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LEAVING CERTIFICATE EXAMINATION, 1977

APPLIED MATHEMATICS - HIGHER LEVEL

FRIDAY, 24 JUNE - MORNING, 9.30 to 12

Six questions to be answered. All questions carry equal marks.
Mathematics Tables may be obtained from the Superintendent.
Take the value of g to be 9·8 metres/second². $\vec{i}$ and $\vec{j}$ are perpendicular unit vectors.

1. A car starts from rest at P and moves with constant acceleration k metres/second². Three seconds later another car passes through P travelling in the same direction with constant speed u metres/second, where $u > 3k$. Draw a velocity/time graph for the two cars, using the same axes and the same scales.

Hence, or otherwise, show that the second car will just catch up on the first if $u = 6k$, and that it will not catch up on it if $u < 6k$.

If $u > 6k$, find the greatest distance the second car will be ahead of the first.

2. Explain, with the aid of a diagram, what is meant by the relative velocity of one body with respect to another.

To a cyclist riding North at 7 m/s the wind appears to blow from the North-West. To a pedestrian walking due West at 1 m/s the same wind appears to come from the South-West. Find the magnitude and direction of the velocity of the wind, by expressing it in the form $u\vec{i} + v\vec{j}$ or otherwise.

3. A particle is projected up a plane which is inclined at an angle α to the horizontal, where $\tan \alpha = \frac{1}{2\sqrt{3}}$. The direction of projection makes an angle of 60° with the inclined plane. The plane of projection is vertical and contains the line of greatest slope. Show that the particle strikes the inclined plane at right angles.

Verify that the total energy of the particle at the moment of striking the plane is the same as when the particle is first projected.

4. A mass of 2 kg is lying on a rough plane inclined at 60° to the horizontal, coefficient of friction $\frac{1}{2}$. The 2 kg mass is connected, by a light inextensible string passing over a smooth fixed pulley at the top of the plane, to a mass of 5 kg hanging freely. When the system is set free the 5 kg mass moves downwards.

Show in separate diagrams the forces acting on each mass, and calculate the common acceleration.

If a mass of 12 kg were used instead of the 2 kg mass, show by considering the forces acting that it would not move up the plane or down the plane.

5. A pump raises water from a depth of 5 m and discharges it horizontally through a nozzle of diameter 0·14 m at a speed of 10 m/s. Calculate

(i) the mass of water raised per second,

(ii) the kinetic energy given to this mass,

(iii) the power at which the pump is working.

If the water strikes a fixed vertical wall directly in front of the nozzle, find the force exerted by the water on the wall, on the assumption that no water bounces back.

[Mass of 1 m³ of water is 1000 kg. Take $\pi = \frac{22}{7}$]

6. One end of a uniform ladder of weight W rests against a smooth vertical wall and the other rests on rough horizontal ground so that it makes an angle $\tan^{-1} \frac{2}{3}$ with the horizontal. Show that the ladder will start to slip outwards if the coefficient of friction μ is less than $\frac{3}{4}$.

When $\mu = \frac{1}{2}$ the ladder is just prevented from slipping by a vertical string attached to the ladder at a point $\frac{1}{3}$ of its length from the top. Calculate the tension in the string in terms of W.

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7. A smooth sphere of mass 3 kg moving at $\sqrt{29}$ m/s collides with a second sphere of mass 6 kg moving at 5 m/s. The directions of motion of the spheres make angles of $\tan^{-1} \frac{2}{5}$ and $\tan^{-1} \frac{4}{3}$, respectively, with the line of centres, both angles being measured in the same sense. The coefficient of restitution is $\frac{3}{4}$.

Find the speeds and directions of motion of the spheres after impact and calculate the kinetic energy lost in the collision.

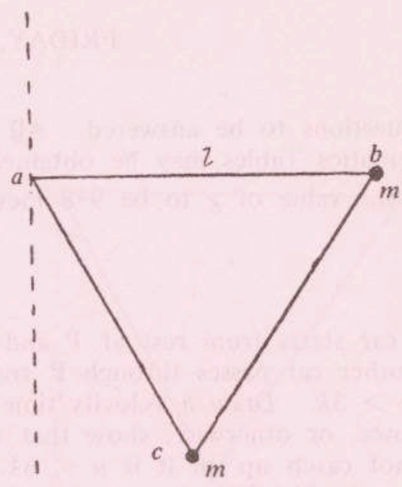
8. The position vector of a particle moving in a circle of radius r with constant angular velocity ω can be expressed in the form

$$r \cos \omega t . \vec{i} + r \sin \omega t . \vec{j}.$$

Find the acceleration of the particle and show that it is directed towards the centre.

Three light rods ab, bc, ca, each of length l, are freely jointed to form a triangle abc. Two particles of mass m are attached, one at b and one at c. The system rotates about a vertical axis through a with constant angular velocity ω such that ab is horizontal and c is vertically below ab. (see diagram)

Show in separate diagrams the forces acting on the particles (the forces exerted by the rods act along the rods). Calculate the forces in the rods and prove that $\omega^2 = 2\sqrt{3}\, g/l$.



9. For a compound pendulum (a rigid body performing small oscillations in a vertical plane about a horizontal axis) prove that the periodic time T is given by

$$T = 2\pi \sqrt{\frac{I}{mgh}}$$

where m is the mass of the pendulum, I the moment of inertia about the axis, and h the perpendicular distance from the centre of gravity to the axis.

If the compound pendulum is a uniform rod of length $2l$, show that $\frac{g}{4\pi^2 l} T^2 = \frac{h}{l} + \frac{1}{3} \frac{l}{h}$ and calculate the value of $\frac{h}{l}$ for which T is a minimum.

10. State the Principle of Archimedes.

A tank contains a layer of water and a layer of oil of relative density 0·8. A uniform rod of relative density $\frac{7}{9}$ is totally immersed with one third of its volume in the water and two thirds in the oil. It is maintained in that position by two vertical strings attached to the ends of the rod and to the bottom of the tank.

Show in a diagram the forces acting on the rod and calculate the tensions in the strings in terms of W, the weight of the rod.

11. Answer any three of (a), (b), (c), (d) below.

(a) Using Taylor's theorem (Mathematics Tables, p. 42) find the first three terms in the Taylor series for $\frac{e^x}{1 - x}$ in the neighbourhood of $x = 0$ i.e. the Maclaurin series for $\frac{e^x}{1 - x}$.

(b) Determine if the series

$$(z - 1) + \tfrac{1}{2}(z - 1)^2 + \ldots + \tfrac{1}{n}(z - 1)^n + \ldots\ldots$$

is absolutely convergent for $z = \frac{1}{4}(1 + 3i)$.

(c) Solve the differential equation

$$\frac{dy}{dx} = y \sin x$$

if $y = \sqrt{e}$ when $x = \frac{\pi}{3}$

(d) Solve the equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$$

if $y = 2$ and $\frac{dy}{dx} = 0$ when $x = 0$.