AN ROINN OIDEACHAIS

LEAVING CERTIFICATE EXAMINATION, 1975

APPLIED MATHEMATICS – HIGHER LEVEL

THURSDAY, 26 JUNE – MORNING, 9.30 to 12

Six questions to be answered. All questions carry equal marks.
Mathematics Tables may be obtained from the Superintendent.
Take the value of g to be $9{\cdot}8$ metres/second^2. $\vec{i}$ and $\vec{j}$ are perpendicular unit vectors.

1. A particle falls freely under gravity from rest at a point p. After it has fallen for one second another particle is projected vertically downwards from p with speed $14{\cdot}7$ m/s. By considering the relative motion of the particles, or otherwise, find the time and distance from p at which they collide. Show the motion of both on a velocity–time graph.

2. A man wishes to swim across a river 60 m wide. The river flows with a velocity of 5 m/s parallel to the straight banks. and the man swims at a speed of 3 m/s relative to the water. If he heads at an angle α to the upstream direction, and his actual velocity is at an angle θ to the downstream direction, show that $\tan\theta = \dfrac{3\sin\alpha}{5 - 3\cos\alpha}$.

   Prove that $\tan\theta$ has a maximum value when $\cos\alpha = \frac{3}{5}$. Deduce that the time taken for the man to cross by the shortest path is 25 s.

3. A particle of mass 4 M rests on a rough horizontal table, where the coefficient of friction between the particle and the table is $\frac{1}{3}$, and is attached by two inelastic strings to particles of masses 3 M and M which hang over smooth light pulleys at opposite edges of the table. The particle and the two pulleys are collinear. Show in separate diagrams the forces acting on each of the three particles when the system is released from rest. Find the distance fallen by the 3 M particle in time t.

4. A particle of mass 5 kg rests on the highest point of a fixed sphere of radius $0{\cdot}25$ m. The particle is slightly displaced from the highest point and slides down the smooth outer surface of the sphere. Show in a diagram the forces acting on the particle when the radius to it has turned through an angle θ. Express the speed v of the particle and the reaction R of the sphere in terms of θ. Find where the particle leaves the sphere.

5. A missile is projected from a point o with speed 21 m/s at an angle α to the horizontal. Express its velocity $\vec{v}$ and its displacement $\vec{r}$ from o after time t seconds in terms of unit vectors along the horizontal and vertical.

   The missile strikes a small target whose horizontal and vertical distances from the point of projection are 30 m and 10 m respectively. Write down two equations in α and t, the time taken to reach the target. Hence find the two possible values of $\tan\alpha$, and the times taken on the corresponding trajectories.

6. a, b are two points 4 m apart on a smooth horizontal table and n is the midpoint of $[ab]$. A particle of mass $0{\cdot}1$ kg is held at n by two elastic strings the other ends of which are attached to a, b respectively. Each of the strings is of natural length 1 m and elastic constant 5 N/m. If the particle is then drawn aside along ab and released from rest when $1{\cdot}5$ m from b, show that it moves with simple harmonic motion.

   Find the period, and the least time taken to reach a point $1{\cdot}75$ m from b.

OVER →

7. A light rod ab of length L connects a smooth ring of mass 8 M, attached to it at a to a particle of mass M attached to it at b. The ring is free to slide along a fixed thin smooth horizontal wire. The rod ab is held in a horizontal position underneath the wire and released from rest. Show in a diagram the forces acting on the ring and the mass M during the motion.

Explain why the conservation of energy and the conservation of linear momentum in a horizontal direction can be applied to the system. Show that the speed of M when b is vertically below a is $4(Lg)^{1/2}/3$.

8. A lamina is rotating with angular velocity ω about an axis perpendicular to its plane. If the moment of inertia of the lamina about the axis is I, prove that its kinetic energy is $\frac{1}{2}I\omega^2$.

Show that the moment of inertia of a uniform square lamina $abcd$, of mass M and side $2l$, about an axis through a perpendicular to the lamina is $8\,Ml^2/3$.

The lamina is free to rotate in a vertical plane under gravity about the axis, which is fixed horizontally. It is released from rest with ab horizontal and above cd. Find the speed of c when ac reaches the vertical.

9. The force of attraction of the earth on a particle of mass M distance x from the centre of the earth is Mga^2/x^2, where a is the radius of the earth $(x \geqslant a)$. Write down the equation of motion for a particle moving under this force alone and calculate the speed of the particle at distance x if it was projected vertically upwards from the earth's surface with speed $(2ga)^{1/2}$.

Prove that the time taken to reach a height $3a$ above the earth's surface is $\frac{7}{3}\left(\frac{2a}{g}\right)^{1/2}$.

10. State the conditions for the equilibrium of a body immersed in a fluid.

A thin uniform rod ab of length $2l$ and weight W, can turn freely about the end a, which is fixed at a height l above the surface of water into which the other end dips. Show in a diagram the forces acting on the rod. If the rod is in equilibrium when inclined at 45° to the vertical, show that the specific weight of the rod is $\frac{1}{2}$.

11. (a) Find the first two terms in the Taylor's series for $\tan x$ in the neighbourhood of $x = 0$, i.e. the Maclaurin series for $\tan x$.

(b) Find the domain of z for which the series of complex terms.

$$1 - 2z + 3z^2 - \ldots + (-)^{n-1}\, nz^{n-1} + \ldots$$

is absolutely convergent.

Is the series absolutely convergent for $z = 1 + 2i$?

(c) By substituting $y = xv$, show that the differential equation

$$\frac{dy}{dx} = \frac{y - x}{y + x}$$

becomes

$$\left(\frac{v}{v^2 + 1} + \frac{1}{v^2 + 1}\right) dv = -\frac{dx}{x}.$$

Solve the equation, given that $x = 1$ when $y = 0$.