ATTEMPT SIX QUESTIONS (50 marks each).

Marks may be lost if all necessary work is not clearly shown.
1. (a) Show that the following simplifies to a constant when \( x \neq 2 \)

\[
\frac{3x - 5}{x - 2} + \frac{1}{2 - x}.
\]

(b) \( f(x) = ax^3 + bx^2 + cx + d \) where \( a, b, c, d \in \mathbb{R} \).

If \( k \) is a real number such that \( f(k) = 0 \), prove that \( x - k \) is a factor of \( f(x) \).

(c) \((x - t)^2\) is a factor of \( x^3 + 3px + c \).

Show that

(i) \( p = -t^2 \)

(ii) \( c = 2t^3 \).

2. (a) Solve for \( x, y, z \)

\[
\begin{align*}
3x - y + 3z &= 1 \\
x + 2y - 2z &= -1 \\
4x - y + 5z &= 4.
\end{align*}
\]

(b) Solve \( x^2 - 2x - 24 = 0 \).

Hence, find the values of \( x \) for which

\[
\left( x + \frac{4}{x} \right)^2 - 2\left( x + \frac{4}{x} \right) - 24 = 0, \quad x \in \mathbb{R}, \ x \neq 0.
\]

(c) (i) Express \( a^4 - b^4 \) as a product of three factors.

(ii) Factorise \( a^5 - a^4b - ab^4 + b^5 \).

Use your results from (i) and (ii) to show that

\[ a^5 + b^5 > a^4b + ab^4 \]

where \( a \) and \( b \) are positive unequal real numbers.
3. \(\textbf{(a)}\) Given that \(A = \begin{pmatrix} 1 & -2 \\ 2 & 3 \end{pmatrix}\) and \(B = \begin{pmatrix} 3 & 1 \\ -5 & -2 \end{pmatrix}\), find \(B^{-1}A\).

\(\textbf{(b)}\) \(\textbf{(i)}\) Simplify \(\left(-\frac{2 + 3i}{3 + 2i}\right)^6\) and hence, find the value of \(\left(-\frac{2 + 3i}{3 + 2i}\right)^9\) where \(i^2 = -1\).

\(\textbf{(ii)}\) Find the two complex numbers \(a + ib\) such that 

\[(a + ib)^2 = 15 - 8i.\]

\(\textbf{(c)}\) Use De Moivre’s theorem

\(\textbf{(i)}\) to prove that \(\cos 3\theta = 4\cos^3 \theta - 3\cos \theta\)

\(\textbf{(ii)}\) to express \((-\sqrt{3} - i)^{10}\) in the form \(2^n(1 - i\sqrt{k})\) where \(n, k \in \mathbb{N}\).

4. \(\textbf{(a)}\) The first three terms of a geometric sequence are \(2x - 4, \ x + 1, \ x - 3\). Find the two possible values of \(x\).

\(\textbf{(b)}\) Given that 

\[u_n = \frac{1}{2}(4^n - 2^n)\]

for all integers \(n\), show that 

\[u_{n+1} = 2u_n + 4^n.\]

\(\textbf{(c)}\) \(\textbf{(i)}\) Given that \(g(x) = 1 + 2x + 3x^2 + 4x^3 \ldots\) where \(-1 < x < 1\), show that 

\[g(x) = \frac{1}{(1-x)^2}.\]

\(\textbf{(ii)}\) \(P(n) = u_1u_2u_3u_4 \ldots u_n\) where 

\[u_k = ar^{k-1}\] for \(k = 1, 2, 3, \ldots, n\) and \(a, r \in \mathbb{R}\).

Write \(P(n)\) in the form \(a^n r^{f(n)}\) where \(f(n)\) is a quadratic expression in \(n\).
5. (a) Express the recurring decimal 1.2 in the form \( \frac{a}{b} \) where \( a, b \in \mathbb{N} \).

(b) Prove by induction that \( n! > 2^n, \ n \in \mathbb{N}, \ n \geq 4 \).

(c) (i) Solve for \( x \)

\[
2 \log_y x = \frac{1}{2} + \log_y (5x + 18), \quad x > 0.
\]

(ii) Solve for \( x \)

\[
3e^x - 7 + 2e^{-x} = 0.
\]

6. (a) Differentiate with respect to \( x \)

(i) \( (1 + 5x)^3 \)

(ii) \( \frac{7x}{x - 3}, \quad x \neq 3 \).

(b) (i) Prove, from first principles, the product rule

\[
\frac{d}{dx} (uv) = u \frac{dv}{dx} + v \frac{du}{dx}
\]

where \( u = u(x) \) and \( v = v(x) \).

(ii) Given \( y = \sin^{-1}(2x - 1) \), find \( \frac{dy}{dx} \) and calculate its value at \( x = \frac{1}{2} \).

(c) \( f(x) = \frac{1}{x + 1} \) where \( x \in \mathbb{R}, \ x \neq -1 \).

(i) Find the equations of the asymptotes of the graph of \( f(x) \).

(ii) Prove that the graph of \( f(x) \) has no turning points or points of inflection.

(iii) If the tangents to the curve at \( x = x_1 \) and \( x = x_2 \) are parallel and if \( x_1 \neq x_2 \), show that

\[
x_1 + x_2 + 2 = 0.
\]
7. (a) Find the slope of the tangent to the curve

\[ x^2 - xy + y^2 = 1 \text{ at the point (1,0)}. \]

(b) The parametric equations of a curve are

\[ x = \cos^3 t \quad \text{and} \quad y = \sin^3 t, \quad 0 \leq t \leq \frac{\pi}{2}. \]

(i) Find \( \frac{dx}{dt} \) and \( \frac{dy}{dt} \) in terms of \( t \).

(ii) Hence, find integers \( a \) and \( b \) such that

\[ \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 = \frac{a}{b} (\sin 2t)^2. \]

(c) \( f(x) = \frac{\ln x}{x} \) where \( x > 0 \).

(i) Show that the maximum of \( f(x) \) occurs at the point \( \left( e, \frac{1}{e} \right) \).

(ii) Hence, show that \( x^e \leq e^x \) for all \( x > 0 \).

8. (a) Find

(i) \( \int (x^2 + 2) \, dx \)  
(ii) \( \int e^{3x} \, dx \).

(b) Evaluate

(i) \( \int_0^{\pi/2} \sin^2 \theta \, d\theta \)  
(ii) \( \int_0^1 \frac{x}{x^2 + 4} \, dx \).

(c) (i) Find the value of the real number \( p \) given that

\[ \int_2^p \frac{dx}{x^2 - 4x + 5} = \frac{\pi}{4}. \]

(ii) The region bounded by the curve \( y = x^2 \) and the line \( y = 4 \) is divided into two regions of equal area by the line \( y = k \).

Show that \( k^3 = 16 \).