1. (i) Using the remainder theorem, or otherwise, find the three roots of the equation

\[ x^3 + 3x^2 + x - 2 = 0. \]

(ii) Express \((\cos 5\pi + i \sin 5\pi)^{\frac{2}{3}}\) in the form \(\frac{a + ib}{c}\), where \(a, b, c \in \mathbb{R}\).

(iii) Evaluate \(\sum_{n=1}^{24} n^2(n + 1)\).

(iv) Test for convergence the series \(\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n + 1}\).

(v) Differentiate \(\cos x\) from first principles.

(vi) Show that \(\int_{1/a}^{1} \frac{dx}{x} = \int_{1}^{e} \frac{dx}{x}\), where \(e\) is the base for \(\log x\).

(vii) The sequence \(u_1, u_2, u_3, \ldots u_r, \ldots\)

is such that

\[ u_1 = 0.1 \]

\[ u_{r+1} = u_r \sqrt{u_r - u_r^2}. \]

Show that \(u_2 = 0.03\)

and noting that \(17^2 = 289\) show that

\(u_3 > 0.005\).

(viii) Evaluate \(\lim_{x \to 0} \frac{\sin (3x^2)}{x \sin x}\).

(ix) The graph of \(y = f(x)\) has a point of inflexion at \(x = a\).

Describe the behaviour of \(\frac{dy}{dx}\) in the neighbourhood of \(x = a\).

(x) If \(z\) is a random variable with normal distribution, use your tables to find the probability that \(z \geq 1.4\).

OR

(x) If \(\vec{u} = 3\vec{i} - 4\vec{j}\), write down a vector \(\vec{v}\) which is perpendicular to \(\vec{u}\) and find scalars \(p\) and \(q\) such that

\[ p\vec{u} + q\vec{v} = \vec{i} + \vec{j}. \]
2. (a) If \( z = x + iy \), where \( i = \sqrt{-1} \),
prove that \( \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2} \)
and deduce that \( \overline{z_1 + z_2 + z_3} = \overline{z_1} + \overline{z_2} + \overline{z_3} \).
If \( z_1 \) is a root of \( z^3 + az^2 + cz + d = 0 \), where \( a, c, d \in \mathbb{R} \), prove that \( \overline{z}_1 \) is also a root.

\((1 + 2i)\) is a root of \( z^3 + az^2 + cz + d = 0 \).
Find the value of \( a \) and the value of \( c \).

(b) Let \( K = |z| \) such that \( |z - 2| = |z - 4| \).
Sketch \( K \) and find its image under the transformation
\[
z \rightarrow \frac{i}{2} (z - \overline{z}).
\]

3. The function
\[
f : x \rightarrow \frac{x + 1}{\sqrt{x}}
\]
is defined for all real \( x \geq 1 \).
Prove that \( f(x) \) increases for \( x > 1 \).
The \( n \)th term of a sequence is given by
\[
T_n = \frac{n + 1}{\sqrt{n}}.
\]
Prove that
\[
(\sqrt{n+1} - \sqrt{n}) + \left( \frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n}} \right) > 0.
\]

4. (a) Let \( y = \left( \frac{x - 1}{x + 1} \right)^{\frac{1}{3}} \).
Evaluate \( \frac{dy}{dx} \) at \( x = 1 \frac{2}{3} \).

(b) If \( y = e^t \cos t \) and \( x = e^t \sin t \), find \( \frac{dy}{dx} \) and express its value at \( t = \frac{\pi}{3} \)
in the form \( a + b\sqrt{3} \).

(c) Find the equation of the tangent to the curve
\[
y = 2 \tan^{-1} \sqrt{x} \text{ at } x = 1.
\]

5. \( h \) and \( k \) are two fixed points on a circle of radius 1 unit and \( \angle hck = \frac{\pi}{3} \).
\( p \) is a point such that \( \angle kcp = \theta \).

Using the cosine formula for the \( \Delta phk \) (Page 9 of Tables),
or otherwise, show that
\[
|ph|^2 = 2 - \cos \theta + \sqrt{3} \sin \theta.
\]
An object at \( p \) is moving away from \( k \) on the circle at a speed given by
\[
\frac{d\theta}{dt} = 10.
\]
Find the rate at which its distance, \( t \), from \( h \) is increasing when \( \theta = \frac{\pi}{3} \).
6. (a) Evaluate

\[ \int_{1}^{2} \left( x^2 + \frac{1}{x} \right)^2 \, dx \]

\[ \int_{0}^{\pi/2} \sin^3 x \, dx \]

\[ \int_{0}^{1} \frac{x}{e^{x^2} \cdot x} \, dx \]

(b) Show that the graph of

\[ y = 1 - \frac{3}{x} + \frac{2}{x^2} \]

has a local minimum at \( x = \frac{4}{3} \)

and a point of inflexion at \( x = 2 \).

Find the area in the 4th quadrant between the curve and the x-axis.

7. (a) Prove that

\[ \frac{1}{\sqrt{n(n+1)}} > \frac{1}{n+1} \]

for all \( n \geq 1 \) and use this result to test the series

\[ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}} \]

for convergence.

(b) \( u_1 + u_2 + u_3 + \ldots + u_n + \ldots \)

is a series of positive terms such that

\[ \lim_{n \to \infty} u_n = k > 0. \]

Say why this series does not converge.

Test for convergence the series

\[ \frac{1}{100} + \frac{4}{103} + \frac{9}{108} + \ldots + \frac{n^2}{n^2 + 99} + \ldots \]

(c) Test for convergence the series

\[ \sum_{n=1}^{\infty} \frac{x^n}{n} \quad \text{for} \quad x > 0. \]

8. (a) Prove that

\[ P(E \cup F) = P(E) + P(F) - P(E \cap F) \]

where \( P(X) \) is the probability of the event \( X \) and singletons are equally likely.

If \( \#(E \setminus F) = 5 \), \( \#(F \setminus E) = 12 \), \( \#(E \cap F) = 7 \), and the sample space has 100 outcomes, calculate \( P(E \cup F) \).
(b) Let $x$ be a random variable denoting the number of householders in a certain town who own a T.V. set. Say why $x$ has a binomial distribution.

If $n$ is the number of householders and $p$ is the probability that a householder owns a T.V. set, write down the expressions in terms of $n$ and $p$ for $\bar{x}$ and $\sigma$.

A random sample of 500 householders is taken. If $p = 0.75$ estimate, with a 90% chance of being correct, the least number of householders in this sample who own a T.V. set.

OR

8. (a) Prove that for any three vectors $\bar{x}$, $\bar{y}$, $\bar{z}$,

$$\bar{x} \cdot (\bar{y} + \bar{z}) = \bar{x} \cdot \bar{y} + \bar{x} \cdot \bar{z}.$$ 

Verify that $|\bar{x}|^2 = \bar{x} \cdot \bar{x}$

and prove that

$$\bar{x} \cdot \bar{y} = \frac{1}{2} (|\bar{x} + \bar{y}|^2 - |\bar{x}|^2 - |\bar{y}|^2).$$

(b) Let $\bar{e}_1 = 3\bar{i} + 4\bar{j}$

and $\bar{e}_2 = 3\bar{i} - 4\bar{j}$.

Write down in terms of $\bar{i}$ and $\bar{j}$ the unit vector, $\bar{u}$, along $\bar{e}_1$.

If $\bar{v} = \bar{e}_2 - (\bar{e}_2 \cdot \bar{u})\bar{u}$, prove that $\bar{v}$ is perpendicular to $\bar{u}$.

LEAVING CERTIFICATE EXAMINATION, 1984

MATHEMATICS - HIGHER LEVEL - PAPER 1 (300 marks)

FRIDAY, 8 JUNE - MORNING 9.45 to 12.15

Attempt QUESTION 1 (100 marks) and FOUR other questions (50 marks each)

Marks may be lost if all your work is not clearly shown

1. (i) Find the one value of $x \in \mathbb{R}$ and the one value of $y \in \mathbb{R}$ which satisfies

$$(2x - 1)^2 + (2y + 1)^2 = 0.$$ 

(ii) Prove

$$\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}.$$ 

(iii) Four distinguishable dice are thrown once. Calculate the number of different ways in which only one six is thrown.

(3, 2, 4, 6 or 5, 6, 5, 5, for example, are possible outcomes).

(iv) Calculate the perpendicular distance between the parallel lines

$$3x - 4y = -10 \quad \text{and} \quad 3x - 4y = 15.$$ 

(v) Verify that the line containing the points of intersection of the two circles

$$x^2 + y^2 + 2x - 4y + 1 = 0 \quad \text{and} \quad x^2 + y^2 - 8x - 9 = 0$$ 

is perpendicular to the line containing their centres.

(vi) Find the coefficient of $xy$ in the quadratic form

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$ 

(vii) Find the matrix of the rotation which maps

$$\begin{pmatrix} 6 \\ 2 \end{pmatrix} \rightarrow \begin{pmatrix} -2 \\ 6 \end{pmatrix}.$$