LEAVING CERTIFICATE EXAMINATION, 1983

MATHEMATICS – HIGHER LEVEL – PAPER II (300 marks)

MONDAY, 13 JUNE – MORNING, 9.30 to 12.00

Attempt **QUESTION 1** (100 marks) and **FOUR** other questions (50 marks each)

Marks may be lost if all your work is not clearly shown

1. (i) Using the remainder theorem, or otherwise, find the three roots of the equation

$$x^3 + 3x^2 + x - 2 = 0.$$

(ii) Express $(\cos 5\pi + i \sin 5\pi)^{\frac{1}{3}}$ in the form $\dfrac{a + ib}{c}$, where $a, b, c \in \mathbf{R}$

(iii) Evaluate $\displaystyle\sum_{n=1}^{24} n^2 (n + 1)$.

(iv) Test for convergence the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n + 1}$.

(v) Differentiate $\cos x$ from first principles.

(vi) Show that $\displaystyle\int_{1/e}^{1} \frac{dx}{x} = \int_{1}^{e} \frac{dx}{x}$

where e is the base for $\log x$.

(vii) The sequence

$$u_1, u_2, u_3, \ldots u_r, \ldots$$

is such that

$$u_1 = 0{\cdot}1$$

$$u_{r+1} = u_r \sqrt{u_r - u_r^2}\,.$$

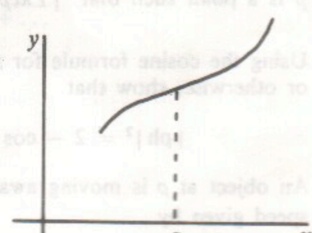
Show that $u_2 = 0{\cdot}03$

and noting that $17^2 = 289$ show that

$$u_3 > 0{\cdot}005.$$

(viii) Evaluate $\displaystyle\lim_{x \to 0} \frac{\sin (3x^2)}{x \sin x}$.

(ix) The graph of $y = f(x)$ has a point of inflexion at $x = a$.



Describe the behaviour of $\dfrac{dy}{dx}$ in the neighbourhood of $x = a$.

(x) If z is a random variable with normal distribution, use your tables to find the probability that $z \geqslant 1{\cdot}4$.

OR

(x) If $\vec{u} = 3\vec{i} - 4\vec{j}$, write down a vector $\vec{v}$ which is perpendicular to $\vec{u}$ and find scalers p and q such that

$$p\vec{u} + q\vec{v} = \vec{i} + \vec{j}.$$

2. (*a*) If $z = x + iy$, where $i = \sqrt{-1}$,

prove that $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

and deduce that $\overline{z_1 + z_2 + z_3} = \overline{z_1} + \overline{z_2} + \overline{z_3}$.

If z_1 is a root of $z^3 + az^2 + cz + d = 0$, where $a, c, d \in \mathbf{R}$, prove that $\overline{z}_1$ is also a root.

$(1 + 2i)$ is a root of $z^3 + az^2 + cz + 5 = 0$.

Find the value of a and the value of c.

(*b*) Let $K = \{z \text{ such that } |z - 2| = |z - 4|\}$

Sketch K and find its image under the transformation

$$z \rightarrow \frac{i}{2}(z - \overline{z}).$$

3. The function

$$f : x \rightarrow \frac{x+1}{\sqrt{x}}$$

is defined for all real $x \geqslant 1$.

Prove that $f(x)$ increases for $x > 1$.

The nth term of a sequence is given by

$$T_n = \frac{n+1}{\sqrt{n}}.$$

Prove that

$$(\sqrt{n+1} - \sqrt{n}) + \left(\frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n}} \right) > 0.$$

4. (*a*) Let $y = \left(\frac{x-1}{x+1} \right)^{\frac{1}{2}}$

Evaluate $\frac{dy}{dx}$ at $x = 1\frac{1}{4}$.

(*b*) If $y = e^t \cos t$ and $x = e^t \sin t$, find $\frac{dy}{dx}$ and express its value at $t = \frac{\pi}{3}$ in the form $a + b\sqrt{3}$.

(*c*) Find the equation of the tangent to the curve

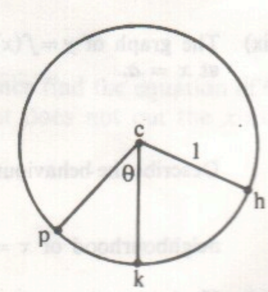
$$y = 2 \tan^{-1} \sqrt{x} \quad \text{at } x = 1.$$

5. h and k are two fixed points on a circle of radius 1 unit and $|\angle hck| = \frac{\pi}{3}$.

p is a point such that $|\angle kcp| = \theta$.

Using the cosine formula for the $\triangle pch$ (Page 9 of Tables), or otherwise, show that

$$|ph|^2 = 2 - \cos\theta + \sqrt{3}\sin\theta.$$



An object at p is moving away from k on the circle at a speed given by

$$\frac{d\theta}{dt} = 10.$$

Find the rate at which its distance, l, from h is increasing when $\theta = \frac{\pi}{3}$.

6. (*a*) Evaluate

(i) $\int_1^2 \left(x^2 + \frac{1}{x}\right)^2 dx$

(ii) $\int_0^{\pi/3} \frac{\sin^3 x \, dx}{\cos^2 x}$

(iii) $\int_0^1 \frac{\sqrt{x}}{e^{\sqrt{x}}} \frac{dx}{x}$.

(*b*) Show that the graph of

$$y = 1 - \frac{3}{x} + \frac{2}{x^2}$$

has a local minimum at $x = \frac{4}{3}$

and a point of inflexion at $x = 2$.

Find the area in the 4th quadrant between the curve and the *x*-axis.

7. (*a*) Prove that

$$\frac{1}{\sqrt{n(n+1)}} > \frac{1}{n+1}$$

for all $n \geqslant 1$ and use this result to test the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}}$$

for convergence.

(*b*) $u_1 + u_2 + u_3 + \ldots + u_n + \ldots$

is a series of positive terms such that

$$\lim_{n \to \infty} u_n = k > 0.$$

Say why this series does not converge.

Test for convergence the series

$$\frac{1}{100} + \frac{4}{103} + \frac{9}{108} + \ldots + \frac{n^2}{n^2 + 99} + \ldots$$

(*c*) Test for convergence the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n} \quad \text{for } x > 0.$$

8. (*a*) Prove that

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

where $P(X)$ is the probability of the event X and singletons are equally likely.

If $\#(E \setminus F) = 5$, $\#(F \setminus E) = 12$, $\#(E \cap F) = 7$, and the sample space has 100 outcomes, calculate

$$P(E \cup F).$$

(b) Let x be a random variable denoting the number of householders in a certain town who own a T.V. set. Say why x has a binomial distribution.

If n is the number of householders and p is the probability that a householder owns a T.V. set, write down the expressions in terms of n and p for $\overline{x}$ and σ.

A random sample of 500 householders is taken. If $p = 0{\cdot}75$ estimate, with a 90% chance of being correct, the least number of householders in this sample who own a T.V. set.

OR

8. (*a*) Prove that for any three vectors $\vec{x}, \vec{y}, \vec{z}$,

$$\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}.$$

Verify that $|\vec{x}|^2 = \vec{x} \cdot \vec{x}$

and prove that

$$\vec{x} \cdot \vec{y} = \tfrac{1}{2}\{|\vec{x} + \vec{y}|^2 - |\vec{x}|^2 - |\vec{y}|^2\}.$$

(*b*) Let $\vec{e}_1 = 3\vec{i} + 4\vec{j}$

and $\vec{e}_2 = 3\vec{i} - 4\vec{j}$.

Write down in terms of $\vec{i}$ and $\vec{j}$ the unit vector, $\vec{u}$, along $\vec{e}_1$.

If $\vec{v} = \vec{e}_2 - (\vec{e}_2 \cdot \vec{u})\vec{u}$, prove that $\vec{v}$ is perpendicular to $\vec{u}$.