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LEAVING CERTIFICATE EXAMINATION, 1967

MATHEMATICS (HONOURS) – PAPER II (300 marks)

TUESDAY, 13th JUNE – MORNING 10 to 12.30

Six questions to be answered. All questions are of equal value. Mathematical Tables may be obtained from the Superintendent.

1. (a) If p and q are real and $i = \sqrt{-1}$, express each of the following in the form $p + iq$:

   (i) $(3 - i)(i + 2)$ (ii) $\frac{i + 2}{i + 3}$ (iii) $\sqrt{5 - 12i}$.

   (b) Show that the recurring decimal $2{\cdot}1\dot{0}\dot{1}$ can be written in the form $\frac{p}{q}$ where p and q are integers and $q \neq 0$.

2. (a) If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$, show that $\alpha + \beta + \gamma = -p$, $\alpha\beta + \beta\gamma + \gamma\alpha = q$, $\alpha\beta\gamma = -r$.

   If α, β, γ are the roots of the equation $x^3 - 3x^2 + x + 2 = 0$, show that $\alpha^3 + \beta^3 + \gamma^3 = 12$ and find the value of $\alpha^4 + \beta^4 + \gamma^4$.

   (b) Prove that the operation of intersection of sets is associative i.e. prove $A \cap (B \cap C) = (A \cap B) \cap C$, where A, B, C are any sets. (Diagram alone not sufficient as a proof).

3. What is meant by the sum of an infinite series ? What is meant by saying that an infinite series is convergent ?

   Show that the sum of the series $\sum_{r=1}^{n} \frac{1}{r(r+1)}$ is $\frac{n}{n+1}$ and hence deduce that the series $\sum_{r=1}^{\infty} \frac{1}{r(r+1)}$ is convergent.

   Prove that $\frac{1}{r^2} \leqslant \frac{2}{r(r+1)}$ for all $r \geqslant 1$ and hence show that the series $\sum_{r=1}^{\infty} \frac{1}{r^2}$ converges.

   Deduce that $\sum_{r=1}^{\infty} \frac{1}{r^t}$ converges for all $t \geqslant 2$.

4. (a) In a recent survey on a certain number of shoppers it was found that at least 82% bought a certain product A and at least 48% bought a certain product B. At least what percentage of shoppers bought both products ?

   If at least 76% of the shoppers of the survey also bought a certain product C, at least how many shoppers bought all three products ?

   (b) Let p, q be any two rational numbers and let the operation $*$ be defined as

   $$p * q = \frac{p + q}{2}.$$

   Show that the set of rational numbers is closed under the operation $*$.

   Say, giving your reasons, whether or not the operation $*$ is (i) Commutative (ii) Associative.

5. (a) Differentiate from first principles $\frac{1}{1 - x}$ with respect to x.

   Differentiate with respect to x:

   $$\frac{x}{1 - x}, \qquad \sin^2 3x, \qquad e^{\frac{1}{1-x}}, \qquad \log_e x^2.$$

   (b) If $y = e^{2x}$, show that $\frac{d^2y}{dx^2} - 4y = 0$.

   Find the values of m for which

   $$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

   when $y = e^{mx}$ (m real and independent of x)

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6. From a thin sheet of tin of area 4π a hollow right circular cone (including base) is made. If the volume of the cone is to be maximum, find the radius of the base.

7. (a) In how many ways can 4 boys and 3 girls occupy 7 seats in a row if no two girls sit next to one another ?

(b) Write down the first three terms and the general term in the expansion of $(1 + x)^n$ (n a natural number) and by differentiation, or otherwise, express

$$\sum_{r=2}^{n} r(r-1)\ {}^nC_r$$

in terms of n.

(${}^nC_r$ denotes the number of combinations of n things taking r at a time.)

8. (a) Evaluate:-

(i) $\int_0^4 (x-1)\sqrt{x}\,dx$; (ii) $\int_0^1 xe^{x^2}dx$; (iii) $\int_0^1 \frac{xdx}{1+x}$.

(b) Show that $\int_{\frac{a}{2}}^{a} f(x)dx = \int_0^{\frac{a}{2}} f(a-x)dx$ and deduce that $\int_0^a f(x)dx = \int_0^{\frac{a}{2}} \left[f(x) + f(a-x)\right] dx$.

Hence, or otherwise, evaluate $\int_0^{\pi} x\sin^3 x\,dx$.

9. (a) If p_1, p_2, p_3 are the probabilities of the events E_1, E_2, $E_1 \cap E_2$, respectively, what relation connects p_1, p_2, p_3 when (i) E_1, E_2 are independent events, (ii) E_1, E_2 are not independent events ?

Two cards are drawn at random from a pack containing 52 cards. Find the probability that both cards are aces if

(i) the first card picked is replaced,
(ii) the first card picked is not replaced.

(b) Two boys A and B toss an unbiased penny and the first to obtain a "head" wins. If A has first toss, find the probability that

(i) A wins on his first toss,
(ii) B wins on his first toss,
(iii) A wins on his second toss.

Calculate the probability that A wins.

10. A wholesaler who supplies sand has two depots A and B and these at present contain 30 tons and 15 tons of sand, respectively. Three builders P, Q, R place orders for 20 tons, 15 tons and 10 tons, respectively. The cost of transporting the sand is a fixed amount £t per ton per mile and the distances in miles from A to P,Q,R are respectively 6, 4, 3 and from B to P,Q,R are respectively 3, 2, 2.

If A supplies x tons to P and y tons to Q and each builder receives his required amount, show that the total cost of transporting the required amounts of sand is £$(140 + 2x + y)t$.

Write down all the inequalities in x, y and $x + y$.

By plotting the set of ordered pairs (x,y) that simultaneously satisfy these inequalities, find (x,y) for which the cost of transport is least.

OR

10. Show that the graph of $y = \frac{x}{x^2-1}$ has no maximum and no minimum and show also that it has only one point of inflexion. Find the coordinates of this one point of inflexion and trace the curve paying special attention to the shape of the graph as x tends to $+1$, as x tends to -1 and as x tends to infinity.