M.50

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LEAVING CERTIFICATE EXAMINATION, 1990

MATHEMATICS - HIGHER LEVEL - PAPER II (300 marks)

FRIDAY, 8 JUNE - MORNING, 9.30 to 12.00

0458890

Attempt **QUESTION 1** (100 marks) and **FOUR** other questions (50 marks each)

Marks may be lost if all your work is not clearly shown
or if you have not indicated where a calculator has been used

1. (i) If $f(x) = \dfrac{x}{x+1}$, find

$$\lim_{h \to 0} \frac{f(3+h) - f(3)}{h}.$$

$$\frac{\frac{3+x}{4+x} - \frac{3}{4}}{x}$$

$$\frac{\frac{12+4x-12-3x}{16+4x}}{x}$$

$$\frac{\left(\frac{x}{16+4x}\right)}{x}$$

$$\frac{1}{16+4x}$$

$$\frac{1}{16}$$

(ii) Evaluate $\left(\dfrac{-1 + i\sqrt{3}}{\sqrt{3} + i}\right)^{99}$, where $i = \sqrt{-1}$.

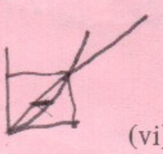
(iii) The first three terms of an arithmetic sequence are

$\log_a p^2, \quad 2\log_a pq, \quad 2\log_a pq^2$

If $pq^2 = a$, find the sum of the first five terms.

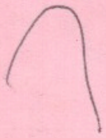
(iv) Find the derivative of $\log_e(\sec x)$ at the point x satisfying $\sqrt{2}\cos x = 1$ and $0 < x < \frac{\pi}{2}$.

(v) Find the maximum distance, measured parallel to the X axis, between the graphs
$y = x, \quad 0 \leqslant x \leqslant 1$ and $y = x^2, \quad 0 \leqslant x \leqslant 1$.



(vi) Evaluate $$\int_0^2 \frac{x\,dx}{4-x}.$$

(vii) Differentiate $\log_a x$ with respect to x.



(viii) The sequence $u_0, u_1, u_2, \ldots, u_n, \ldots$ is such that

$u_0 = \frac{1}{5}$

$u_n = \frac{5}{2}u_{n-1} + \frac{3}{2}$ for $n \in \mathbf{N}_0$.

Investigate if $2(u_1 + u_2) > u_3$.

(ix) Write down the equation of one asymptote to the curve $x^2 - y^2 = 1$.

(x) 300 throws of a die resulted in 62 sixes. At the 5% level of significance would you conclude that the die was biased in favour of sixes ?

OR

(x) oab is a triangle. c and d are points on $[oa]$ and $[ob]$, respectively, such that $\vec{a} \cdot \vec{b} = |\vec{c}|\,|\vec{a}| = |\vec{d}|\,|\vec{b}|$
where o is taken as origin.

Show that the circle on $[ab]$ as diameter contains the points c and d.

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OVER →

2. (a) Determine the real numbers p and q so that

$(p + iq)^2 = 15 - 8i$, where $i = \sqrt{-1}$.

Solve the equation

$$(1 + i)z^2 + (-2 + 3i)z - 3 + 2i = 0.$$

$(p+iq)^2 = 15 - 8i$

$p^2 + 2ipq - q^2 =$

$p^2 - q^2 = 15$

$2pq = -8$

$pq = -4$

$p = \frac{-4}{q}$

(b) Let $z = x + iy$ where $x, y \in \mathbf{R}$ and $i = \sqrt{-1}$.

Indicate clearly on an Argand diagram in the z-plane the set K of z given by

$$K = \left\{ z \mid 1 \leqslant x \leqslant 2 \right\}.$$

$\frac{16}{q^2} - q^2 = 15$

On an Argand diagram in the w-plane plot f(K), the image of K, under the transformation

$16 - q^4 = 15q^2$

$$w = f(z) = 5iz - 2.$$

$q^4 - 15q^2 - 16 = 0$

$(q^2+1)(q^2-16) = 0$

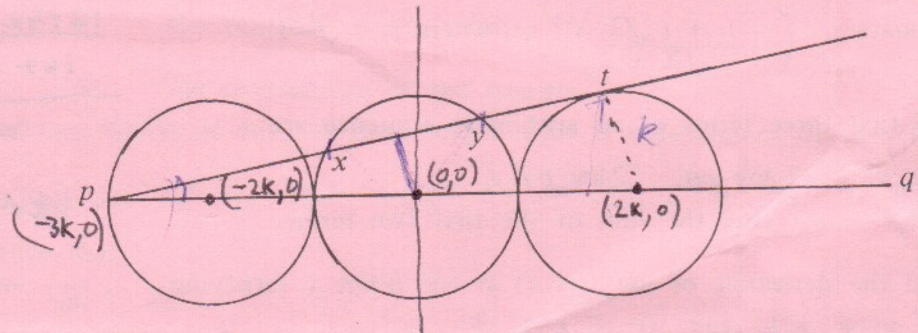
$q = 1, -1, 4, -4$

$p = \frac{-4}{q}$

$p = -4, 4, -1, 1$

3.

$x - 16$
$+ -15$
$1 \quad -16$



The diagram shows three touching circles, each of radius length k units, having their centres in the line pq. The tangent pt to the third circle cuts the middle circle at x and y.

Calculate $|xy|$ in terms of k.

4. (a) Show that

$$\log_e a \,.\, \log_a e = 1.$$

$\log_e a \,.\, \log_a e = 1$

$e^{\log_e a \,.\, \log_a e} = e^1$

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If $x = a^y$, show that $x \dfrac{dy}{dx} = \log_a e$.

$a^{\log_a e} = e^1$

$e = e^1$

(b) Given that

$$x = \sin\theta + \cos\theta$$

$$y = 1 + \sin\theta \cos\theta$$

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find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

13 (c) If $y = \sqrt{\dfrac{e^x}{1 + e^x}}$, find the value of $\dfrac{dy}{dx}$ when $x = 0$.

(d) Find the equation of the tangent to the curve

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$$x^2 y + xy^2 = 6$$

at the point (2, 1).

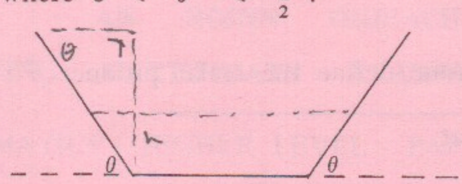
$y = x^3$

$\boxed{\frac{dy}{dx}} = 3x^2$

$\cancel{A = k^2 + k^2 \sin\theta\cos\theta}$

$\cancel{\frac{dA}{dx} = k^2(\sin^2\theta + \cos^2\theta)}$

5. The diagram shows the cross-section of a water channel. The bottom and the sides are a fixed k cm in length while the sides make an angle θ with the horizontal, where $0 < \theta < \frac{\pi}{2}$.



$$\frac{h}{k\sin\theta} = \frac{x}{k\cos\theta}$$

$$\frac{h\cos\theta}{\sin\theta} = x$$

60.

Express the area of the cross-section in terms of k and θ and show that $\frac{\pi}{3}$ is the value of θ which makes this area a maximum.

While θ has this value the channel begins to fill with water. If the cross-section area of the water is increasing at the rate of $\frac{k}{2}$ cm² per minute, find the rate of increase in the vertical height, h, of the water when

$$h = \frac{k}{\sqrt{3}}.$$

6. (a) Evaluate (i) $\displaystyle\int_0^1 \frac{x^3 + 1}{x + 1}\,dx$ (ii) $\displaystyle\int_0^1 \frac{x^2 + 1}{x + 1}\,dx$

(b) Evaluate $\displaystyle\int_0^{\log_e 2} \frac{e^x(e^x - 1)}{e^x + 1}\,dx$

(c) The curve $y = \sin\theta$, $0 \leqslant \theta \leqslant 2\pi$, is rotated about the θ axis. Calculate the volume generated.

7. (a) Given that $\displaystyle\sum_{n=1}^{\infty} u_n$ and $\displaystyle\sum_{n=1}^{\infty} v_n$ are convergent series show that

$$\sum_{n=1}^{\infty} (u_n + v_n) \text{ is also convergent.}$$

Find the range of values of $x > 0$ for which the series

$$\sum_{n=1}^{\infty} \frac{3^n + 4^n}{5^n} x^n$$

is convergent.

(b) Prove that

$$\frac{1}{n} - \frac{1}{(n+1)!} \geqslant \frac{1}{n+1} \quad \text{for } n \in N_0.$$

(c) Test for convergence the series

$$\sum_{n=1}^{\infty} \frac{n\,x^n}{(n+1)\cos x} \quad \text{for } 0 < x < \frac{\pi}{2}.$$

OVER→

8. (a) If $\vec{p}$ and $\vec{q}$ are any vectors, define the scalar product $\vec{p} \cdot \vec{q}$.

If $\vec{r}$ is a vector such that

$$\vec{r} = |\vec{q}|\vec{p} + |\vec{p}|\vec{q},$$

show that

$$\cos\theta_1 = \cos\theta_2$$

where θ_1, θ_2 are the measures of the angles between $\vec{p}$, $\vec{r}$ and between $\vec{q}$, $\vec{r}$, respectively.

(b) In a $\Delta\, xyz$ the circumcentre is taken as origin.

Write down in terms of $\vec{x}, \vec{y}, \vec{z}$

(i) the centroid of the triangle

(ii) the orthocentre of the triangle

and say which, if any, of these expressions is independent of the choice of origin.

In a $\Delta\, abc$ $\vec{a} = -2\vec{i} + 3\vec{j}$, $\vec{b} = -3\vec{j}$, $\vec{c} = 7\vec{i} + 6\vec{j}$.

Prove that

$$\vec{ab} \perp \vec{ac}$$

and express the circumcentre of the triangle in terms of $\vec{i}$ and $\vec{j}$.

Hence, or otherwise, find the distance between the orthocentre and the centroid of the triangle.

OR

8. (a) In a population of 50 boys and 450 girls exactly 5 boys and 15 girls wear glasses. A person is picked at random from the population. Find the probability that the person

$B \cap G = \frac{5}{500} = \frac{1}{100}$

$\frac{50}{500}$ $\frac{1}{10}$ $\frac{20}{500}$ $\frac{1}{25}$ B if $G = \frac{1}{4}$

(i) is a boy (ii) wears glasses (iii) is a boy, given that the person wears glasses.

Five people are chosen at random, with replacement, from the population. Find correct to three places of decimals the probability that at most two wear glasses.

(b) Z is a random variable having a standard normal distribution. Use your tables, page 36, to estimate

$$P(Z \leqslant -1{\cdot}565).$$

The probability that a seed will germinate is 0·9. A gardener planted 500 seeds. When only 439 seeds germinated he concluded that the seeds were sub-standard.

At the 5% level of significance, was the gardener's conclusion justified ?

(Note: Use $(\mu - 0{\cdot}5)$ for the expected value (mean) of the random variable X, the number of germinating seeds).