LEAVING CERTIFICATE EXAMINATION, 1985

MATHEMATICS - HIGHER LEVEL - PAPER II (300 marks)

MONDAY, 17 JUNE - MORNING, 9.30 to 12.00

Attempt **QUESTION 1** (100 marks) and **FOUR** other questions (50 marks each)

Marks may be lost if all your work is not clearly shown

1. (i) If $x^2 - 6x + 2 \leqslant 0$, show that

$$3-\sqrt{7} \leqslant x \leqslant 3+\sqrt{7}.$$

10.

(ii) Show that $z_1 . \bar{z}_2 + \bar{z}_1 . z_2 = 2\,\mathrm{Re}(z_1 . \bar{z}_2)$, where z_1, z_2 are complex numbers and Re means real.

(iii) Differentiate from first principles the function

$$x \rightarrow \frac{1}{x^2}.$$

(iv) Let $S_n = 1 + 2 + 2^2 + \ldots + 2^{n-1}$. Write down a formula for S_n and evaluate

$$1 + (1 + 2) + (1 + 2 + 2^2) + \ldots + (1 + 2 + 2^2 + \ldots + 2^{19}).$$

(v) A semicircle of radius r is rotated about the x-axis to form a sphere. Show that the volume generated is $\frac{4}{3}\pi r^3$.

(vi) Write down a series

$$\Sigma u_n = u_1 + u_2 + u_3 + \ldots + u_n + \ldots$$

for which Σu_n is divergent while $\Sigma u_n{}^2$ is convergent. Give a reason for your answer in each case.

(vii) Write down the equations of the two asymptotes to the curve

$$y = \frac{1 - x}{1 - x^2}$$

(viii) The function f is defined for $n \in \mathbf{N}$ as follows:

$$f(n) = n + 2 \qquad \text{for } n \text{ odd}$$
$$f(n) = f(f(n - 1)) \qquad \text{for } n \text{ even.}$$

Evaluate $f(10)$.

(ix) Find the slope of the tangent to the curve

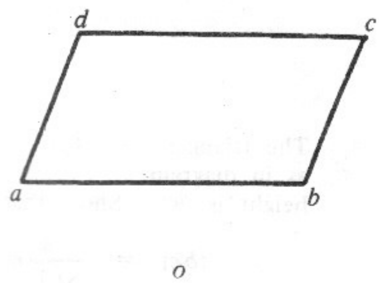
$$x^2 - xy + y^2 = 1$$

at the point $(1, 0)$.

(x) Show that in 5 throws of an unbiased die the probability of throwing one "six" is the same as the probability of throwing no "six".

OR

(x) The origin, o, is outside the parallelogram $abcd$. Express the vector $\vec{b}$ in terms of the vectors $\vec{a}, \vec{d}, \vec{c}$.



2. (a) State De Moivre's Theorem and use it to express $\cos 4A$ as a polynomial in $\cos A$.

(b) Complete the sentence:

"If the complex roots of the equation $z^3 + bz^2 + cz + d = 0$ are $p + iq$ and $p - iq$ where $p, q \in \mathbf{R}$, then the coefficients b, c, d belong to the set of .... numbers"

Show that i is a root of the equation

$$z^3 - iz^2 - z + i = 0$$

and find the other two roots.

(c) Let $\omega = u + iv$ and $z = x + iy$ where $u, v, x, y \in \mathbf{R}$. If

$$\omega = 3 - 2z$$

express u and v in terms of x and y. Find an equation in u and v which corresponds to $x = y$.

On an Argand diagram with axes x and iy plot the set

$$K = \{z \mid x = y\}.$$

On another Argand diagram with axes u and iv plot the set of points into which K is mapped under the transformation

$$z \rightarrow 3 - 2z \; (= \omega).$$

3. (a) If $y^4 = y^3 - y^2 + y - 1$, show that $y^5 + 1 = 0$.

(b) What is meant by a monotonic decreasing (m.d.) sequence ?

$$u_1, \; u_2, \; u_3, \; \ldots, \; u_n, \; \ldots \quad \text{is a m.d. sequence.}$$

Prove that

$$u_n < \frac{u_1 + u_2 + u_3 + \ldots + u_n}{n} < u_1.$$

Deduce that

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} > \sqrt{n}.$$

4. (a) If $x = e^{t+1}$ and $y = e^t$,

find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

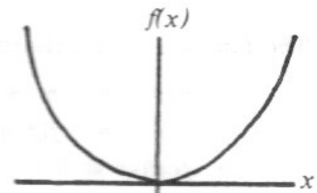
(b) (i) Show that the derivative of $\log x$ is $\frac{1}{x}$ and evaluate the derivative of $\log(\cos x)$ at $x = \frac{\pi}{4}$.

(ii) If $y = \sin^{-1} \frac{x-1}{x+1}$,

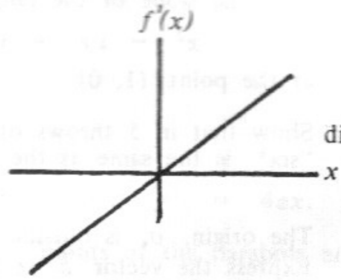
show that $\frac{dy}{dx} = \frac{1}{(x+1)\sqrt{x}}$.

(c) $f'(x)$ is the derivative of a function $f(x)$, $x \in \mathbf{R}$.

Say why diag. (2) does not necessarily follow from diag. (1).



diag. (1)

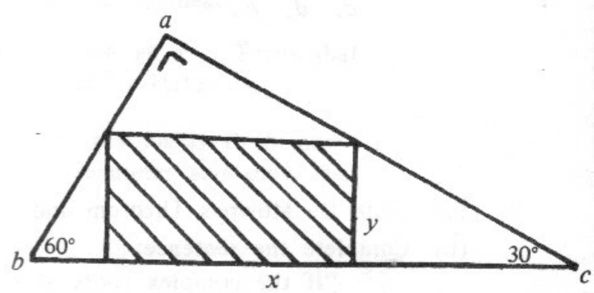


diag. (2)

5. The triangle abc has angles 30°, 60°, 90°, as in diagram. Taking $[bc]$ as base, the height is h. Show that

$$|bc| = \frac{4}{\sqrt{3}} h.$$

A rectangular piece, measuring x by y, as in diagram, is to be cut from the triangle. Show that the maximum area of this piece is half the area of the triangle abc.



6. (a) A tangent is drawn to the graph of $x \rightarrow f(x)$ at $x = t$ and the slope of this tangent is

$$6t^2 - 2t + 3.$$

If the graph contains the point $(-1, -6)$, find $f(x)$.

(b) Evaluate

(i) $$\int_0^1 (1 + x^2)^3 \, dx$$

(ii) $$\int_0^{\frac{\pi}{2}} \sin^3 x \, (1 + \cos^3 x) \, dx$$

(iii) $$\int_1^4 \frac{dx}{x + \sqrt{x}}$$

7. (a) Test for convergence:

(i) $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n} + 1}$

(ii) $\sum_{n=1}^{\infty} \frac{2^n}{n(n + 1)}$

(iii) $\sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!}$.

(b) Let $f(x) = x^2 + x + 2$ and let $0 < k < 1$.

Prove that

$$f(x) - 4 > -k$$

when

$$0 > x - 1 > -\frac{k}{10}.$$

8. (a) Define when two events E and F are independent.

If E and F are independent, prove that E and F' are also independent. (F' is the complement of F)

Three pupils X, Y, Z are attempting to solve a problem and the probability of success for each pupil is given in the table:

	X	Y	Z
Probability of success	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{5}$

Find the probability that all three are unsuccessful and calculate the probability that the problem is solved by at least one of them.

(b) The mean height of a set of 2000 people is 169 cm with standard deviation 5·3 cm. Find the probability that the height of one person chosen at random from the 2000 lies in the range

$$163{\cdot}7 \text{ cm} \leqslant \text{height} \leqslant 174{\cdot}3 \text{ cm}$$

and estimate the number of people from the 2000 who are not smaller that 174·3 cm.

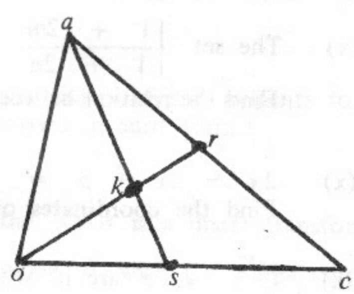
OR

8. (a) If r is any point in the line ab, prove that

$$\vec{r} = t\vec{b} + (1 - t)\vec{a} \text{ for } t \in \mathbf{R}.$$

Deduce that if r divides $[ab]$ internally in the ratio $m : n$, then

$$\vec{r} = \frac{m\vec{b} + n\vec{a}}{m + n}.$$



r and s are the midpoints of the two sides of the triangle in the diagram.

Taking o as the origin and assuming that

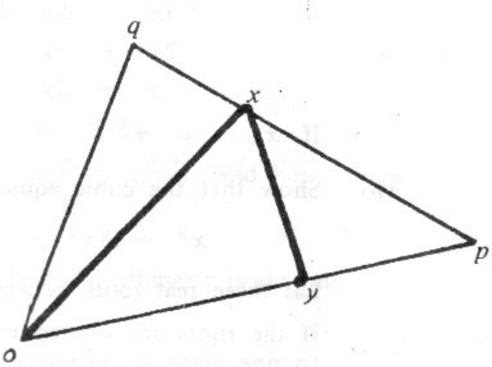
$$|ok| : |kr| = m : n = |ak| : |ks|$$

prove that

$$m : n = 2 : 1.$$

(b) x and y are points on two sides of the Δopq, as shown, such that

$$|qx| : |xp| = 1 : 2 = |py| : |yo|.$$



If it is necessary to have $|\vec{ox}| = |\vec{xy}|$, prove that

$$\vec{q} \perp \vec{p}$$

where o is the origin.