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LEAVING CERTIFICATE EXAMINATION, 1977

MATHEMATICS—HIGHER LEVEL—PAPER II (300 marks)

TUESDAY, 14 JUNE—MORNING 9.30 to 12

Six questions to be answered.
All questions carry equal marks.
Mathematics Tables may be obtained from the Superintendent.

1. (*a*) The function f is defined for $x \in \mathbf{R}$, $x > 0$ by

$$f : x \to 2\pi \sqrt{\frac{x^2 + k^2}{gx}} \; ,$$

where k and g are known constants and $g > 0$. If $f(x_1) = f(x_2)$ for $x_1 \neq x_2$, prove that $x_1 x_2 = k^2$.

(*b*) The volume of a segment of height x cm cut from a sphere of radius r cm is given by

$$\frac{\pi}{3} x^2 (3r - x).$$

If from a sphere of radius 3 cm a segment of height x cm is cut off so that the volume of the segment is 1/6 the volume of the sphere, show that x is a root of

$$x^3 - 9x^2 + 18 = 0$$

and hence, or otherwise, calculate this value of x correct to two significant figures. [See Tables page 7]

2. (*a*) Find the total number of combinations of 10 objects taken 5 at a time.
Show that there are 126 ways in which 10 children can be divided into two groups of 5.
Find the number of ways this can be done if the two youngest children are always
(i) in the same group (ii) in different groups.

(*b*) Find the first three terms in ascending powers of x in the expansion of

$$(1 + 3x)^{\frac{1}{2}} (1 - 2x)^{-\frac{1}{3}}.$$

Hence, or otherwise, evaluate $\dfrac{\sqrt{1{\cdot}006}}{\sqrt[3]{0{\cdot}996}}$ correct to six places of decimals.

3. (*a*) Sketch roughly the function

$$x \to \frac{x + 1}{x - 1}$$

defined for $x \in \mathbf{R}$, $x \neq 1$, paying special attention to the asymptotes.

(*b*) A function $y = f(x)$ is defined for different intervals of its domain $\mathbf{R}$ as follows:

$y = -x$	for $-\infty < x \leqslant 0$
$y^2 = x, y > 0$	for $0 < x \leqslant 1$
$(y - 1)^2 = x - 1, y > 1$	for $1 < x \leqslant 2$
$(y - 2)^2 = x - 2, y > 2$	for $x > 2$.

Draw a rough graph of the function and investigate if
(i) $f(x_1 + x_2) = f(x_1) + f(x_2)$ for all $x_1, x_2 \in \mathbf{R}$,
(ii) its derivative is defined for all $x \in \mathbf{R}$.

4. (*a*) Verify that

$$\frac{1}{(r+3)(r+5)} = \frac{1}{2(r+3)} - \frac{1}{2(r+5)}$$

and hence investigate if the series

$$\sum_{r=1}^{\infty} \frac{1}{(r+3)(r+5)}$$

is convergent.

(*b*) Examine for convergence

(i) $2 + \frac{3}{4} + \dots + \dfrac{n+1}{n^2} + \dots$

(ii) $2x + \frac{3}{8}x^2 + \dots + \dfrac{n+1}{n^3} x^n + \dots$

for $x > 0$ including the value $x = 1$.

[P.T.O.

5. (*a*) Evaluate:

(i) $\lim_{n\to\infty} \frac{2n-3}{3n-2}$ (ii) $\lim_{x\to 0} \frac{\sin 3x}{\sin 4x}$ (iii) $\lim_{x\to 0} \frac{1-\cos x}{\sin x}$.

(*b*) From first principles, differentiate with respect to x the function

$$x \to \frac{1}{x} \qquad \text{for } x \in \mathbf{R}_0.$$

(*c*) Differentiate with respect to x:

(i) $\sqrt{1+\frac{1}{x^2}}$ (ii) $e^{-\frac{1}{2}(x-5)^2}$ (iii) $\log_e \sin^3 2x$.

6. (*a*) A rectangular sheet of paper for a poster has an area of 18 m². The margins at the top and bottom are each 1 m wide and the margins at the two sides are each 50 cm wide. If the remaining rectangular piece is to have a maximum area, find the dimensions of the sheet of paper.

(*b*) Water is being poured at the rate of 19 cm³ per second into a hemispherical bowl of radius 10 cm. Find the rate at which the depth of water is increasing when the depth is 1 cm.
[*The volume of a segment of a sphere is given in question* 1(*b*)].

7. A function f is defined for $x \in \mathbf{R}$ as

$$f : x \to \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

(i) Evaluate $f(0)$ and show that $f(-x) = -f(x)$.
(ii) Prove that f is an increasing function.
(iii) Prove that f has a point of inflexion at $x = 0$.
(iv) Draw a rough sketch of f and indicate its shape as $x \to \pm\infty$.

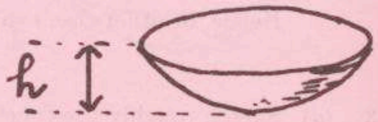
8. Evaluate

(i) $\int_4^9 \frac{(1+\sqrt{x})^2}{\sqrt{x}}\,dx$ (ii) $\int_0^{\pi/2} \cos 2x \cos 3x\,dx$ (iii) $\int_0^{2/3} \frac{dx}{9x^2+4}$

(iv) $\int_1^2 \frac{x\,dx}{(2x+1)(2x-1)}$ (v) $\int_0^{\pi/3} \frac{\sin 3x}{\cos x}\,dx$ (Hint: Express $\sin 3x$ in terms of $\sin x$)

9. The diagram shows a segment of height h cm cut from a sphere of radius r cm. Show by integration that the volume of this segment is

$$\frac{\pi}{3}h^2(3r-h).$$



10. Let

$$u_1, u_2, \ldots, u_n, \ldots \qquad (n \in \mathbf{N}_0)$$

be a sequence of terms such that

$$u_1 = 1, \quad u_{n+1} = 2 + \frac{1}{2+u_n}.$$

(i) Prove by induction that $u_n > 0$ for all n and deduce that $u_n > 2$ for all $n > 1$.
(ii) Write out the first four terms of the sequence and verify that $|u_4 - \sqrt{5}| < 0{\cdot}001$.
(iii) If

$$k = 2 + \frac{1}{2+k}$$

and if

$$\frac{u_{n+1}-k}{u_n-k} = \frac{-1}{(2+k)(2+u_n)},$$

prove that

$$|u_{n+1}-k| < \frac{1}{12}|u_n-k|.$$

(iv) Deduce that $|u_{n+1}-k| \to 0$ as $n \to \infty$.