OPEN PROBLEMS INSPIRED BY ADAM ALLAN'S VISIT

ADAM ALLAN
JOHN MURRAY

Here p is a rational prime and k is an algebraically closed field of characteristic p. Also G is a finite group and H is a subgroup of G. All algebras and modules are finite dimensional.

Lemma 1 (Morita?)**.** *Let A, B be k-algebras. Then $A \otimes_k B$ is quasi-Frobenius iff both A and B are quasi-Frobenius.*

Question 2. *Let A, B be k-algebras. Then $A \otimes_k B$ is symmetric iff both A and B are symmetric?*

Lemma 3. *$G = N \otimes P$ where N is a p'-group and P is a p-group. Let b be a block of N and let B be the unique block of G covering b. Suppose that $D \leq P$ is a defect group of B. Then*

$$B^P \cong b^D \otimes_k kP.$$

So B^P is q.f. iff b^D is q.f.

Question 4. *Suppose that G is p-nilpotent, and that B is block of kG with a defect group P that is a Sylow p-subgroup of G. Is B^P symmetric or quasi-Frobenius?*

Lemma 5 (Allan)**.** *Let P be a cyclic p-group, and let M be a kP-module. Then $End_{kP}(M)$ is symmetric iff M is 'isotypic' as kP-module.*

Proof. Jordan form theory. □

Example 6. *Let N be an extra-special group of order 5^{1+2} and exponent 5. Then $Aut(N)$ contains a cyclic group P of order 3 which centralizes $Z(N)$. Set $G = N \rtimes P$ and let $\zeta \in Irr(N)$ lie over a non-trivial irreducible character of $Z(N)$. Then $\zeta(1) = 5$ and there is a unique 3-Brauer character θ of G which extends ζ. Moreover, the 3-block B of G containing θ contains all irreducible characters $Irr(G \mid \zeta)$ of G lying over ζ. Let M be the irreducible kG-module whose Brauer character is θ. So $M{\downarrow_N}$ affords ζ. Also $B \cong End_k(M)$ as k-algebras. It can be checked that $M{\downarrow_P} \cong kP \oplus kP/P^+kP$, as right kP-modules. In particular, $B^P = End_{kP}(M)$ is not quasi-Frobenius.*

Lemma 7. *Suppose that G is p-nilpotent. Let B_0 be the principal p-block of G. Then B_0^H is symmetric iff $[H, G] \leq O_{p'}(G)$.*

Question 8. *Suppose that $p \nmid [G : H]$. Is it true that if kG^H is q.f. then G is p-nilpotent and the Sylow p-subgroups of G are abelian? This is true if G is p-nilpotent. If we remove the assumption on $[G : H]$, and kG^H is q.f., is it true that the Sylow p-subgroups of H are abelian?*

Lemma 9 (Allan). *Suppose that $p \not| |H|$. Then kG^H is symmetric.*

Proof. The symmetrizing form on kG^H is just the restriction of the usual form on kG to kG^H. □

Lemma 10 (Allan). *Let B be a block of kG that has defect 0. Then B^H is symmetric.*

Proof. Let M be the unique irreducible B-module. Then M is projective and $B \cong End_k(M)$, as k-algebras. Moreover, $B^H \cong End_{kH}(M\downarrow_H)$. But $M\downarrow_H$ is projective. So $End_{kH}(M\downarrow_H)$ is a symmetric algebra, by a standard result. □

Question 11. *Let b be a block of kH that has defect 0. Is $(kGe_b)^H$ symmetric?*

Lemma 12 (Allan). *Let G be a dihedral group. If $p = 2$ then kG^H is q.f. iff $[H,G] \leq O_{2'}(G)$. If $p \neq 2$, then kG^H is q.f. iff $p \not| |H|$.*

$kS_4^{S_3}$ is symmetric, when $p = 2$, despite the fact that $|S_3|$ is even.

Question 13. *Let e be a block idempotent of kS_n and let f be a block idempotent of kS_ℓ with $ef \neq 0$, when is $kS_n^{S_\ell}ef$ a symmetric algebra? When does this algebra have tame/finite representation type? Special case: $\ell = n-1$?*

Question 14. *Suppose that $|H| = 2$ and $p = 2$. Is $Z(kG^H)$ generated by $Z(kG)$ and kH? What if we replace the hypothesis by $|H| = p$ and $p \neq 2$?*

Note that kG is self-dual as kG^H-module.

Question 15. *When is kG a projective kG^H-module?*

Lemma 16. *Suppose that kG is projective as kG^H-module. Then kG^H is a self-injective algebra.*

Proof. Two facts: the dual of a projective module is injective, and each pim is a direct summand of the regular module.

We need to show that each projective indecomposable kG^H-module P is injective. Now P is a submodule of kG^H, and hence a submodule of kG. So P^* is a submodule of $kG^* = kG$. As P^* is injective, this implies that P^* is even a direct summand of kG. But each indecomposable direct summand of kG is projective, by hypothesis. We deduce from the fact that P^* is projective that P is injective. □

Note: if kG is a projective kG^H-module, it follows from the proof above that the regular kG^H-module is a direct summand of kG.

Note: kG is a projective $Z(kG)$-module iff G is p-nilpotent and $G/O_{p'}(G)$ is abelian (Müller's theorem).

Question 17. *If kG^H is q.f., is it necessarily symmetric? a Hopf algebra? Note that Hopf algebras are symmetric (reference?). Note that $Z(kS_3)$ is not a Hopf subalgebra of kS_3, $p = 2$, but it is q.f.*

Question 18. *When is it the case that every irreducible kG^H-module has the form $Hom_{kH}(S_H, S_G)$, where S_X is an irreducible kX-module?*

Question 19. *When is $End_{kG^H}(kG) = End_{kH\times G}(kG)$?*

Question 20. *Formulate and prove a first main theorem for the blocks of kG^H.*

Question 21. *Describe the blocks and irreducible modules of kG^H when G is p-solvable.*