LEAVING CERTIFICATE EXAMINATION, 1987

MATHEMATICS – HIGHER LEVEL – PAPER II (300 marks)

Attempt **QUESTION 1** (100 marks) and **FOUR** other questions (50 marks each)

Marks may be lost if all your work is not clearly shown
or if you have not indicated where a calculator has been used

1. (i) Prove that when a positive real number is added to its reciprocal the sum is greater than or equal to 2.

(ii) Find the complex number z such that

$$z\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = 1.$$

(iii)) Differentiate from first principles the function

$$x \rightarrow \frac{1}{1+x^2}$$

(iv) Find the local minimum of the function

$$x \rightarrow x + \frac{4}{x^2}, \quad x > 0$$

and draw a rough graph of the function.

(v) Verify that the point $(1, \sqrt{3})$ is common to the circles

$$x^2 + y^2 = 4 \quad \text{and} \quad (x-2)^2 + y^2 = 4$$

and find the volume generated by rotating the region common to both circles about the X axis.

(vi) Express in terms of x,

$$\sum_{n=0}^{\infty}\left(\frac{x}{x+1}\right)^n \quad \text{for } x > 0.$$

(vii) The sequence

$$u_1, \ u_2, \ u_3, \ \ldots\ldots, \ u_n, \ \ldots$$

is such that

$$u_1 = 1$$

$$u_{n+1} = \frac{u_n}{2} + \frac{1}{u_n} \quad \text{for } n \geqslant 1.$$

If the sequence converges to k, find k.

(viii) Test for convergence the series

$$\sum_{n=1}^{\infty}\frac{n^2+1}{n^3+1}.$$

(ix) Evaluate

$$\lim_{x \to 3}\frac{x-3}{1-\sqrt{4-x}}.$$

(x) If 256 tosses of a coin resulted in 142 heads, would you conclude, at the 5% level of significance, that the coin was biased in favour of heads ?

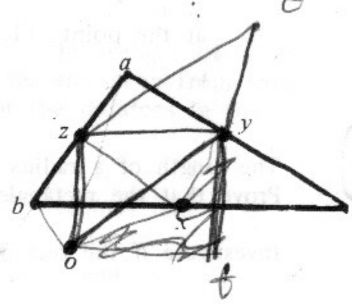
6

OR

(x) x, y, z are the midpoints of the sides of the Δabc and t is such that $oytz$ is a parallelogram.

By expressing $\vec{a}$ in terms of $\vec{x}, \vec{y}, \vec{z}$, or otherwise, where $\vec{o}$ is the origin, prove that

$$ta \parallel xo$$



2. (a) (i) Let z_1 and z_2 be the two complex roots of the equation

$$z^3 + 8 = 0.$$

Find these roots and write down a cubic equation in z for which $z_1 \cdot z_2$ is one of the roots.

(ii) Evaluate $\left(\frac{1 - i\sqrt{3}}{4}\right)^{12}$.

(b) On an Argand diagram plot the set K of z for which

$$|z - i| = 2.$$

Let $w = 3z - 2$.

If $w = u + iv$, express u and v in terms of x and y.

On an Argand diagram in the u, v plane plot the image of K under the transformation $w = 3z - 2$.

3. Let

$$u_1, \; u_2, \; u_3, \; \ldots \; u_n, \; \ldots$$

be a sequence in which

$$S_n = \text{sum of the first } n \text{ terms} = \text{product of the first } n \text{ terms}.$$

Express u_{n+1} in terms of S_n.

$S_n(u_{n+1}) = S_n + u_{n+1}$

$(u_{n+1} - 1)\, S_n = u_{n+1}$

If $S_1 = 3$, show that

$$S_1 \cdot S_2 \cdot S_3 < 3^7 (S_1 - 1)(S_2 - 1)(S_3 - 1).$$

4. (a) If

$$y = 2^x x^2$$

show that

$$\frac{dy}{dx} = \frac{y}{x}(2 + x \log 2).$$

(b) If

$$y = (\sin^{-1} x)^2$$

and

$$v = \sqrt{1 - x^2}\,\frac{dy}{dx},$$

evaluate

$$\frac{dv}{dx} \quad \text{when} \quad x = \frac{1}{\sqrt{5}}.$$

(c) Find the equation of the tangent to the curve

$$y^3 - xy - 6x^3 = 0$$

at the point $(1, 2)$.

5. The length of a radius of a circle is r. A rectangle is inscribed in the circle. Prove that the rectangle of maximum area is a square.

Investigate if the rectangle of maximum perimeter is also a square.

6. Evaluate each of the following:

(a) $\displaystyle\int_0^1 (1 - x^2)^3 \, dx$

(b) $\displaystyle\int_2^4 \frac{x\,dx}{\sqrt{x - 1}}$

(c) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^3 x \; dx}{1 + \cos x}$

(d) $\displaystyle\int \frac{dt}{e^t + e^{-t}}$.

$\frac{e^t}{e^{2t}+1}\,dt$ $\quad u = e^{2t}+1$ $\quad \sqrt{u-1}$

$\frac{du}{dt} = 2e^{2t}$

$\int \frac{e^t}{u} \, \frac{du}{2e^{2t}}$

$\frac{1}{2}\int \frac{\cancel{e^{2t}}}{u\,e^t}$

$\frac{1}{2}\int u^{-1}(u-1)^{-\frac{1}{2}}\,du$

7. (a) Test for convergence

$$\sum_{n=2}^{\infty} \frac{(n - 1)!}{2^n}$$

(b) Find the range of values of $x > 0$ for which the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2 + n}$$

converges.

(c) Test for convergence the series

$$\sum_{n=1}^{\infty} \frac{5^n}{4^n + 5^n} .$$

8. (a) Prove that

$$P(E \cup F) \;=\; P(E) + P(F) - P(E \cap F)$$

where $P(X)$ is the probability of the event X.

A and B are mutually exclusive events such that

$P(A) = \frac{1}{4}$ and $P(B) = \frac{1}{5}$.

Evaluate

(i) $P(A \cup B)$ (ii) $P(A' \cup B')$

where X' means the complement of event X.

(b) A factory exports large consignments of potatoes. On average 2% of the potatoes are bad. Random samples of 40 potatoes are taken from each consignment. A consignment is rejected if 5% of the potatoes in the sample are bad.

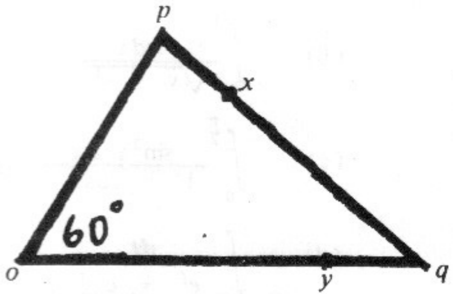
Find

(i) the probability correct to two places of decimals that a consignment is rejected

(ii) the expected percentage of rejected consignments.

OR

8. x and y are points on the sides of the Δopq such that $|\angle poq| = 60°$ and $|px| : |xq| = 1 : 3 = |qy| : |yo|$.



Taking $\vec{o}$ as the origin express $\vec{x}$ and $\overrightarrow{xy}$ in terms of $\vec{p}$ and $\vec{q}$ and prove that

if $|\vec{x}| = |\overrightarrow{xy}|$

then $|\vec{q}| = 3\,|\vec{p}|$.

If z is the midpoint of $[oq]$ and t is the midpoint of $[oz]$, prove that $\angle oxz$ is a right angle and that the Δxtz is equilateral.