \prod_{n=0}^{\infty}(1-x^{2n-1})^{-1}(1-x^n)^{-1}.$$

A 2-*modular* diagram is a partition λ filled by 2's and 1's such that a 1 can only appear at the end of a row, and no 2 can occur below a 1. There is a bijection between 2-modular diagrams and partition diagrams: replace each 2 by a horizontal domino, and each 1 by a box. So the 1's document the odd rows of the partition. Also there is a bijection between MacMahon diagrams and 2-modular diagrams. Setting $o(\lambda)$ as the number of odd parts in λ, we get

$$\prod_{n=0}^{\infty}\frac{(1+x^n)}{(1-x^n)} = \sum_{\lambda} x^{(|\lambda|+o(\lambda))/2}.$$

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1.1. **Odd or Distinct parts.** Let $\mathcal{O}$ be the partitions of with all parts odd. Then $|\mathcal{O}(n)| = |\mathcal{D}(n)|$. This is equivalent to:

$$\prod_{n=1}^{\infty}(1+x^n)(1-x^{2n-1}) = 1,$$

or $(-x;x)_\infty(x;x^2)_\infty = 1$. Also

$$O(x) = \prod_{n=1}^{\infty}(1-x^{2n-1})^{-1} = \prod_{n=1}^{\infty}(1+x^n) = D(x).$$

Another form of this is

$$\prod_{n=1}^{\infty}(1+x^{2n-1})^{-1} = \prod_{n=1}^{\infty}(1-x^{2n-1})(1+x^{2n}).$$

Also $(1+x^n)(1-x^{2n})^{-1} = (1-x^n)^{-1}$. So

$$P(x) = D(x)P(x^2), \quad D(x) = \frac{P(x)}{P(x^2)} = O(x).$$

Then

$$O(x)^{-1} = D(x)^{-1} = \prod_{n=1}^{\infty}(1+x^n)^{-1} = \sum_{\lambda}(-1)^{l(\lambda)}x^{|\lambda|}.$$

In fact (G.E. Andrews) for each $k \geq 0$, the number of partitions of n into odd parts (with repetition), such that exactly k distinct parts appear equals the number of partitions of n into distinct parts such that exactly k sequences of consecutive integers occur. Now let $|\lambda| = \sum \lambda_i$ be sum and $|\lambda|_a := \sum(-1)^{i-1}\lambda_i$ the alternating sum of parts of a partition λ. Then (Bousquet-Mélou, Eriksson):

$$\sum_{\lambda \in D(n)} x^{|\lambda|}y^{|\lambda|_a} = \sum_{\lambda \in O(n)} x^{|\lambda|}y^{l(\lambda)}.$$

So for each k the number of partitions of n into distinct parts with alternating sum k is equal to the number of partitions of n into k distinct parts.

For $m > 0$, write $m = \sum 2^j$, distinct powers of 2. Set $G(i^m) = (2^j i)$ and define the Glaisher bijection: $G : \mathcal{O}(n) \to \mathcal{D}(n)$ by applying G to the distinct parts of λ, and concatenating.

Sylvester bijection: $\mathcal{D}(n) \to \mathcal{O}(n)$. Let $[\lambda_1, \ldots, \lambda_{2k}] \in \mathcal{D}(n)$ (where λ_{2k} may be 0). Lay out the boxes of λ in rows, with λ_1-boxes in the first row. Place λ_2-boxes on top, starting on the left of row 1. Next place λ_3-boxes on top, starting on the right of row 2 e.t.c. Divide the resulting diagram vertically, so that $|\lambda|_a$ boxes lie to the right in row 1. Consider the column lengths to the rights. Each one is odd. The rows to the left come in pairs of equal length. Take the pair in rows $2i-1, 2i$ and extend one column of length $2i-1$ by the sum of these rows (an even number). Now take the resulting column lengths (all odd!) as the partition in $\mathcal{O}(n)$.

1.2. **Odd and Different parts.** Let $p^o(n)$ be the number of partitions of n with all parts odd and different. Then Olsson:

$$P^o(x) = \frac{P(x)P(x^4)}{P(x^2)^2}.$$

Let $\Delta_e(n)$ be the number of e-core partitions of n. Then

$$\Delta_e(x) = \frac{P(x)}{P(x^e)^e}.$$

Let $K_e(n)$ be the number of partitions in an e-block of weight n. Then $P(x) = \Delta(x)_e K_e(x^e)$. So

$$K_e(x) = P(x)^e.$$

Let $L_e(n)$ be the number of e-regular partitions in an e-block of weight n. Then

$$L_e(x) = P(x)^{e-1}.$$

2. Partition Identities

2.1. **Gauss Identities.** First Gauss identity, the inverse of the generating function for MacMahon diagrams:

$$\prod_{n=1}^{\infty} \frac{(1-x^n)}{(1+x^n)} = \sum_{k=-\infty}^{\infty} (-1)^k x^{k^2}.$$

Here the lhs is the sum of $(-1)^{l(\lambda)}$ over all standard MacMahon diagrams. This is proved by Igor Pak using a sign-reversing involution: compare the length of the last row $h(\lambda)$ of λ and the last (rightmost) column $v(\lambda)$ (also the multiplicity of λ_1). If $h(\lambda) > v(\lambda)$, move the last column to form the row below the last row.If $h(\lambda) < v(\lambda)$, move the last row to form the column to the right of the last column. If $h(\lambda) = v(\lambda)$, and there is a marked node in the last column, unmark this node, and move the last column to form a row between the penultimate and last row. If $h(\lambda) = v(\lambda)$, and there is no marked node in the last column, unmark this node, and move the last column to form a row between the penultimate and last row. problem cases: $k \times (k+1)$ and $(k+1) \times k$ and $k \times k$-rectangles.

Special case $e = 2$ on 2-cores (see below), called here the *second Gauss Identity*:

$$\prod_{n=1}^{\infty}(1+x^{2n-1})(1-x^{4n}) = \prod_{n=1}^{\infty} \frac{(1-x^{2n})}{(1-x^{2n-1})}$$
$$= \frac{P(x)}{P(x^2)^2} = \frac{D(x)}{P(x^2)} = \sum_{k=0}^{\infty} x^{k(k+1)/2}.$$

In terms of q-series $(x^2;x^2)(x;x^2)^{-1} = \sum_{k=0}^{\infty} x^{k(k+1)/2}$. Another sign-reversing involution: if λ_1 is even, move one node below last row, to get a partition with first part odd, and last part 1. If λ_1 is odd, but $\lambda_l > 1$, consider the last row, as before, and the last pair of columns. The length of the last row is $h(\lambda) = \lambda_l$. If $\lambda_1 - 1$ is not a part length of λ, the length of the last pair of columns is $v(\lambda) := 2m_1$, where m_1 is the multiplicity of λ_1; otherwise it is $v(\lambda) := 2m_1 + 1$. The only problem case is the $(2k+1) \times k$-rectangle.

2.2. **Euler Pentagonal Number theorem.** Euler:

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} = \prod_{n=1}^{\infty} (1-x^n)^{-1}.$$

Euler Pentagonal Number Theorem:

$$P(x)^{-1} = \prod_{n=1}^{\infty}(1-x^n) = \sum_{m=-\infty}^{\infty} (-1)^m x^{(3m^2+m)/2}.$$

Thus $(x;x)_\infty = \sum_{m=-\infty}^{\infty} (-1)^m x^{(3m^2+m)/2}$.

Jacobi (Theorem 357 in Hardy & Wright):

$$\prod_{n=1}^{\infty}(1-x^n)^3 = \sum_{m=0}^{\infty}(-1)^m(2m+1)x^{m(m+1)/2}.$$

So $(x;x)_\infty^3 = \sum_{m=0}^{\infty}(-1)^m(2m+1)x^{m(m+1)/2}$.

Ramanujan's Identity (Hardy & Wright, pxxxv):

$$\sum_{n\geq 0} p(5n+4)x^n = 5\frac{(x^5;x^5)^5}{(x;x)^6}.$$

Rogers-Ramanujan identities (Wikepedia):

$$1+\sum_{n\geq 1}\frac{x^{n^2}}{(1-x)(1-x^2)\dots(1-x^n)} = \prod_{n\geq 1}\frac{1}{(1-x^{5n-1})(1-x^{5n-4})}$$
$$1+\sum_{n\geq 1}\frac{x^{n^2+n}}{(1-x)(1-x^2)\dots(1-x^n)} = \prod_{n\geq 1}\frac{1}{(1-x^{5n-2})(1-x^{5n-3})}.$$

In q-series notation:

$$\sum_{n=0}^{\infty}\frac{x^{n^2}}{(x;x)_n} = \frac{1}{(x;x^5)(x^4;x^5)}$$
$$\sum_{n=0}^{\infty}\frac{x^{n^2+n}}{(x;x)_n} = \frac{1}{(x^2;x^5)(x^3;x^5)}.$$

There is a bijection between partitions and the product set of partitions with all parts distinct $\mathcal{D}$ and partitions with all parts even $\mathcal{E}$. Alternatively, $(1+x^n)(1-x^{2n})^{-1} = (1-x^n)^{-1}$. Thus

$$\prod_{n=1}^{\infty}(1-x^n)^{-1} = \prod_{n=1}^{\infty}(1+x^n)(1-x^{2n})^{-1}$$

2.3. **Jacobi-Triple product.** Various forms:

$$\prod_{n=1}^{\infty}(1+st^{2n-1})(1+s^{-1}t^{2n-1})(1-t^{2n}) = \sum_{k=-\infty}^{\infty} s^k t^{k^2}.$$
$$\prod_{n=1}^{\infty}(1+st^{n})(1+s^{-1}t^{n-1})(1-t^{n}) = \sum_{k=-\infty}^{\infty} s^k t^{k(k+1)/2}.$$
$$\prod_{n=1}^{\infty}(1+x^{2n-1}y^{2n-2})(1+x^{2n-1}y^{2n})(1-x^{2n}y^{2n}) = \sum_{k=-\infty}^{\infty} x^{k^2}y^{k^2+k}.$$

In q-series notation:

$$(-ax;x^2)(-a^{-1}x;x^2)(x^2;x^2) = \sum_{k=-\infty}^{\infty} a^k x^{k^2}$$

$$(a^{-1}x;x^3)(ax^2;x^3)(x^3;x^3) = \sum_{k=-\infty}^{\infty} (-1)^k a^k x^{(3k^2+k)/2}$$

$$(a^{-1};x)(ax;x)(x;x) = \sum_{k=-\infty}^{\infty} (-1)^k a^k x^{(k^2+k)/2}$$

Special case: $y = x^{-1/2}$. Then

$$\prod_{n=1}^{\infty}(1+x^n)^2(1-x^n) = \prod_{n=1}^{\infty}(1+x^{2n-1})(1-x^{4n}) = \sum_{k=0}^{\infty} x^{k(k-1)/2}.$$

Thus $(-x;x)_\infty^2(x;x)_\infty = (-x;x^2)_\infty(x^4;x^4)_\infty = \sum_{k=0}^{\infty} x^{k(k-1)/2}$.

Other general cases (see (19.9.1) and (19.9.2) in Hardy & Wright, for any a, b:

$$\prod_{n=0}^{\infty}(1-x^{2an+a-b})(1-x^{2an+a+b})(1-x^{2an+2a}) = \sum_{k=-\infty}^{\infty}(-1)^k x^{ak^2+bk},$$

$$\prod_{n=0}^{\infty}(1+x^{2an+a-b})(1+x^{2an+a+b})(1-x^{2an+2a}) = \sum_{k=-\infty}^{\infty} x^{ak^2+bk}.$$

In particular, taking $a = \frac{3}{2}$ and $b = \frac{1}{2}$ we obtain Euler's theorem:

$$\prod_{n=0}^{\infty}(1-x^n) = \prod_{n=0}^{\infty}(1-x^{3n+1})(1-x^{3n+2})(1-x^{3n+3}) = \sum_{k=-\infty}^{\infty}(-1)^k x^{(3k^2+k)/2}.$$

2.4. **Cilann Boulet 4-parameter identity.** Cilanne E. Boulet. Let $\equiv$ be congruence modulo 2. The box with upper left hand vertex (i, j) has one of four colours, depending on the parities of i and j:

$$c(i,j) = \begin{cases} a, & \text{if } i \equiv 0, j \equiv 0. \\ b, & \text{if } i \equiv 0, j \equiv 1. \\ c, & \text{if } i \equiv 1, j \equiv 0. \\ d, & \text{if } i \equiv 1, j \equiv 1. \end{cases}$$

Note that $c(i,j) = a$ or d if $j-i \equiv 0$, while $c(i,j) = b$ or c if $j-i \equiv 1$. Set $\alpha(\lambda), \beta(\lambda), \gamma(\lambda), \delta(\lambda)$ as the numbers of nodes of colour a, b, c, d, respectively in $[\lambda]$. These numbers refine $c_0(\lambda)$ and $c_1(\lambda)$, because

$$c_0(\lambda) = \alpha(\lambda) + \delta(\lambda), \quad c_1(\lambda) = \beta(\lambda) + \gamma(\lambda).$$

Then we have

$$\sum_{\lambda\in\mathcal{P}} a^{\alpha(\lambda)}b^{\beta(\lambda)}c^{\gamma(\lambda)}d^{\delta(\lambda)} = \prod_{n=1}^{\infty} \frac{(1+a^nd^{n-1}b^{n-1}c^{n-1})(1+a^nd^{n-1}b^nc^n)}{(1-a^nd^nb^nc^n)(1-a^nd^{n-1}b^nc^{n-1})(1-a^nd^{n-1}b^{n-1}c^n)}.$$

The proof is based on a bijection $\mathcal{P} \leftrightarrow \mathcal{R}\times\mathcal{S}$. Here $\mathcal{R}$ is the set of partitions α with $\alpha_{2i-1}-\alpha_{2i}\leq 1$ for all i and $\mathcal{S}$ is the set of partitions where all parts are odd and of even multiplicity. The map $(\lambda)\to(\alpha,\beta)$ is obtained via: β is got by stripping all pairs of odd length columns from λ, then transposing the resulting parts. α is got by pushing together the remaining columns of λ. Then pairs of rows $\alpha_{2i-1},\alpha_{2i}$ of α come in one of two types, depending on whether $\alpha_{2i-1}=\alpha_{2i}$ or $\alpha_{2i}+1$. In the latter case the pair of parts occurs at most once in α.

If we set $x=a=d$ and $y=b=c$ we get

$$\sum_{\lambda\in\mathcal{P}} x^{c_0(\lambda)}y^{c_1(\lambda)} = \prod_{n=1}^{\infty}\frac{(1+x^{2n-1}y^{2n-2})(1+x^{2n-1}y^{2n})}{(1-x^{2n}y^{2n})(1-x^{2n-1}y^{2n-1})^2} = P(xy)^2\sum_{k=-\infty}^{\infty} x^{k^2}y^{k^2+k}.$$

A partition λ with all parts odd has 2-core [] or [1], as $|\lambda|$ is odd or even. So

2.5. **Derivative of partition generating function.** Let $\sigma(n)$ be the sum of the divisors of m. Then

$$\frac{P'(x)}{P(x)} = \sum_{n=0}^{\infty}\sigma(n+1)x^n$$

Set $\sigma_k(n) := \sum_{d|n} d^k$ and B_k as the k-th Bernoulli number. Kiming & Olsson: for an even integer $k\geq 4$ the following series is a modular form of weight k

$$E_k = 1-\frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)x^n.$$

Define

$$T(x) = 1-24\sum_{n=1}^{\infty}\sigma_1(n)x^n.$$

In fact Kiming and Olsson denote this by P, contrary to our notation for the partition generating function. The following is a modular form of weight 4:

$$Q(x) = E_4 = 1+240\sum_{n=1}^{\infty}\sigma_3(n)x^n.$$

Define the operator θ by $\theta(\sum a_nx^n) := \sum na_nx^n$. Then (Ramanujan)

$$\theta(T) = \frac{1}{12}(T^2-Q).$$

2.6. **Hook-length statistics.** Recall the set $\mathcal{H}(\lambda)$ of hook-lengths of a partition λ. A vast generalization, due to Nekrasov-Okounkov:

$$\prod_{n=1}^{\infty}(1-x^n)^{z-1} = \sum_{\lambda} x^{|\lambda|} \prod_{h\in\mathcal{H}(\lambda)} (1-\frac{z}{h^2}).$$

To get Jacobi's result, take $z=4$. The only partitions that contribute on the rhs are the 2-cores (no hooks equal to 2). The 'hook product' for $[m, m-1, 2, 1]$ is equal to $2m+1$.

Let $r(\lambda) := \#\{\lambda_i\}$ be the number of part lengths or removable nodes of λ. Then $a(\lambda) = r(\lambda)+1$ and we have

$$\sum_{\lambda} r(\lambda)x^{|\lambda|} = \frac{x}{(1-x)}P(x), \quad \sum_{\lambda} a(\lambda)x^{|\lambda|} = \frac{1}{(1-x)}P(x).$$

According to Guoniu Han,

$$\sum_{\lambda} x^{|\lambda|} z^{r(\lambda)} = \prod_{n=1}^{\infty} \frac{1+(z-1)x^n}{1-x^n}.$$

To see this, interpret the rhs as enumerating all standard MacMahon diagrams, where each mark is $z-1$. Now suppose that λ has r removable boxes. Then we get a contribution $\binom{r}{j}(z-1)^j$ to the lhs by considering the $\binom{r}{j}$-ways of marking exactly j of the removable boxes in λ. So the sum of the contributions of all MacMahon diagrams for λ to the lhs is $\sum_{j=0}^{r}\binom{r}{j}(z-1)^j = z^r$.

More generally, if $h>0$ and $r_h(\lambda)$ is the number of hook-lengths in λ of length h, he proves that

$$\sum_{\lambda} x^{|\lambda|} z^{r_h(\lambda)} = \prod_{n=1}^{\infty} \frac{(1+(z-1)x^{hn})^h}{1-x^n}.$$

Variant: set $r_0(\lambda) := \#\{\lambda_i \equiv 0(\mod 2)\}$ and $r_1(\lambda) := \#\{\lambda_i \equiv 1(\mod 2)\}$ as the number of even, respectively odd, parts lengths of λ. Then

$$\sum_{\lambda} x^{|\lambda|} y^{r_0(\lambda)} z^{r_1(\lambda)} = \prod_{n=1}^{\infty} \frac{(1+(y-1)x^{2n})(1+(z-1)x^{2n-1})}{(1-x^n)}.$$

To see this, interpret the rhs as enumerating all standard MacMahon diagrams, where each mark is $z-1$. Now suppose that λ has r removable boxes. Then we get a contribution $\binom{r}{j}(z-1)^j$ to the lhs by considering the $\binom{r}{j}$-ways of marking exactly j of the removable boxes in λ. So the sum of the contributions of all MacMahon diagrams for λ to the lhs is $\sum_{j=0}^{r}\binom{r}{j}(z-1)^j = z^r$.

3. Enumerating Skew Diagrams

3.1. **Hooks of a given arm length.** Set $r_h(\lambda)$ as the number of removable h-hooks and $a_h(\lambda)$ as the number of addable h-hooks in λ, respectively. We write $r(\lambda) := r_1(\lambda)$ for the number of removable nodes of λ, and $a(\lambda)$ for the number of addable nodes of λ. Now each h-hook in λ has an arm length m between 0 and $h-1$ inclusive. Refine the numbers r_h and a_h to $r_h(\lambda, m)$ and $a_h(\lambda, m)$, according to arm-length m. C. Bessenrodt has shown that

$$a_h(\lambda, m) = r_h(\lambda, m) + 1,$$

for each m. So

$$a_h(\lambda) = r_h(\lambda) + h.$$

It follows from this that

$$\sum_\lambda r_h(\lambda, m) x^{|\lambda|} = \frac{x^h}{(1-x^h)} P(x), \quad \sum_\lambda r_h(\lambda) x^{|\lambda|} = \frac{hx^h}{(1-x^h)} P(x).$$

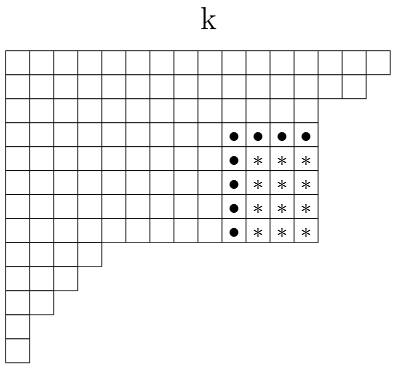
For integers $a \geq b \geq 0$, the q-binomial coefficient $\binom{a}{b}_q$ is the following polynomial in q:

$$\binom{a}{b}_q := \prod_{i=0}^{b-1} \frac{1-q^{a-i}}{1-q^{b-i}}.$$

If $b = 0$, set $\binom{a}{0}_q := 1$. Then it is *well-known* that the generating function for partitions in an $a \times b$-rectangle is $\binom{a+b}{b}_q$.

Consider all h-hooks with arm length $m \in \{0, 1, \ldots, h-1\}$ and leg length $h - m - 1$. The generating function for partitions marked by one h-hook of arm length m equals $\frac{x^h}{(1-x^h)} P(x)$.

Now let us 'manually' enumerate all h-hooks of arm length m. Consider the following construction of partitions λ. This illustrates the case of an $h = 8$-hook of arm length $m = 3$.



The 8 bulleted boxes form an 8-hook of arm length 3. We can insert any partition μ into the 4×3-rectangle of $*$'s and the rim of μ, together with the SW and NE bullet, also forms an 8-hook of arm length 3. The generating function for all these 5-bulleted rows is equal to

$$\sum_{k=0}^{\infty} x^{5k+8}\binom{7}{3}_x = \sum_{k=0}^{\infty} x^{5k+8}\frac{(1-x^7)(1-x^6)(1-x^5)}{(1-x^3)(1-x^2)(1-x)}$$
$$= \frac{x^8(1-x^7)(1-x^6)}{(1-x^3)(1-x^2)(1-x)}$$

Removing the 5-bulleted rows from λ produces a partition with no rows of length $k+1, k+2, k+3$. The generating function of such partitions is

$$(1-x^{k+1})(1-x^{k+2})(1-x^{k+3})P(x).$$

Moreover, any 5-bulleted rows can be inserted into such a partition to get λ containing an 8-hook of arm length 3. Thus the generating function of 8-hooks of arm length 3 is

$$\sum_{k=0}^{\infty} x^{5k+8}\frac{(1-x^7)(1-x^6)(1-x^5)}{(1-x^3)(1-x^2)(1-x)}(1-x^{k+1})(1-x^{k+2})(1-x^{k+3})P(x) = \frac{x^8}{(1-x^8)}P(x).$$

We can rewrite this as:

$$\sum_{k=0}^{\infty} x^{5k}(1-x^{k+1})(1-x^{k+2})(1-x^{k+3}) = \frac{(1-x^3)(1-x^2)(1-x)}{(1-x^8)(1-x^7)(1-x^6)(1-x^5)}.$$

Generalizing this to arbitrary $h > m$, we get the following identity:

$$\boxed{\sum_{k=0}^{\infty} x^{(h-m)k}\prod_{i=1}^{m}(1-x^{k+i}) = \frac{1}{(1-x^h)\binom{h-1}{m}_x} = \frac{(1-x^m)(1-x^{m-1})\dots(1-x)}{(1-x^h)(1-x^{h-1})\dots(1-x^{h-m})}}.$$

We can rewrite this, for arbitrary $0 \le m < n$ as:

$$\sum_{k=0}^{\infty} x^{nk}\prod_{i=1}^{m}(1-x^{k+i}) = \frac{(1-x^m\quad)(1-x^{m-1}\quad)\dots(1-x\quad)}{(1-x^m x^n)(1-x^{m-1}x^n)\dots(1-xx^n)} \times \frac{1}{(1-x^n)}$$

Example: For 3-hooks of arm length 1, we have $\binom{2}{1}_x = \frac{1-x^2}{1-x}$. So

$$\sum_{k=0}^{\infty} x^{2k}(1-x^{k+1}) = \frac{(1-x)}{(1-x^3)(1-x^2)}.$$

Example: For 4-hooks of arm length 2, we have $\binom{3}{2}_x = \frac{(1-x^3)}{(1-x)}$ and

$$\sum_{k=0}^{\infty} x^{2k}(1-x^{k+1})(1-x^{k+2}) = \frac{(1-x)}{(1-x^4)(1-x^3)}.$$

3.2. **Skew diagrams with at most 3 boxes.** In this section we give the generating function $R_k(x)$ for skew-diagrams with k boxes, for $k = 1, 2, 3$. Thus

$$R_k(x) = \sum_{\alpha,\beta} x^{|\alpha|},$$

where α ranges over all partitions, and for each α, β ranges over the partitions of $|\alpha| - k$ contained in α. We will show that

$$\begin{aligned} R_1(x) &= \frac{x}{(1-x)}P(x) \\ R_2(x) &= \frac{x^2(2-x)}{(1-x)(1-x^2)}P(x) \\ R_3(x) &= \frac{x^3(3-x-x^2)}{(1-x)(1-x^2)(1-x^3)}P(x). \end{aligned}$$

Actually, the first of these is well known.

Note that the generating function for partitions with k distinct parts is

$$x^{k(k+1)/2}\prod_{j=1}^{k}(1-x^j)^{-1}.$$

Let $r(\lambda)$ denote the number of removable nodes in λ. Then

$$\sum_{\lambda}\binom{r(\lambda)}{k}x^{|\lambda|} = x^{k(k+1)/2}\prod_{j=1}^{k}(1-x^j)^{-1}P(x).$$

This is the generating function for standard MacMahon diagrams with exactly k marked boxes. Recall that the generating function for standard MacMahon diagrams is $P(x)D(x)$

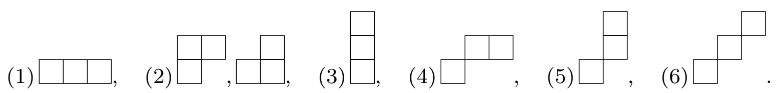
Now

$$\sum_{\lambda} r_2(\lambda)x^{|\lambda|} = \frac{2x^2}{(1-x^2)}P(x), \quad \sum_{\lambda}\binom{r(\lambda)}{2}x^{|\lambda|} = \frac{x^3}{(1-x)(1-x^2)}P(x).$$

We deduce that

$$\begin{aligned} R_2(x) &= \frac{x^2(2-x)}{(1-x)(1-x^2)}P(x) \\ &= \left(\frac{1}{(1-x)}+1\right)\frac{x^2}{(1-x^2)}P(x). \end{aligned}$$

Here are the types of skew-tableaux with 3-boxes:



The generating function for the three types of 3-hooks (1),(2) and (3) is

$$\sum_\lambda r_3(\lambda)x^{|\lambda|} = \frac{3x^3}{(1-x^3)}P(x),$$

while that for type (6) is

$$\sum_\lambda \binom{r(\lambda)}{3} x^{|\lambda|} = \frac{x^6}{(1-x)(1-x^2)(1-x^3)}P(x).$$

We enumerate skew diagrams of type (5). By conjugation, those of type (4) have the same generating function. If the 2-hook occurs in a long row we get

$$\frac{x^5}{(1-x^2)(1-x^3)}P(x).$$

If instead the 2-hook occurs in a short row, we get

$$\frac{x^4}{(1-x)(1-x^3)}P(x).$$

So the generating function for skew diagrams of types (4) and (5) is

$$\frac{2x^4}{(1-x)(1-x^3)} + \frac{2x^5}{(1-x^2)(1-x^3)}.$$

Thus

$$\begin{aligned} R_3(x) &= \left(\frac{3x^3}{(1-x^3)} + \frac{x^6}{(1-x)(1-x^2)(1-x^3)} + \frac{2x^4}{(1-x)(1-x^3)} + \frac{2x^5}{(1-x^2)(1-x^3)}\right)P(x) \\ &= \frac{x^3(3-x-x^2)}{(1-x)(1-x^2)(1-x^3)}P(x) \\ &= \left(\frac{1}{(1-x)(1-x^2)} + \frac{1}{(1-x^2)} + \frac{1}{(1-x)}\right)\frac{x^3}{(1-x^3)}P(x) \end{aligned}$$

The general formula cannot be

$$\begin{aligned} R_k(x) &= \frac{x^k(k-x-\cdots-x^{k-1})}{(1-x)(1-x^2)\dots(1-x^k)}P(x) \\ &= \left(\sum_{i=0}^{k-1}\prod_{j\neq i, j=1}^{k-1}(1-x^j)^{-1}\right)\frac{x^k}{(1-x^k)}P(x) \end{aligned}$$

This is because the first few terms of $R_k(x)$ are

(1) $\quad R_k(x) = p(k)x^k + p(k+1)x^{k+1} + (2p(k+2)-2)x^{k+2} + (3p(k+3)-\epsilon_k)x^{k+3} + \dots.$

Here

$$\epsilon_k = \begin{cases} k+7, & \text{if } k \text{ is even,} \\ k+6, & \text{if } k \text{ is odd.} \end{cases}$$

For, removing k nodes from a partition of k leaves the empty partition; removing them from a partition of $k+1$ leaves the partition [1]. For a partitions λ of $k+2$, we get both partitions [2] and $[1^2]$, unless $\lambda = [k+2]$ and we only get [2], or $\lambda = [1^{k+2}]$ and we only get $[1^2]$.

If we assume that $R_k(x) = (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots) x^k P(x) \prod_{i=1}^{k} (1 - x^i)^{-1}$, we can compare with (1) and solve to get

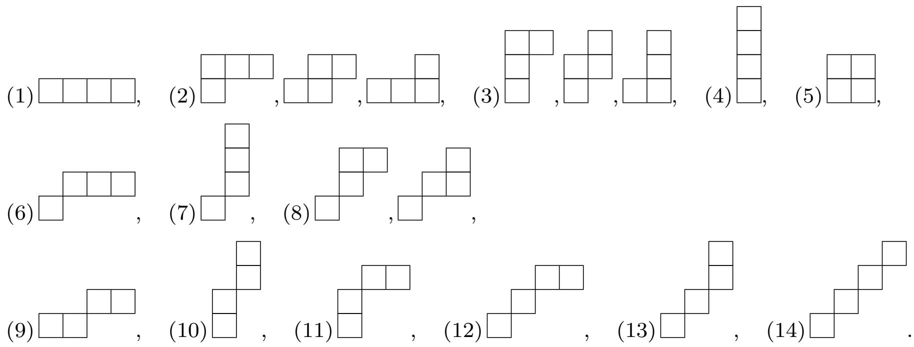
$$(2) \qquad a_0 = p(k), \quad , a_1 = p(k+1) - 2p(k), \quad a_2 = 2p(k+2) - 2p(k+1) - p(k) - 2.$$

3.3. **Skew diagrams with 4 boxes.** In this section we show that

$$R_4(x) = \frac{x^4(5 - 3x + x^2 - 2x^3 - x^4 + x^5)}{(1-x)(1-x^2)(1-x^3)(1-x^4)} P(x).$$

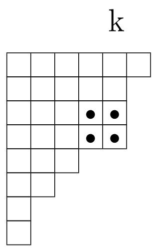
Of course, $a_0 = p(4) = 5$ and $a_1 = p(5) - 2p(4) = -3$ and $a_2 = 2p(6) - 2p(5) - p(4) - 2 = 22 - 14 - 5 - 2 = 1$, using (2).

Here are the types of skew-tableaux with 4-boxes:



The generating function for each of (1),(2),(3) and (4) is $\frac{x^4}{(1-x^4)} P(x)$.

Let λ be a partition of type (5), in which two rows of length $k \geq 2$ contain a removable 2×2 block. We illustrate λ here:

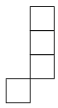
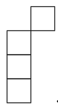


Removing the two bulleted rows from λ leaves a partition μ which has no parts of length $k-1$. Such partitions have generating function $(1 - x^{k-1})P(x)$. Moreover, we can recover λ

from μ and $[k, k]$. So the generating function for partitions of type (5) is

$$\sum_{k=2}^{\infty} x^{2k}(1-x^{k-1})P(x) = \sum_{k=2}^{\infty}(x^{2k}-x^{3k-1})P(x)$$
$$= \left(\frac{x^4}{1-x^2} - \frac{x^5}{1-x^3}\right) x^4 P(x)$$
$$= \frac{x^4(1-x)}{(1-x^2)(1-x^3)} P(x).$$

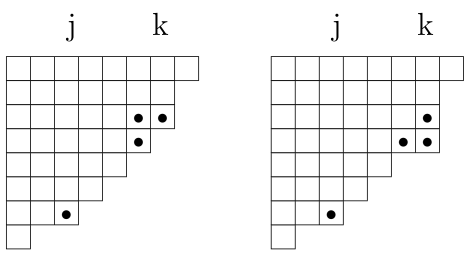
Types (6) and (7) have the same generating function (by transposition). We consider the two subcases for type (7):

 and  .

The generating functions are got by considering rows for each case. This gives

$$\left(\frac{x^7}{(1-x^3)(1-x^4)} + \frac{x^5}{(1-x)(1-x^4)}\right) P(x)$$

We enumerate partitions λ of type (8). Suppose that the 3-hook occurs at the end of two rows of lengths $k, k-1$ (on the left above) or at the end of two rows of lengths k, k (on the right above). The removable node occurs in a row of length $j \neq k, k-1$. Here is an illustration of the two possibilities:



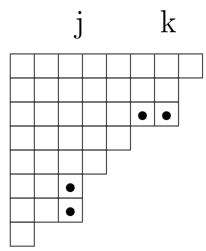
Removing the three bulleted rows from λ leaves a partition μ which has no parts of length $k-1$. Such partitions have generating function $(1-x^{k-1})P(x)$. Moreover, we can recover λ from μ and the three part length, either $k, k-1, j$ or k, k, j. It follows that the generating function for partitions of type (8) is

$$
\begin{aligned}
&\sum_{k=2}^{\infty}(x^{2k-1}+x^{2k})\left(\frac{x}{1-x}-x^{k-1}-x^{k}\right)(1-x^{k-1})P(x)\\
&\qquad=\left(\frac{x}{(1-x)}\sum_{k=2}^{\infty}(x^{2k-1}-x^{3k-2})-(1+x)\sum_{k=2}^{\infty}(x^{3k-2}-x^{4k-3})\right)(1+x)P(x)\\
&\qquad=\left(\frac{1}{(1-x)}\left(\frac{1}{1-x^2}-\frac{x}{1-x^3}\right)-(1+x)\left(\frac{1}{1-x^3}-\frac{x}{1-x^4}\right)\right)x^4(1+x)P(x)\\
&\qquad=\left(\frac{1}{(1-x^2)(1-x^3)}-\frac{(1-x^2)}{(1-x^3)(1-x^4)}\right)x^4(1+x)P(x)\\
&\qquad=\frac{2x^6(1+x)}{(1-x^3)(1-x^4)}P(x).
\end{aligned}
$$

Types (9) and (10) have the same generating function (by transposition). The generating function for (10) is easily seen to be

$$\frac{x^6}{(1-x^2)(1-x^4)}P(x).$$

Let λ be a partition of type (11), in which a row of length $k \geq 2$ contains the horizontal hook, and two rows of length $j \neq k, k-1$ contain the vertical hook. We illustrate λ here:

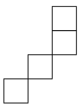
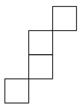
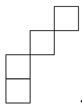


Removing the three bulleted rows from λ leaves a partition μ which has no parts of length $k-1$. Such partitions have generating function $(1-x^{k-1})P(x)$. Moreover, we can recover λ

from μ and $[k, j, j]$. So the generating function for partitions of type (11) is

$$\begin{aligned}
&\sum_{k=2}^{\infty} x^k \left(\frac{x^2}{1-x^2} - x^{2k} - x^{2(k-1)} \right) (1 - x^{k-1}) P(x) \\
&\quad = \left(\frac{x^2}{1-x^2} \sum_{k=2}^{\infty} (x^k - x^{2k-1}) - (1+x^2) \sum_{k=2}^{\infty} (x^{3k-2} - x^{4k-3}) \right) P(x) \\
&\quad = \left(\frac{1}{1-x^2} \left(\frac{1}{1-x} - \frac{x}{1-x^2} \right) - (1+x^2) \left(\frac{1}{1-x^3} - \frac{x}{1-x^4} \right) \right) x^4 P(x) \\
&\quad = \left(\frac{1}{(1-x^2)^2} - \frac{(1-x)}{(1-x^2)(1-x^3)} \right) x^4 P(x) \\
&\quad = \frac{(1-x)\ (1+2x)}{(1-x^2)^2(1-x^3)} x^5 P(x).
\end{aligned}$$

Types (12) and (13) have the same generating function (by transposition). Type (13) splits into 3 subcases, depending on the relative lengths of the rows:

 and  and  .

The generating function for (13) can now be computed as:

$$\left(\frac{x^9}{(1-x^2)(1-x^3)(1-x^4)} + \frac{x^8}{(1-x)(1-x^3)(1-x^4)} + \frac{x^7}{(1-x)(1-x^2)(1-x^4)} \right) P(x)$$

Finally, type (14) has generating function:

$$\frac{x^{10}}{(1-x)(1-x^2)(1-x^3)(1-x^4)} P(x).$$

The work above shows that the function $R_4(x)/P(x)$ is

$$
\begin{array}{ll}
(1),(2),(3),(4) & \dfrac{4x^4}{(1-x^4)} \\
(5) & +\dfrac{x^4(1-x)}{(1-x^2)(1-x^3)} \\
(6),(7) & +\dfrac{2x^7}{(1-x^3)(1-x^4)}+\dfrac{2x^5}{(1-x)(1-x^4)} \\
(8) & +\dfrac{2x^6(1+x)}{(1-x^3)(1-x^4)} \\
(9),(10) & +\dfrac{2x^6}{(1-x^2)(1-x^4)} \\
(11) & +\dfrac{x^5(1-x)(1+2x)}{(1-x^2)^2(1-x^3)} \\
(12),(13) & +\dfrac{2x^9}{(1-x^2)(1-x^3)(1-x^4)}+\dfrac{2x^8}{(1-x)(1-x^3)(1-x^4)} \\
 & \qquad +\dfrac{2x^7}{(1-x)(1-x^2)(1-x^4)} \\
(14) & +\dfrac{x^{10}}{(1-x)(1-x^2)(1-x^3)(1-x^4)}
\end{array}
$$

The sum of the functions of types (5) and (11) simplies to

$$
\begin{aligned}
\frac{x^4(1-x)}{(1-x^2)(1-x^3)}+\frac{x^5(1-x)(1+2x)}{(1-x^2)^2(1-x^3)} &= \frac{x^4(1-x)}{(1-x^2)^2(1-x^3)}\big((1-x^2)+x(1+2x)\big) \\
&= \frac{x^4(1-x)(1+x+x^2)}{(1-x^2)^2(1-x^3)} \\
&= \frac{x^4}{(1-x^2)^2}.
\end{aligned}
$$

Also

$$
\begin{aligned}
&\frac{x^9}{(1-x^2)(1-x^3)(1-x^4)}+\frac{x^{10}}{(1-x)(1-x^2)(1-x^3)(1-x^4)} \\
&= \frac{x^9}{(1-x)(1-x^2)(1-x^3)(1-x^4)}\left((1-x)+x\right) \\
&= \frac{x^9}{(1-x)(1-x^2)(1-x^3)(1-x^4)}.
\end{aligned}
$$

So $R_4(x)/P(x)$ can be simplified as the *positive* power series

$$\frac{4x^4}{(1-x^4)} + \frac{x^4}{(1-x^2)^2} + \frac{2x^5}{(1-x)(1-x^4)} + \frac{2x^6}{(1-x^3)(1-x^4)} + \frac{2x^6}{(1-x^2)(1-x^4)}$$
$$+\frac{4x^7}{(1-x^3)(1-x^4)} + \frac{2x^7}{(1-x)(1-x^2)(1-x^4)} + \frac{2x^8}{(1-x)(1-x^3)(1-x^4)}$$
$$+\frac{x^9}{(1-x^2)(1-x^3)(1-x^4)} + \frac{x^9}{(1-x)(1-x^2)(1-x^3)(1-x^4)}$$

We then calculate:

$$\begin{aligned} R_4(x) &= \frac{-x^{11}+x^{10}+3x^9-2x^8+x^7-4x^6-3x^5+5x^4}{(1-x)(1-x^2)^2(1-x^3)(1-x^4)}P(x) \\ &= \frac{x^4(5-3x+x^2-2x^3-x^4+x^5)}{(1-x)(1-x^2)(1-x^3)(1-x^4)}P(x) \end{aligned}$$

Here are the first 22 terms of the power series $R_4(x)$:

$$\begin{aligned} R_4(x) =& 5x^4+7x^5+20x^6+35x^7+70x^8+114x^9+202x^{10}+314x^{11} \\ &+511x^{12}+772x^{13}+1185x^{14}+1737x^{15}+2571x^{16}+3673x^{17} \\ &+5271x^{18}+7384x^{19}+10343x^{20}+14228x^{21}+19550x^{22}+\dots. \end{aligned}$$

Here are the first 22 terms of the power series $R_4(x)/P(x)$:

$$\begin{aligned} R_4(x)P(x)^{-1} =& 5x^4+2x^5+8x^6+8x^7+15x^8+14x^9+25x^{10}+23x^{11} \\ &+37x^{12}+37x^{13}+51x^{14}+52x^{15}+72x^{16}+71x^{17} \\ &+93x^{18}+96x^{19}+120x^{20}+123x^{21}+152x^{22}+\dots. \end{aligned}$$

We list the generating functions for $k=1,\dots,7$.

$$R_1(x) = \frac{x}{(1-x)}P(x)$$

$$R_2(x) = \frac{x^2(2-x)}{(1-x)(1-x^2)}P(x)$$

$$R_3(x) = \frac{x^3(3-x-x^2)}{(1-x)(1-x^2)(1-x^3)}P(x)$$

$$R_4(x) = \frac{x^4(5-3x+x^2-2x^3-x^4+x^5)}{(1-x)(1-x^2)(1-x^3)(1-x^4)}P(x)$$

$$R_5(x) = \frac{x^5(7-3x-x^2+x^3-2x^4-5x^5+3x^6+x^7-x^8+2x^9-x^{10})}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)}P(x)$$

$$R_6(x) = \frac{x^6(11-7x+x^2+x^3+x^4-11x^5+3x^6-2x^7+2x^8+x^{10}+4x^{11}-4x^{12}+2x^{13}-x^{14})}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)}P(x)$$

$$R_7(x) = \frac{x^7(15-8x-x^2+4x^3-7x^5-3x^6-14x^7+12x^8+2x^9-9x^{10}+7x^{11}+5x^{12}+x^{13}+x^{14}-4x^{15}-x^{16}+5x^{17}-7x^{18}+2x^{19}+2x^{20}-x^{21})}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7)}P(x$$

3.4. **Enumerating the number of connected skew shapes.** For $n \geq m > 0$, set $T_{n,m}$ as the number of pairs of $0,1$ sequences $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ which satisfy $\sum_{j=1}^{i}(x_j - y_j) > 0$, for $i = 1, \ldots, n-1$ and $\sum_{j=1}^{n}(x_j - y_j) = m$. Then $T_{n,n}$ equals the number of connected skew shapes of hook length n. Moreover, the T's satisfy the following recursion relation, for $n > 1$, where $T_{1,1} = 1$:

$$T_{n,m} = T_{n-1,m-1} + 2T_{n-1,m} + T_{n-1,m+1}.$$

It can be shown that $T_{n,n}$ is the n-th Catalan number

$$T_{n,n} = C_n := \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n-1}.$$

Lemma 1. *For all $n > 0$ and $m = 1, \ldots, n$ we have*

$$T_{n,m} = \binom{2n-1}{n+m-1} - \binom{2n-1}{n-m-1}.$$

Proof. When $n = 1$, the right hand side equals $-\binom{2}{0} + \binom{2}{1} = 1$. Suppose that $n > 1$ and assume that the result holds for all $T_{n-1,m}$, with $m = 1, \ldots, n-1$. Then we have

$$\begin{aligned} T_{n,m} &= T_{n-1,m-1} + 2T_{n-1,m} + T_{n-1,m+1} \\ &= \left[\tbinom{2n-3}{n+m-3} + 2\tbinom{2n-3}{n+m-2} + \tbinom{2n-3}{n+m-1}\right] - \left[\tbinom{2n-3}{n-m-1} + 2\tbinom{2n-3}{n-m-2} + \tbinom{2n-3}{n-m-3}\right] \\ &= \tbinom{2n-2}{n+m-2} + \tbinom{2n-2}{n+m-1} - \tbinom{2n-2}{n-m-2} - \tbinom{2n-2}{n-m-1}. \\ &= \tbinom{2n-1}{n+m-1} + \tbinom{2n-1}{n-m-1}. \end{aligned}$$

□

4. Other Topics

4.1. **Fairy sequences and partitions.** Due to Fomin and Stanton. This appears to be nothing more than the e-abacus representation of a partition.

A fairy sequence is a map $f : \mathbb{Z} \to \mathbb{Z}$ such that f is non-decreasing on $(-\infty, 0]$ and non-increasing on $[0, \infty]$ and $|f(n) - f(n-1)| \leq 1$ for all n and $f(n) = 0$, for all but finite number of $n \in \mathbb{Z}$.

There is a bijection between fairy sequences and partitions: if λ is a partition, the corresponding fairy sequence f is $f(n) = \#$ boxes on n-th diagonal of λ.

4.2. **Involution on partitions of Joichi-Stanton.** Consider a 2-dimensional array of boxes indexed by $\mathbb{Z}^2$. Place some partitions in boxes as follows: the partition $[1]$ is placed in $(1,1)$. Each partition λ of an integer n with no part 1 is placed in (λ_1, n).

Let $f(m,n)$ be the number of partitions in (m,n). Then

$$f(m,n) = f(m, n-m) + f(m-1, n-1).$$

To see this, consider $\lambda \in (m,n)$. If $\lambda_1 = \lambda_2$, then $[\lambda_2, \ldots, \lambda_l]$ belongs to $(m, n-m)$. If instead $\lambda_1 > \lambda_2$, then $[\lambda_1 - 1, \lambda_2, \ldots, \lambda_l]$ belongs to $(m-1, n-1)$.

Claim: $p(n) = \sum_{i=1}^{n} \sum_{j=1}^{i} f(j,i)$. For, we can identify $[1^n]$ with $[1]$ and otherwise $[\lambda_1, \dots, \lambda_r > 1, 1^{m_1}]$ with $[\lambda_1, \dots, \lambda_r]$ in the box $(\lambda_1, n - m_1)$.

Given a partition in (m,n), there is a unique path along other boxes back to $(1,1)$ that is determined as follows: remove the largest part if it repeats. Otherwise delete one from the largest part. This gives a sequence of letters L (left) and D (diagonally northwest) that identifies λ.

If $f(m,n)$ is odd, then exactly one $f(m, n-m)$ and $f(m-1, n-1)$ is odd. In this way, each box (m,n) with $f(m,n)$ odd corresponds top a unique path of boxes back to $(1,1)$. This path defines a unique partition inside (m,n). Denote it $\lambda(m,n)$.

Involution ϕ on partitions: say $\lambda \in (m,n)$. Consider the path for λ back to $(1,1)$. If f is odd along this path, then λ is fixed by ϕ. Otherwise, let (i,j) be the last box with $f(i,j)$ even. Then both $f(i, i-j)$ and $f(i-1, j-1)$ are odd. One belongs to the path of λ. The other is the start of a unique odd path back to $(1,1)$. Then ϕ transposes λ with the partition got by keeping the path $(m,n) \to (i,j)$ but transposing the tails $(i,j) \to (1,1)$.

4.3. **Guo-Zhang Binomial identity.**

$$\sum_{k=0}^{\infty} \binom{3n}{2k} (-3)^k = (-8)^n.$$

Special case of:

$$\sum_{k=0}^{\infty} \binom{n}{2k} (-3)^k = \begin{cases} (-2)^n, & \text{if } n \equiv 0 \pmod 3. \\ (-2)^{n-1}, & \text{if } n \equiv 1 \pmod 3. \\ (-2)^{n-2}, & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Proof: expand $(\sqrt{3} + i)^n$ using the binomial theorem and then using the fact that

$$\sqrt{3} + i = 2e^{i\pi/6}.$$

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