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LEAVING CERTIFICATE EXAMINATION, 1974

MATHEMATICS—HIGHER LEVEL—PAPER II (300 marks)

TUESDAY, 18 JUNE—MORNING 9.30 to 12

Six questions to be answered.
All questions carry equal marks.
Mathematics Tables may be obtained from the Superintendent.

1. (*a*) The roots of the equation $x^2 + 2x + 3 = 0$ are α and β. Find

 (i) the equation whose roots are $1 - \frac{1}{\alpha}$, $1 - \frac{1}{\beta}$,

 (ii) the value of $\alpha^3 + \beta^3$.

 (*b*) Solve the equation

$$32x^3 - 14x + 3 = 0,$$

given that one root is twice another root.

2. (*a*) A test consists of seven questions, to each of which a candidate must give one of three possible answers. If the candidate must score 1, 2 or 3 points for each of the seven questions, in how many different ways can a candidate score exactly 18 points in the test?

 (*b*) Write down the first three terms of the binomial $(1 + ax)^b$ in ascending powers of x.

 If the first three terms are $1 + 3x + \frac{27}{2}x^2$, find a and b.

 Using these values for a and b in $(1 + ax)^b$, find the percentage error, correct to two decimal places, if the sum of the first three terms is used as an approximate value of the binomial when $x = 0{\cdot}06$.

3. (*a*) Examine for convergence the following series:

 (i) $\sum_{n=0}^{\infty} \frac{n^2}{n!}$ (ii) $\sum_{n=1}^{\infty} \frac{2n+1}{n^4+3n}$.

 (*b*) State the domain of values of x for which $|x| < 1$.

 The first term of a geometric series is 2 and the common ratio is $\left(2k - \frac{1}{k}\right)$, where $k > 0$. Find

 (i) the domain of values of k for which the series converges,

 (ii) the sum to infinity of this series when $k = \frac{3}{4}$.

4. (*a*) Evaluate $\lim_{x \to 2} \frac{x-2}{x^2-4}$.

 What value, if any, has $\frac{x-2}{x^2-4}$ when $x = 2$?

 Sketch the graph of the function $x \to \frac{x-2}{x^2-4}$ in the domain $-4 \leqslant x \leqslant 4$, $x \in R$.

 (*b*) Show that the sum to n terms of the series

$$\frac{1}{2.4} + \frac{1}{3.5} + \dots + \frac{1}{(r+1)(r+3)}$$

 is

$$\frac{5}{12} - \frac{1}{2}\left(\frac{1}{n+2} + \frac{1}{n+3}\right),$$

 and hence, or otherwise, show that the series

$$\sum_{r=1}^{\infty} \frac{1}{(r+1)(r+3)}$$

 is convergent.

 Deduce that the series $\sum_{r=1}^{\infty} \frac{1}{(r+1)(r+3)} \frac{x^r}{2^r}$ converges for $0 < x \leqslant 2$, $x \in R$.

5. (*a*) From first principles, differentiate with respect to x the function

$$x \to x^2 + \frac{1}{x^2}, \ (x \neq 0).$$

(*b*) Differentiate with respect to x

(i) $\sqrt{\dfrac{x}{x+1}}$, $x > 0$ (ii) $\sin^{-1}(\cos x)$ (iii) $\log_e \tan^2 x$.

(*c*) If $x = \dfrac{2}{1+3t}$, $(t \neq -\frac{1}{3})$, show that $x\dfrac{d^2x}{dt^2} = 2\left(\dfrac{dx}{dt}\right)^2$.

6. The total length of the twelve edges of a rectangular box is 64 cm and the total surface area is 104 cm². If x cm is the length of any one edge and if V cm³ is the volume of the box, show that

$$V = 52x - 16x^2 + x^3.$$

Find the maximum volume of the box and the lengths of the edges in this maximum case.

7. f and g are functions defined as follows:

$$f : x \to \frac{e^x + e^{-x}}{2} = f(x), \ x \in R$$

$$g : x \to \frac{e^x - e^{-x}}{2} = g(x), x \in R.$$

(i) Show that $f(-x) = f(x)$ and $g(-x) = -g(x)$.

(ii) Show that $f'(x) = g(x)$ where f' means $\dfrac{df}{dx}$ and hence, or otherwise, evaluate

$$\int_0^{1/2} 2 f(x) g(x)\, dx.$$

(iii) Show that $f(x)$ is positive for all values of x and hence prove that g is an increasing function of x for all x.

(iv) Prove that g has a point of inflexion at (0, 0) and sketch the graph of g.

8. (*a*) Evaluate

(i) $\displaystyle\int_1^4 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx$ (ii) $\displaystyle\int_0^8 \frac{x\,dx}{9-x}$ (iii) $\displaystyle\int_0^{\pi/2} \sin 2x(1 + \sin^2 x)\, dx$.

(*b*) Write down the first five terms of the expansion of $(1 + x^2)^{-1}$ and taking the integral of these between 0 and 1 as an approximation for $\displaystyle\int_0^1 \frac{dx}{1+x^2}$ find an approximate value of $\frac{\pi}{4}$.

(See tables p. 41).

9. Sketch the graph of the function

$$f : R \to R : x \to \frac{1 - x^2}{x^2 - 4}$$

in the domain $-2 < x < 2$.

Indicate the region bounded by the graph and the x-axis and calculate the volume generated by rotating this region about the y-axis.

10. The tangent at the point (x_n, y_n) on the graph of the function $x \to f(x) = y$ cuts the x-axis at $(x_{n+}\,, 0)$ as in diagram.

Show that

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \text{ where } f' \text{ means } \frac{df}{dx}.$$

If $f(x) = x^3 - 2$, prove that

$$x_{n+1} = \frac{2}{3}x_n + \frac{2}{3x_n^2}.$$

Verify that one root of the equation $x^3 - 2 = 0$ lies between $1\frac{1}{4}$ and $1\frac{1}{2}$.

Taking $1\frac{1}{4}$ as an approximation for $\sqrt[3]{2}$, use the above formula to find a further approximation.