MATT FAYERS' IDEAS ABOUT HOMOMORPHISMS BETWEEN SPECHT MODULES

JOHN MURRAY

1. Generalities: subgroups and double cosets

Suppose that k is a field, G is a group, and H is a subgroup of G. Let $H\backslash G$ be the set of right cosets of H in G, considered as G-set. The corresponding kG-permutation module is isomorphic to the right ideal H^+kG of kG.

Let K be another subgroup of G. Then $\mathrm{Hom}(H^+kG, K^+kG)$ has as basis the (K,H)-double cosets in G. Here a coset KgH gives rise to the Schur homomorphism θ_{KgH} that sends H^+ to $(KgH)^+$. Note that $(KgH)^+ \in (kG\,H^+)\cap(K^+kG)$. For, let L be a set of right coset representatives for $K\cap gHg^{-1}$ in K, and let R be a set of left coset representatives for $g^{-1}Kg\cap H$ in K. Then

$$(KgH)^+ = L^+gH^+ = K^+gR^+.$$

The double cosets index the G-orbits in the product G-set $K\backslash G\times H\backslash G$. In fact KgH corresponds to the G-orbit containing the pair (Kg,H). It has point stabilizer $G_{KgH} := g^{-1}Kg\cap H$ in G, and hence has size $[G : g^{-1}Kg\cap H]$.

There is a bijection between the (K,H)-double cosets and the (H,K)-double cosets. This reflects the G-set isomorphism $K\backslash G\times H\backslash G\cong H\backslash G\times K\backslash G$. The bijection is $(KgH)\leftrightarrow(Hg^{-1}K)$ and is induced by sending each group element to its inverse.

Let KgH be a (K,H)-double coset. Suppose also that G has a non-trivial linear character sgn whose kernel contains G_{KgH}. We define the orbit sum $O(KgH)^+$ and the signed orbit sum $O(KgH)^-$ in $k(K\backslash G\times H\backslash G)$ as follows:

$$O(KgH)^+ = \sum_{x\in G_{KgH}\backslash G} (Kgx, Hx), \qquad \text{and}$$
$$O(KgH)^- = \sum_{x\in G_{KgH}\backslash G} \mathrm{sgn}(x)(Kgx, Hx).$$

2. Generalities: Young subgroups and double cosets

Now let α be a partition of n. Recall that the α-tabloids index the right S_α-cosets in S_n. The TL-BR (TopLeft-BottomRight) path in the Young diagram $[\alpha]$ of α is the sequence of nodes

$$(1,1),(1,2),\dots(1,\alpha_1),(2,1),\dots(2,\alpha_2),(3,1),\dots$$

Fix t^α as the most dominant α-tableau. This is obtained by placing the symbols $1,2,\dots,n$ in turn into the TL-BR path of $[\alpha]$.

Date: October 19, 2009.

Let β be another partition of n and let T be an α-tableau of type β. If t is an α-tableau, then $T(t)$ is the β-tableau constructed as follows: scan through the TL-BR path of $[\alpha]$; if $t_{(r,c)} = i$ and $T_{(r,c)} = j$, place i in the first empty node in row j of $T(t)$. In this way T induces a bijection between α-tableaux and β-tableaux.

There is a bijection between the α-tableaux of type β and (S_β, S_α)-double cosets or S_n-orbits of (β,α)-tabloid pairs. Write $T(t^\alpha) = t^\beta\sigma(T)$, for a uniquely determined $\sigma(T) \in S_n$. Then the (S_β, S_α)-double coset corresponding to T is $S_\beta\sigma(T)S_\alpha$ and the S_n-orbit on pairs of tabloids contains $(\{T(t^\alpha)\}, \{t^\alpha\})$. The point stabilizer of this pair is $R_T := R_{\{T(t^\alpha)\}} \cap R_{\{t^\alpha\}}$. Note that the orbit is regular if and only if no row of T contains a repeated symbol.

We can construct the β-tableau T^{-1} of type α that labels the S_n-orbit containing the reverse pair of tabloids $(\{t^\alpha\}, \{T(t^\alpha)\})$ as follows: run through the TL-BR path of $[\alpha]$; if $T_{(i,c)} = j$, then place an i at the first empty space in row j of T^{-1}. As the (S_α, S_β)-double coset corresponding to T^{-1} is $S_\alpha\sigma(T)^{-1}S_\beta$, it follows that $T^{-1}(t^\beta) = t^\alpha\sigma(T)^{-1}$ and hence that $\sigma(T^{-1}) = \sigma(T)^{-1}$.

With T an α-tableau of type β, we set

$$\left.\begin{array}{rcl} O_T^- & := & \sum_{\rho\in R_T\backslash S_n} \operatorname{sgn}(\rho)(\{t^\alpha\}\rho, \{T(t^\alpha)\}\rho), \\ O_{T^{-1}}^- & := & \sum_{\rho\in R_T\backslash S_n} \operatorname{sgn}(\rho)(\{T^{-1}(t^\beta)\}\rho, \{t^\beta\}\rho), \end{array}\right. \quad \in M^\alpha \otimes_k M^\beta.$$

Then by the previous paragraph,

$$O_{T^{-1}}^- = \operatorname{sgn}(\sigma(T))\, O_T^-. \tag{1}$$

Note 1. *Let T be an α-tableau of type β. Then $\operatorname{sgn}(\sigma(T))$ can be computed efficiently as follows. Write down the symbols in the TL-BR-sequence of T as $r_1, r_2, \ldots, r_n$. For $1 \le i \le n$, set $e_i := |\{1 \le j < i \mid r_j > r_i\}|$. Then $\operatorname{sgn}(\sigma(T)) = (-1)^{\sum e_i}$. To see this, consider $\alpha = (5,4,2), \beta = (3^2, 2^2, 1)$ and $T = \begin{array}{ccccc} 1&2&3&4&5\\ 2&1&3&4&\\ 1&2&&& \end{array}$. Then the r-sequence is $1,2,3,4,5,2,1,3,4,1,2$. So the e-sequence is $0,0,0,0,0,3,5,2,1,7,5$. As the sum is odd, $\operatorname{sgn}(\sigma(T)) = -1$. The e-sequence reflects the factorization*

$$\begin{aligned} \sigma(T) &= (6,5,4,3)(7,5,4,3,6,2)(8,5,4)(9,5)(10,5,9,4,8,3,6,2)(11,5,9,4,8,3) \\ &= (2,7,3,10,9,4)(5,6,11). \end{aligned}$$

3. Duals of Specht modules

If M is a kS_n-module, then M^* will denote its linear dual. If α is a partition of n, then α' will be the transpose partition. For each α-tableau t, let σ_t be the unique permutation in S_n such that $t = t^\alpha\sigma_t$. Also set $\operatorname{sgn}(t) := \operatorname{sgn}(\sigma_t)$. Let $\langle\,,\,\rangle$ be the standard S_n-invariant form on M^α. Then $X^\perp$ will denote the subspace of M^α orthogonal to a subspace X, with respect to this form. Letting $s = 12\ldots n$ be the unique standard (n)-tableau, we use the notation

$$\begin{array}{rcccll} s^+ & := & e_s & = & \{12\ldots n\} & \in M^{(n)}, \quad \text{and} \\ s^- & := & e_{s'} & = & \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\{s\sigma\} & \in M^{(1^n)}. \end{array}$$

We refer the reader to G.D. James monograph *The Representation theory of the Symmetric Groups,* Lecture Notes in Math. 682, Springer Verlag, 1978, for any undefined notation.

Lemma 1. *$S^{\alpha *} \cong \mathrm{sgn} \otimes_k S^{\alpha'}$, as right kS_n-modules.*

Proof. Define a map $\phi : M^\alpha \to \mathrm{sgn} \otimes S^{\alpha'}$ by $\phi(\{t\}) = \mathrm{sgn}(t)\, s^- \otimes e_{t'}$, for each α-tableau t. Then ϕ is a surjective kS_n-homomorphism. James shows that $\ker(\phi)$ contains $(S^\alpha)^\perp$ (by lifting to characteristic zero). As $M^\alpha/(S^\alpha)^\perp \cong S^{\alpha *}$ (naturally), coincidence of dimensions implies that $(S^\alpha)^\perp = \ker(\phi)$. The lemma follows. □

Transposition establishes a bijection between the set $\mathrm{std}(\alpha)$ of standard α-tableaux and the set $\mathrm{std}(\alpha')$ of standard α'-tableaux. Now $\{\phi(\{t\}) \mid t \in \mathrm{std}(\alpha)\}$ is a basis for $\mathrm{sgn} \otimes S^{\alpha'}$. So $\{\langle \{t\},\ \rangle \mid t \in \mathrm{std}(\alpha)\}$ is a basis for $(S^\alpha)^*$. Here $\langle \{t\},\ \rangle$ is the linear map $e_s \to \langle \{t\}, e_s\rangle$, for each α-tableau s. We use $\{d_t \mid t \in \mathrm{std}(\alpha)\}$ to denote the dual basis of S^α i.e.

$$\langle \{s\}, d_t\rangle \;=\; \delta_{st} 1_k, \quad \text{for all } s, t \in \mathrm{std}(\alpha).$$

Corollary 2. *The map $e_t \to \mathrm{sgn}(t)\, s^- \otimes \langle \{t'\},\ \rangle$, for each α-tableau t, extends to an isomorphism of right kS_n-modules $S^\alpha \cong \mathrm{sgn} \otimes_k (S^{\alpha'})^*$.*

Proof. We tensor both sides of $(S^{\alpha'})^* \cong \mathrm{sgn} \otimes_k S^\alpha$ with sgn, and use the fact that $\mathrm{sgn} \otimes_k \mathrm{sgn} \cong k$, the trivial kG-module. □

The isomorphism $S^{\alpha *} \cong \mathrm{sgn} \otimes S^{\alpha'}$ also furnishes us with a generating set for $(S^\alpha)^\perp$. The standard polytabloids form a basis for $S^{\alpha'}$. So any identity involving polytabloids can be proved by rewriting each polytabloid as a linear combination of standard polytabloids, and checking that the coefficient of each standard polytabloid is zero. James proves that one can write any polytabloid in terms of standard polytabloids using a finite sequence of relations of the form $e'_t G^-_{X,Y} = 0$. Here X, Y are subsets of adjacent columns of t' with the property that $|X \cap Y| = 1$ and $|X \cup Y|$ is one more than the leftmost of the two columns. Also $G^-_{X,Y}$ is a signed sum of coset representatives for $S_X \times S_Y$ in $S_{X\cup Y}$ (called a Garnir element). Let $G^+_{X,Y}$ be the sum of these representatives. Then $(S^\alpha)^\perp$ is generated by $\{t\}G^+_{X,Y}$, as submodule of M^α.

Example 3. *The permutation module M^{2^2} has as basis the 6 tabloids (curly brackets removed):*

$$\begin{matrix}1&2\\3&4\end{matrix},\ \begin{matrix}1&3\\2&4\end{matrix},\ \begin{matrix}1&4\\2&3\end{matrix},\ \begin{matrix}2&3\\1&4\end{matrix},\ \begin{matrix}2&4\\1&3\end{matrix},\ \begin{matrix}3&4\\1&2\end{matrix}.$$

Then the 4 'dual Garnir elements' given by the above remark are:

$$\begin{matrix}1&2\\3&4\end{matrix}+\begin{matrix}1&3\\2&4\end{matrix}+\begin{matrix}1&4\\2&3\end{matrix},\quad \begin{matrix}1&2\\3&4\end{matrix}+\begin{matrix}2&3\\1&4\end{matrix}+\begin{matrix}2&4\\1&3\end{matrix},$$
$$\begin{matrix}1&2\\3&4\end{matrix}+\begin{matrix}1&3\\2&4\end{matrix}+\begin{matrix}2&3\\1&4\end{matrix},\quad \begin{matrix}1&2\\3&4\end{matrix}+\begin{matrix}1&4\\2&3\end{matrix}+\begin{matrix}2&4\\1&3\end{matrix}.$$

4. Identifications of homomorphism spaces

We begin with a result on homomorphisms between modules for a group algebra.

Lemma 4. *Let G be a group and let X, Y, Z be right kG-modules. Then*

$$\mathrm{Hom}_{kG}(X \otimes_k Y^*, Z) = \mathrm{Hom}_{kG}(X, Y \otimes_k Z),$$

as k-vector spaces. Let $y_1, \ldots, y_n$ be any k-basis for Y, and let $y_1^*, \ldots, y_n^*$ be the dual basis for Y^*. The above equality sends a map $f : X \otimes Y^* \to Z$ to the map $\hat{f} : X \to Y \otimes Z$, where

$$\hat{f}(x) := \sum_{i=1}^{n} y_i \otimes f(x \otimes y_i^*).$$

Proof. Here Y^* is a right kG-module in the usual manner. If $f : Y \to k$ is a k-linear map, and $g \in G$, then fg is the k-linear map with $(fg)(y) := f(yg^{-1})$, for all $y \in Y$. Also for right kG-modules X, Y, the tensor product $X \otimes_k Y$ is a kG-module via the diagonal action $(x \otimes y)g := xg \otimes yg$, for all $x \in X, y \in Y$ and $g \in G$.

Now recall that $\mathrm{Hom}_k(X, Y) \cong X^* \otimes_k Y$, as k-vector spaces. Here if $x_1, \ldots, x_m$ is a basis of X, and $x_1^*, \ldots, x_m^*$ is the dual basis of X^*, then the isomorphism maps $f : X \to Y$ to $\sum_{i=1}^m x_i^* \otimes f(x_i)$. A calculation shows that this restricts to a k-linear isomorphism $\mathrm{Hom}_{kG}(X, Y) \cong X^* \otimes_{kG} Y$.

By the previous paragraph, we have

$$\mathrm{Hom}_k(X \otimes_k Y^*, Z) = (X \otimes_k Y^*)^* \otimes_k Z = X^* \otimes (Y \otimes_k Z) = \mathrm{Hom}_k(X, Y \otimes_k Z).$$

Tracing the isomorphisms through, a map $f : X \otimes_k Y^* \to Z$ gets sent to the map $\hat{f} : X \to Y \otimes_k Z$, where $\hat{f}(x) = \sum_{i=1}^n y_i \otimes f(x \otimes y_i^*)$, for all $x \in X$.

Now suppose that $g \in G$. If $y_i g = \sum a_{ij} y_j$, then $y_i^* g = \sum b_{ij} y_j^*$, where $[b_{ij}]$ is the inverse-transpose matrix of $[a_{ij}]$. Suppose that f is G-invariant. Then for $x \in X$ we have

$$\begin{aligned}
\hat{f}(xg) &= \textstyle\sum_{i=1}^n y_i \otimes f(xg \otimes y_i^*) \\
&= (\textstyle\sum_{i=1}^n y_i g^{-1} \otimes f(x \otimes y_i^* g^{-1}))g \\
&= (\textstyle\sum_{i,j,k} b_{ij} a_{ik} y_j \otimes f(x \otimes y_k^*))g \\
&= (\textstyle\sum_{j=1}^n y_j \otimes f(x \otimes y_j^*))g \\
&= \hat{f}(x)g.
\end{aligned}$$

So $\hat{f}$ is also G-invariant. We can reverse the argument to show that $\hat{f}$ is G-invariant, for any map f, only if f is G-invariant. So we get a restricted isomorphism $\mathrm{Hom}_{kG}(X \otimes_k Y^*, Z) = \mathrm{Hom}_{kG}(X, Y \otimes_k Z)$, as required. ☐

We apply this and the results of the last section:

Theorem 5. *As k-vector spaces*

$$Hom_{kS_n}(S^\alpha, S^\beta) = Hom_{kS_n}(sgn, S^{\alpha'} \otimes_k S^\beta).$$

Proof. Corollary 2 shows that $S^\alpha \cong sgn \otimes (S^{\alpha'})^*$. So

$$Hom_{kS_n}(S^\alpha, S^\beta) = Hom_{kS_n}(sgn \otimes_k (S^{\alpha'})^*, S^\beta) = Hom_{kS_n}(sgn, S^{\alpha'} \otimes_k S^\beta).$$

□

Using $\{\mathrm{sgn}(t)\langle\{t'\},\ \rangle \mid t \in \mathrm{std}(\alpha)\}$ as a basis for $(S^{\alpha'})^*$ with dual basis $\{\mathrm{sgn}(t)d_{t'} \mid t \in \mathrm{std}(\alpha)\}$ for $S^{\alpha'}$, and keeping track of the isomorphisms, we get

$$f : S^\alpha \to S^\beta \quad \text{gets sent to} \sum_{t\in\mathrm{std}(\alpha)} sgn(t)\, d_{t'} \otimes f(e_t).$$

The right hand side spans a 1-dimensional submodule of $S^{\alpha'} \otimes_k S^\beta$ that affords the sign representation.

Example 6 (Trivial example)**.** *If $\alpha = \beta$ and f is the identity map, then $\sum_{t\in\mathrm{std}(\alpha)} sgn(t)d'_t \otimes e_t$ spans the unique submodule of $S^{\alpha'} \otimes S^\alpha$ that affords the sign representation. Writing this in terms of the tabloid basis of $M^{\alpha'} \otimes M^\alpha$, we get*

$$\sum_{t\in\mathrm{std}(\alpha)} sgn(t)\, d'_t \otimes e_t = \sum_s \mathrm{sgn}(s)\, \{s'\} \otimes \{s\},$$

where s runs over all $n!$ α-tableau.

5. Semistandard homomorphisms

From now on $\mathrm{char}(k) \neq 2$. Then the sign representation is isomorphic to a submodule of M^α, for α a partition of n, if and only if $\alpha = (1^n)$, and hence $M^\alpha \cong kS_n$, the regular S_n-module. Moreover, sgn appears with multiplicity 1 in kS_n, as the right ideal generated by S_n^-. Now for any partitions α, β of n, $M^\alpha \otimes M^\beta$ is a sum of permutation modules M^γ. This can be used to show that the sum of all submodules of $M^\alpha \otimes M^\beta$ affording sgn coincides with the span of the signed regular orbits in the corresponding S_n-set of pairs of α, β-tabloids.

Using the ideas of the previous section, we also have:

$$Hom_{kS_n}(S^\alpha, M^\beta) \cong Hom_{kS_n}(sgn, S^{\alpha'} \otimes_k M^\beta).$$

Let T be a semi-standard α-tableau of type β. The the above isomorphism identifies the semi-standard homomorphism $\hat{\theta}_T$ with a certain linear combination of signed regular orbit sums in $M^{\alpha'} \otimes M^\beta$. The point of the next theorem is that we can compute this image fairly easily. Note that the transpose T' is a row standard α'-tableaux of type β.

Theorem 7. *Let T be a semi-standard α-tableau of type β. Then $\hat{\theta}_T$ is identified with the following element of $M^{\alpha'} \otimes_k M^\beta$:*

$$\sum \left\{ O_U^- \mid \begin{array}{l} U \text{ is column equivalent to } T' \text{ and} \\ U \text{ has no row repeats} \end{array} \right\}.$$

Proof. Via the equality of Theorem 5, the semi-standard homomorphism $\hat{\theta}_T$ is identified with

$$\sum_{t \in \mathrm{std}(\alpha)} \mathrm{sgn}(t)\, d_{t'} \otimes \hat{\theta}_T(e_t).$$

This can be written as a linear combination of signed regular orbit sums in $M^{\alpha'} \otimes_k M^{\beta}$. Fix a standard α-tableau s. Then every regular orbit contains a pure tensor with first coordinate $\{s'\}$. But $\langle \{s'\}, d_{t'} \rangle = \delta_{st}$, for all $t \in \mathrm{std}(\alpha)$. So we can compute the expansion of $\hat{\theta}_T$ into signed regular orbit sums by examining the multiplicity of each orbit in $\{s'\} \otimes \hat{\theta}_T(e_s)$. We have

$$\begin{aligned}
&\{s'\} \otimes \hat{\theta}_T(e_s) = \{s'\} \otimes \theta_T(\{s\})C_s^- \\
&= \{s'\} \otimes \sum \{\{U(s')\} \mid U \text{ is column equivalent to } T'\}C_s^-. \\
&= \sum \left\{ \{s'\} \otimes \{U(s')\} \mid \begin{array}{l} U \text{ is column equivalent to } T' \text{ and} \\ U \text{ has no row repeats} \end{array} \right\} C_s^-
\end{aligned}$$

as $\{s'\}\sigma = \{s'\}$, for all $\sigma \in C_s$, and $\{U(s')\}C_s^- = 0$, whenever U has a row repeat. The assertion of the theorem now follows from the fact that $\{U(s')\}\sigma \neq \{U(s')\}$, if $\sigma \in C_s$ and U has no row repeats. □

Now recall that

$$S^{\alpha} = \bigcap_{\nu_{i,r}} \ker(\nu_{i,r}).$$

Here $0 \le r \le \alpha_{i+1}$ and $\nu_{i,r}$ is a homomorphism that maps a tabloid $\{t\}$ into the sum of all those tabloids whose $i, i+1$-th rows are identical to $\{t\}$, and whose $i+1$-th row is a subset of size of r of the $i+1$-th row of $\{t\}$. So $\nu_{i,r} : M^{\alpha} \to M^{\nu}$, where $\nu = (\alpha_i \ldots, \alpha_i + r, \alpha_{i+1} - r, \ldots)$. We call each such homomorphism a *test function* for M^{α}. Each $\nu_{i,r}$ defines a homomorphism $\nu_{i,r} \otimes 1 : M^{\alpha} \otimes M^{\beta} \to M^{\nu} \otimes M^{\beta}$ via $(\nu_{i,r} \otimes 1)(x \otimes y) := \nu_{i,r}(x) \otimes y$, for all $x \in M^{\alpha}, y \in M^{\beta}$. We call this a *left* test function for $M^{\alpha} \otimes M^{\beta}$. We can also define the right test functions, in the obvious way.

Lemma 8. *$S^{\alpha'} \otimes_k S^{\beta} = (S^{\alpha'} \otimes_k M^{\beta}) \bigcap (M^{\alpha'} \otimes_k S^{\beta})$ is the intersection of the kernels of all left and right test functions for $M^{\alpha'} \otimes_k M^{\beta}$.*

Proof. We prove a more general fact: let $U \le V$ and $W \le X$ be k-vector spaces. Then we claim that

$$U \otimes_k W = (U \otimes_k X) \cap (V \otimes_k W).$$

Clearly the left hand side is contained in the right hand side. For the opposite containment, extend a basis $\{u_i\}$ of U to a basis $\{v_i\}$ of V, and extend a basis $\{w_j\}$ of W to a basis $\{x_j\}$ of X. Then $\{v_i \otimes x_j\}$ is a basis for $V \otimes_k X$ and $\{u_p \otimes x_q\}$ is a basis of $U \otimes_k X$, and $\{v_r \otimes w_s\}$ is a basis for $V \otimes_k W$. Now suppose that $\sum_{p,q} \lambda_{pq} u_p \otimes x_q = \sum_{r,s} \mu_{rs} v_r \otimes w_s$ is an element in $(U \otimes_k X) \cap (V \otimes_k W)$. Equating coefficients of $v_i \otimes x_j$ on both sides, we get $\lambda_{pq} = 0$, if $x_q \notin W$, and $\mu_{rs} = 0$, if $v_r \notin U$. So the element is $\sum_{p,q} \lambda_{pq} u_p \otimes w_q \in U \otimes W$. □

Corollary 9. $\mathrm{Hom}(S^\alpha, S^\beta)$ *can be identified with the intersection of the subspace of* $M^{\alpha'} \otimes M^\beta$ *spanned by the images of semi-standard homomorphisms* $S^\alpha \to M^\beta$ *and the subspace of spanned by the images of semi-standard homomorphisms* $S^{\beta'} \to M^{\alpha'}$

6. Two examples

Example 10. *Let* $\alpha = (4,3)$ *and* $\beta = (3^2, 1)$*. So* $\alpha' = (2^3, 1)$ *and* $\beta' = (3, 2^2)$*. Then there are* 2 *semi-standard* α*-tableau* T *of type* β*. Their images in* $M^{\alpha'} \otimes M^\beta$ *are:*

$$\begin{array}{l}1112\\223\end{array} \leftrightarrow \begin{array}{l}12\\12\\13\\2\end{array}^{-} + \begin{array}{l}12\\13\\12\\2\end{array}^{-} + \begin{array}{l}13\\12\\12\\2\end{array}^{-} + \begin{array}{l}12\\12\\23\\1\end{array}^{-} + \begin{array}{l}12\\23\\12\\1\end{array}^{-} + \begin{array}{l}23\\12\\12\\1\end{array}^{-},$$

$$\begin{array}{l}1113\\222\end{array} \leftrightarrow \begin{array}{l}12\\12\\12\\3\end{array}^{-} + \begin{array}{l}12\\12\\32\\1\end{array}^{-} + \begin{array}{l}12\\32\\12\\1\end{array}^{-} + \begin{array}{l}32\\12\\12\\1\end{array}^{-}$$

$$= \begin{array}{l}12\\12\\12\\3\end{array}^{-} - \begin{array}{l}12\\12\\23\\1\end{array}^{-} - \begin{array}{l}12\\23\\12\\1\end{array}^{-} - \begin{array}{l}23\\12\\12\\1\end{array}^{-}$$

Now if $U = \begin{array}{l}123\\134\\2\end{array}$ *, then* $U^{-1} = \begin{array}{l}12\\13\\12\\2\end{array}$ *and* $\mathrm{sgn}(\sigma(U)) = -1$, (1) *gives* $O^-_U = -O^-_{U^{-1}}$*. Similarly:*

$$\begin{array}{l}134\\123\\2\end{array}^{-} = \begin{array}{l}12\\23\\12\\1\end{array}^{-}, \quad \begin{array}{l}123\\234\\1\end{array}^{-} = \begin{array}{l}13\\12\\12\\2\end{array}^{-}, \quad \begin{array}{l}234\\123\\1\end{array}^{-} = -\begin{array}{l}23\\12\\12\\1\end{array}^{-},$$

$$\begin{array}{l}123\\124\\3\end{array}^{-} = \begin{array}{l}12\\12\\13\\2\end{array}^{-}, \quad \begin{array}{l}124\\123\\3\end{array}^{-} = -\begin{array}{l}12\\12\\23\\1\end{array}^{-}, \quad \begin{array}{l}123\\123\\4\end{array}^{-} = -\begin{array}{l}12\\12\\12\\3\end{array}^{-}.$$

There are 3 *semi-standard* β'*-tableaux of type* α'*. We compute their images in* $M^{\alpha'} \otimes M^\beta$*, using the identities above. In some cases we must apply a premutation to order the rows; the* *orbit sum is multiplied by the sign of the permutation.*

$$\begin{array}{l}112\\23\\34\end{array} \leftrightarrow \begin{array}{l}123\\134\\2\end{array}^{-} + \begin{array}{l}134\\123\\2\end{array}^{-} + \begin{array}{l}123\\234\\1\end{array}^{-} + \begin{array}{l}234\\123\\1\end{array}^{-}$$

$$= -\begin{array}{l}12\\13\\12\\2\end{array}^{-} + \begin{array}{l}12\\23\\12\\1\end{array}^{-} + \begin{array}{l}13\\12\\12\\2\end{array}^{-} - \begin{array}{l}23\\12\\12\\1\end{array}^{-}$$

$$\begin{array}{l}113\\22\\34\end{array} \leftrightarrow \begin{array}{l}123\\124\\3\end{array}^{-} + \begin{array}{l}124\\123\\3\end{array}^{-} + \begin{array}{l}123\\324\\1\end{array}^{-} + \begin{array}{l}324\\123\\1\end{array}^{-},$$

$$= \begin{array}{l}12\\12\\13\\2\end{array}^{-} - \begin{array}{l}12\\12\\23\\1\end{array}^{-} - \begin{array}{l}13\\12\\12\\2\end{array}^{-} + \begin{array}{l}23\\12\\12\\1\end{array}^{-}$$

$$\begin{array}{l}114\\22\\33\end{array} \leftrightarrow \begin{array}{l}123\\123\\4\end{array}^{-} + \begin{array}{l}123\\423\\1\end{array}^{-} + \begin{array}{l}423\\123\\1\end{array}^{-} = -\begin{array}{l}12\\12\\12\\3\end{array}^{-} + \begin{array}{l}13\\12\\12\\2\end{array}^{-} - \begin{array}{l}23\\12\\12\\1\end{array}^{-}.$$

We now form a matrix whose rows are indexed by the semi-standard homomorphisms, and whose columns are indexed by the signed regular orbits in $M^{\alpha'} \otimes M^{\beta}$:

	12 12 13 2	12 13 12 2	13 12 12 2	12 12 23 1	12 23 12 1	23 12 12 1	12 12 12 3
1112 223	1	1	1	1	1	1	0
1113 222	0	0	0	−1	−1	−1	1
112 23 34	0	−1	1	0	1	−1	0
113 22 34	1	0	−1	−1	0	1	0
114 22 33	0	0	1	0	0	−1	−1

It can be checked that

$$\begin{array}{l}1112\\223\end{array} - 3\cdot \begin{array}{l}1113\\222\end{array} = -\begin{array}{l}112\\23\\34\end{array} + \begin{array}{l}113\\22\\34\end{array} - 2\cdot \begin{array}{l}114\\22\\33\end{array} \pmod 5.$$

Moreover, the transpose of the matrix has elementary divisors $[1,1,1,1,5]$*. It follows from this that* $\mathrm{Hom}(S^{(4,3)}, S^{(3^2,1)})$ *and* $\mathrm{Hom}(S^{(3,2^2)}, S^{(2^3,1)})$ *are each* 1*-dimensional, if char*$(k) = 5$*, and* 0*-dimensional otherwise.*

Example 11. *Let* $\alpha = (4,2)$ *and* $\beta = (2^3)$*. So* $\alpha' = (2^2,1^2)$ *and* $\beta' = (3,2^2)$*. Then there are* 3 *semi-standard* α*-tableaux* T *of type* β*. Their images in* $M^{\alpha'} \otimes M^{\beta}$ *are:*

$$\begin{array}{l}1122\\33\end{array} \leftrightarrow \begin{array}{l}13\\13\\2\\2\end{array}^{-} + \begin{array}{l}13\\23\\1\\2\end{array}^{-} + \begin{array}{l}23\\13\\1\\2\end{array}^{-} + \begin{array}{l}13\\23\\2\\1\end{array}^{-} + \begin{array}{l}23\\13\\2\\1\end{array}^{-} + \begin{array}{l}23\\23\\1\\1\end{array}^{-},$$

$$\begin{array}{l}1123\\23\end{array} \leftrightarrow \begin{array}{l}12\\13\\2\\3\end{array}^{-} + \begin{array}{l}13\\23\\2\\3\end{array}^{-} + \begin{array}{l}12\\13\\3\\2\end{array}^{-} + \begin{array}{l}13\\12\\3\\2\end{array}^{-} +$$

$$\begin{array}{l}12\\23\\1\\3\end{array}^{-} + \begin{array}{l}23\\12\\1\\3\end{array}^{-} - \begin{array}{l}13\\23\\1\\2\end{array}^{-} - \begin{array}{l}23\\13\\1\\2\end{array}^{-} +$$

$$\begin{array}{l}12\\23\\3\\1\end{array}^{-} + \begin{array}{l}23\\12\\3\\1\end{array}^{-} - \begin{array}{l}13\\23\\2\\1\end{array}^{-} - \begin{array}{l}23\\13\\2\\1\end{array}^{-} - 2\cdot \begin{array}{l}23\\23\\1\\1\end{array}^{-},$$

$$\begin{array}{l}1133\\22\end{array} \leftrightarrow \begin{array}{l}12\\12\\3\\3\end{array}^{-} + \begin{array}{l}23\\23\\1\\1\end{array}^{-} - \begin{array}{l}12\\23\\1\\3\end{array}^{-} - \begin{array}{l}23\\12\\1\\3\end{array}^{-} - \begin{array}{l}12\\23\\3\\1\end{array}^{-} - \begin{array}{l}23\\12\\3\\1\end{array}^{-}.$$

Recall that the signed regular orbit sums in $M^{\alpha'} \otimes M^{\beta}$ *are labelled by the row the regular* α'*-tableaux of type* β*, and also by row regular* β*-tableaux of type* α'*. We equate these two* *labellings:*

$$\begin{array}{l}12\\34\\12\end{array}^{-} = -\begin{array}{l}13\\13\\2\\2\end{array}^{-}, \quad \begin{array}{l}13\\24\\12\end{array}^{-} = \begin{array}{l}13\\23\\1\\2\end{array}^{-}, \quad \begin{array}{l}23\\14\\12\end{array}^{-} = -\begin{array}{l}23\\13\\1\\2\end{array}^{-}, \quad \begin{array}{l}24\\13\\12\end{array}^{-} = \begin{array}{l}23\\13\\2\\1\end{array}^{-}, \quad \begin{array}{l}14\\23\\12\end{array}^{-} = -\begin{array}{l}13\\23\\2\\1\end{array}^{-},$$

$$\begin{array}{l}34\\12\\12\end{array}^{-} = -\begin{array}{l}23\\23\\1\\1\end{array}^{-}, \quad \begin{array}{l}12\\13\\24\end{array}^{-} = \begin{array}{l}12\\13\\2\\3\end{array}^{-}, \quad \begin{array}{l}12\\23\\14\end{array}^{-} = -\begin{array}{l}13\\12\\2\\3\end{array}^{-}, \quad \begin{array}{l}12\\14\\23\end{array}^{-} = -\begin{array}{l}12\\13\\3\\2\end{array}^{-}, \quad \begin{array}{l}12\\24\\13\end{array}^{-} = \begin{array}{l}13\\12\\3\\2\end{array}^{-},$$

$$\begin{array}{l}13\\12\\24\end{array}^{-} = -\begin{array}{l}12\\23\\1\\3\end{array}^{-}, \quad \begin{array}{l}23\\12\\14\end{array}^{-} = \begin{array}{l}23\\12\\1\\3\end{array}^{-}, \quad \begin{array}{l}14\\12\\23\end{array}^{-} = \begin{array}{l}12\\23\\3\\1\end{array}^{-}, \quad \begin{array}{l}24\\12\\13\end{array}^{-} = -\begin{array}{l}23\\12\\3\\1\end{array}^{-}, \quad \begin{array}{l}12\\12\\34\end{array}^{-} = -\begin{array}{l}12\\12\\3\\3\end{array}^{-},$$

There are 2 *semi-standard* β'*-tableaux of type* α'*. We compute their images in* $M^{\beta} \otimes M^{\alpha'}$*:*

$$\begin{aligned}
\begin{array}{l}112\\234\end{array} \leftrightarrow\ & \begin{array}{l}12\\13\\24\end{array}^{-} + \begin{array}{l}12\\14\\23\end{array}^{-} + \begin{array}{l}13\\12\\24\end{array}^{-} + \begin{array}{l}14\\12\\23\end{array}^{-} + \begin{array}{l}12\\23\\14\end{array}^{-} + \begin{array}{l}12\\24\\13\end{array}^{-} \\
& + \begin{array}{l}14\\23\\12\end{array}^{-} + \begin{array}{l}13\\24\\12\end{array}^{-} + \begin{array}{l}23\\12\\14\end{array}^{-} + \begin{array}{l}24\\12\\14\end{array}^{-} + \begin{array}{l}23\\14\\12\end{array}^{-} + \begin{array}{l}24\\13\\12\end{array}^{-} + \\
=\ & \begin{array}{l}12\\13\\2\\3\end{array}^{-} - \begin{array}{l}12\\13\\3\\2\end{array}^{-} - \begin{array}{l}12\\23\\1\\3\end{array}^{-} + \begin{array}{l}12\\23\\3\\1\end{array}^{-} - \begin{array}{l}13\\12\\2\\3\end{array}^{-} + \begin{array}{l}13\\12\\3\\2\end{array}^{-} \\
& - \begin{array}{l}13\\23\\2\\1\end{array}^{-} + \begin{array}{l}13\\23\\1\\2\end{array}^{-} + \begin{array}{l}23\\12\\1\\3\end{array}^{-} - \begin{array}{l}23\\12\\3\\1\end{array}^{-} - \begin{array}{l}23\\13\\1\\2\end{array}^{-} + \begin{array}{l}23\\13\\2\\1\end{array}^{-},
\end{aligned}$$

$$\begin{matrix}113\\224\end{matrix} \leftrightarrow \begin{matrix}12\\12\\34\end{matrix}^{-} - \begin{matrix}12\\14\\23\end{matrix}^{-} - \begin{matrix}14\\12\\23\end{matrix}^{-} - \begin{matrix}12\\23\\14\end{matrix}^{-} + \begin{matrix}12\\34\\12\end{matrix}^{-} - \begin{matrix}14\\23\\12\end{matrix}^{-}$$

$$- \begin{matrix}23\\12\\14\end{matrix}^{-} - \begin{matrix}23\\14\\12\end{matrix}^{-} + \begin{matrix}34\\12\\12\end{matrix}^{-}$$

$$= - \begin{matrix}12\\12\\3\\3\end{matrix}^{-} + \begin{matrix}12\\13\\3\\2\end{matrix}^{-} - \begin{matrix}12\\23\\3\\1\end{matrix}^{-} + \begin{matrix}13\\12\\2\\3\end{matrix}^{-} - \begin{matrix}13\\13\\2\\2\end{matrix}^{-} + \begin{matrix}13\\23\\2\\1\end{matrix}^{-}$$

$$- \begin{matrix}23\\12\\1\\3\end{matrix}^{-} + \begin{matrix}23\\13\\1\\2\end{matrix}^{-} - \begin{matrix}23\\23\\1\\1\end{matrix}^{-} .$$

We now form a matrix whose rows are indexed by the semi-standard homomorphisms, and whose columns are indexed by the row regular α'-tableaux of type β:

	13 13 2 2	13 23 1 2	23 13 1 2	13 23 2 1	23 13 2 1	23 23 1 1	12 13 2 3	13 12 2 3	12 13 3 2	13 12 3 2	12 23 1 3	23 12 1 3	12 23 3 1	23 12 3 1	12 12 3 3
1122 33	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
1123 23	0	−1	−1	−1	−1	−2	1	1	1	1	1	1	1	1	0
1133 22	0	0	0	0	0	1	0	0	0	0	−1	−1	−1	−1	1
112 234	0	1	−1	−1	1	0	1	−1	−1	1	−1	1	1	−1	0
113 224	−1	0	1	1	0	−1	0	1	1	0	0	−1	−1	0	−1

It can be checked that

$$-2 \cdot \begin{matrix}1122\\33\end{matrix} + \cdot \begin{matrix}1123\\23\end{matrix} - 2 \cdot \begin{matrix}1133\\22\end{matrix} = \begin{matrix}112\\234\end{matrix} + 2 \cdot \begin{matrix}113\\224\end{matrix}, \quad (\mathrm{mod}\, 4).$$

The transpose of the matrix has elementary divisors $[1,1,1,1,4]$*. It follows from this that* $\mathrm{Hom}(S^{(4,2)}, S^{(3^2)}) \neq 0$ *and* $\mathrm{Hom}(S^{(2^3)}, S^{(2^2,1^2)}) \neq 0$*, if char*$(k) = 2$*, and both spaces are* 0*-dimensional otherwise.*

Department of Mathematics, National University of Ireland, Maynooth, Co. Kildare, Ireland
E-mail address: John.Murray@maths.nuim.ie