LEAVING CERTIFICATE EXAMINATION, 1981

MATHEMATICS - HIGHER LEVEL - PAPER I (300 marks)

WEDNESDAY, 10 JUNE - MORNING 9.45 to 12.15

Attempt **QUESTION 1** (100 marks) and **FOUR** other questions (50 marks each)

Marks may be lost if all your work is not clearly shown

1. (i) Solve the simultaneous equations

$$\frac{1}{x} = 1\tfrac{1}{4}$$

$$\frac{1}{x} + \frac{1}{y} = 2\tfrac{1}{3}$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 3\tfrac{1}{2}$$

(ii) Find the coefficient of x^3 in the product

$$(a_0 + a_1x + a_2x^2 + a_3x^3)\,(a_0x^3 + a_1x^2 + a_2x + a_3).$$

(iii) How many natural numbers greater than 3000 can be formed from the digits

1, 3, 5, 9

if no digit is repeated in any number.

(iv) Find the equation of the line which passes through the point of intersection of the two lines

$$5x = 3y$$
$$3x + 5y = 2$$

and is perpendicular to the line $x + 2y = 0$.

(v) The line segment joining $(-3, 4)$ and $(1, 2)$ is the diameter of a circle. Write the equation of the circle in the form

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

(vi) If $A = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix}$, find a matrix $C \neq B$ such that $AB = AC$.

(vii) Describe the linear transformation of the plane defined by the matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

(viii) If $7\cos\frac{\theta}{2} = 20\cos\theta$, find the two values of $\cos\frac{\theta}{2}$.

(ix) Find $\displaystyle\lim_{x\to 0} \frac{\tan 2x}{\sin 3x}$.

(x) e is the identity element of the group G under multiplication.
Suppose $g \in G$ such that $g^{11} = g^3$.
Prove that $g^{16} = e$.

OR

(x) The line $y = x$ cuts the parabola $y^2 = 8x$ at p and at the origin o. The tangent at p to the parabola cuts the y-axis at q. Prove that

area of Δopq = area of the square on $[oq]$.

2. (a) Solve the simultaneous equations

$$x + y = z$$
$$3x + 2y - 4z = -1$$
$$x - 3y + 3z = 2 .$$

(b) If α, β are the roots of

$$x^2 - x - 1 = 0 ,$$

find the equation having α^2, β^2 as roots.

(c) Show that $x^3 - 3x^2 - x + 2 = 0$ has one negative real root and two positive real roots and find the smaller of the two positive roots correct to one place of decimals.

3. (a) Find in the form $\frac{a}{b}$ where $a, b \in \mathbf{N}$ the term independent of x in the expansion of

$$(2x^3 - \frac{1}{3x})^8 .$$

(b) If x is so small that its square and higher powers may be neglected, find an approximation of the form $a + bx$ for the expression

$$\frac{1 + 8x}{(1 - 3x)(1 + 2x)}$$

and hence estimate the value of the expression when $x = \frac{13}{9000}$.

(c) Use a binomial expansion to find an approximation for $(0{\cdot}98)^{2{\cdot}5}$ to four places of decimals.

4. (a) Prove that the angles between the lines of slopes m_1 and m_2 are given by

$$\tan\theta = \pm \frac{m_1 - m_2}{1 + m_1 m_2}$$

Find the equations of the lines through the point (4, 3) which make an angle of 45° with the line $6x + y = 5$.

(b) A line through the point (4, 3) forms a triangle in the first quadrant with the axes. If the area of that triangle is 24, find the slope of the line.

5. $S = 0$ represents the equation of a circle.
$K = 0$ represents the equation of a line which intersects the circle.
Prove that $S + \lambda K = 0$ for $\lambda \in \mathbf{R}$

(i) represents a circle and that
(ii) this circle passes through the points of intersection of the line and the circle.

Find the equation of the circle C which contains the points of intersection of the circle

$$x^2 + y^2 - x - y - 2 = 0$$

and the line

$$2x - y + 2 = 0$$

and which has its centre on the line

$$2x - 2y - 9 = 0.$$

This circle C intersects the y-axis at a and b.
Find $|ab|$.

6. Let f be the projection of the plane on the line $x - y = 0$ where the projection is taken parallel to the line $x + y = 0$.
Show that the image of the vector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ under f is $\begin{pmatrix} \frac{1}{2}(x_1 + x_2) \\ \frac{1}{2}(x_1 + x_2) \end{pmatrix}$

Prove that f is a linear transformation and write down the matrix for f.

Let g be the projection of the plane on the line $x + y = 0$ where the projection is taken parallel to the line $x - y = 0$. Write down the matrix for g and investigate if g^{-1} is a linear transformation.

Find the image of the vector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ under the composition $f \circ g$.

7. (a) Find the value of x to the nearest degree between 0° and 360° for which

$$3 \cos x - 4 \sin x = 5 .$$

(b) Using the usual notation, prove that in any triangle

$$\frac{a}{c^2 - b^2} = \frac{1}{c \cos B - b \cos C} .$$

(c) Let f be the real valued function

$$x \rightarrow \sin x \sin 3x .$$

Show that $f(x + 2\pi) = f(x)$

and find the least value of k for which

$$f(x + k) = f(x) .$$

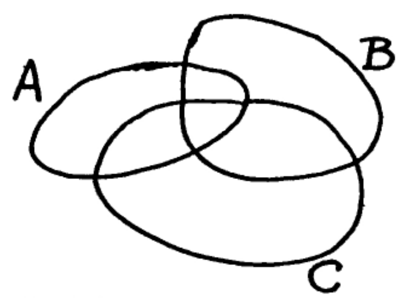
8. (a) Say, giving a reason, whether or not each of the following sets under the indicated operation is a group.

(i) **R**, + (ii) **Z**, − (iii) **R**, × .

(b) Copy this diagram into your answer book and indicate on it the set

$$A \,\Delta\, (B \,\Delta\, C)$$

where Δ denotes symmetric difference.



Use another diagram to show the set

$$(A \,\Delta\, B) \,\Delta\, C .$$

Prove that the set of subsets of $\{a, b\}$ under Δ is a group.

(c) The set $\{e, t, u, v\}$ is a group under multiplication as defined in the Cayley table:

	e	t	u	v
e	e	t	u	v
t	t	u	v	e
u	u	v	e	t
v	v	e	t	u

Each element of the set can be expressed in the form t^n and so the group is said to be generated by t. Find a value of n for each of u, v, e. Is the group generated by any other element of the set ? Give your reason.

OR

8. $x - y + 1 = 0$ is the equation of the directrix of a parabola having its focus at (6, 3).

Find the equation of the parabola. Verify that the parabola does not cut the y-axis and show that the distance between the points where it cuts the x-axis is $8\sqrt{5}$.

Find the equation of the axis of the parabola and call it $A = 0$. Find the equation of the tangent of the vertex of the parabola and call it $T = 0$.

Verify that the equation of the parabola can be put into the form

$$A^2 = pT.$$