Reversibility

Anthony G. O'Farrell

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1 What is the question?

The big question is: Classify those mappings that are conjugate to their own inverses in a given group of mappings.

One could rephrase this as the classification of those discrete dynamical systems that admit time-reversal symmetries of the same degree of smoothness, or "type" (such as holomorphic, polynomial, PL).

The general problem is a very large research programme. A nontrivial answer to the general problem is not in sight, but sub-problems make for reasonable, interesting and challenging projects.

The main focus at present is on two-dimensional systems, where there are many open problems. Specifically, we will try to understand reversibility in interesting groups of plane and surface homeomorphisms.

2 Why is this problem significant?

The interest in reversibility began in physics, but eventually links appeared to complex analysis, geometry, functional equations, discrete dynamics, approximation, number theory, algebraic geometry and operator theory.

Reversibility encapsulates the idea of a time-reversal symmetry in a system. (A system with a time-reversal symmetry is one for which the past is obtained as the future of an alternative present; the symmetry maps each state to its alternative present.) For that reason, it first attracted notice in classical dynamics, in the work of people such as Poincaré and Birkhoff [B]. For example, the dynamics of the solar system, or the galaxy, are reversible, to a high approximation. Experts in Dynamical Systems are well-aware of the special character of reversible systems [D,AS,S]. But reversibility connects to a wider range of pure and applied mathematics.

An involution is a map which equals its own inverse. Pairs of non-commuting involutive maps play central roles in problems in many areas of Mathematics. The composition of two involutions is an example of a reversible map (– we call such maps *strongly reversible*). So for such problems one can make a link to dynamics, and employ dynamical methods to good effect.

Jürgen Moser was probably the first person to make this connection explicitly, and it is exploited brilliantly in his paper with Webster [MW] on the biholomorphic classification of surfaces and hypersurfaces in several complex variables. They were able to exploit the same mathematics developed by Siegel and Moser for the study of celestial meachnics, to solve this problem in several complex variables, a special case of a famous problem posed by Poincaré.

I picked up the idea, and it is exploited in my work with Sanabria, [OS] solving De Paepe's problem on polynomial approximation. This problem had resisted attack by all known methods for 20 years, but yielded when we recognised that there were two holomorphic involutions in the story, and used them to construct a discrete dynamical system, then imbed it in a flow, and complexify the time variable. This work used deep results from classical one-dimensional complex dynamics. De Paepe was astonished by this connection.

In earlier, unrelated work, Marshall and I [MO1, MO2] realised that there was a dynamical connection to another approximation problem. This work was about the problem of approximating a function $f(x, y)$ of two variables by sums $g(x) + h(y)$. This problem connects to Hilbert's 13th problem about "superposition" of functions (work of Kolmogoroff, Arnol'd, Vitushkin, Kahane), to dimension theory (work of Sternfeld), to numerical analysis and functional equations, and even to economics (work of my colleague Paddy Geary), and had been studied intensively, notably by S. Ya. Havinson. We made a major advance by applying Birkhoff's ergodic theorem to the associated dynamical system. Havinson said he was amazed to learn that he needed to learn ergodic theory to understand his problem.

Webster has pointed out [W1] a surprising application to classical conformal mapping, arising from two-valued reflections. A class of quartic curves admit such reflections, and one can construct the conformal map to the disc from the dynamics of an associated map. Further examples of reversible elements occur in connection with the areas mentioned above, including such remote-looking things as the foliations of three-manifolds [G] and the integral quadratic forms [SA].

Voronin called this phenomenon “hidden dynamics” (–when a nondynamical problem has an essential invariant associated to some invisible dynamical system.) He has listed half-a-dozen examples related to aspects of complex analysis, including some other cases of Poincarés problem.

Mathematics is full of hidden connections: you can never tell at the beginning of an investigation what areas will be involved by the end. I have spent a lifetime avidly learning, listening to ideas, fitting them together, making connections. I was struck by the repeated occurrence of reversibility across a wide range of problems, and focussed on it. Gradually, it came to light that many of the great minds of the past century were there ahead of me, well aware of the importance of this concept. Apart from those mentioned, the Fields medallist Smale gave the relation between hamiltonian and continuous reversible systems to Devaney as a thesis topic [D]. Arnol'd's student M. B. Sevryuk wrote a monograph in 1986 [SE] about the occurrence of KAM-like structures in the phase space of continuous reversible systems, near a periodic orbit.

I decided to embark on a systematic study, with a view to classifying the possibilities and elucidating the general structure. The first thing to do, is to examine examples, i.e. to understand reversibility in varied specific groups of maps, and look for patterns.

When one considers the general reversibility problem, the various cases are concrete and interesting problems in varied areas of mathematics. For instance, Jarzcyk [J1,J2] came to the case of one-dimensional homeomorphisms through a connection with functional equations. Some cases call for expertise in point-set topology, some for expertise in differential, or analytic, or algebraic geometry, some are essentially combinatorics, or classical group problems. The whole subject may be viewed as a branch of nonabelian (typically infinite) group theory. But it can also be viewed in completely different ways. For instance, it may be viewed as a branch of functional analysis, by identifying the mappings with the associated composition operators. It may also be viewed as pure dynamical systems.

The theory of single involutions is not simple, but is well-developed. For instance, the differential topology of involutions (and other smooth maps of finite order) was much studied in the wake of the Atiyah-Singer index theorem. In abstract algebra, group theorists have studied involutions since the beginning, and they play a basic structural role. Groups generated by exactly two involutions are called dihedral (because they include the classical symmetry groups of the regular polygons). They are very much more special than the groups generated by three involutions — for instance, every finite group is a quotient of a subgroup of a group generated by three involutions. At the same time, the group generated by two involutive maps displays rather complex and varied behaviour, and there is a rich field of possibilities.

I have spoken about aspects of reversibility to about a dozen audiences over the past couple of years. These talks included colloquia for nonspecialists, general mathematical audiences, and specialists in complex analysis, harmonic analysis, functional analysis, geometric analysis, operator theory and dynamics. In all cases, the response has been enthusiastic. The simply-stated and apparently-innocent question strikes a chord in people, and they want to get involved. Many have a favourite group, or have encountered important involutions in their work. Some are already clued-in to the significance of reversibility.

3 How will the question be answered?

In each full permutation group, all elements are reversible, and in fact strongly-reversible. In other words, the problem is trivial at the level of unstructured bijections. For other groups of maps, the difficult thing is to understand the interaction between the algebraic reversibility condition and the topological (or differential, or analytic, or formal) properties of the maps.

The question may be asked for each group of bijective maps. In general, one expects that a non-tautological answer will be harder to come by as the dimension of the underlying space increases, and as its topology becomes more complicated. One also expects that the successful methodology will be quite different in groups of analytic, differentiable, and merely continuous maps.

The investigation has been under way for a few years now, and we already have some useful results to hand.

At the level of pure algebra, pure group theory, we have worked out various equivalent ways to characterise reversibility.

The investigation proceeds, broadly speaking, by exploring cases, seeking patterns, and trying to prove that they really are patterns.

We have located substantial previous work on classical groups, homeomorphisms of the line, invertible polynomial maps, and made contact with the groups that worked on these.

Our own results to hand include the characterization of reversibility in the following groups:

- The group of real-analytic invertible germs in one variable (Exploratory work with two undergraduates, Dymphna Graham and Siobhan Keane (published in the Irish Systems and Signals Conference Proceedings, 2001).

- The linear fractional group over the quaternions (with Roman Lavicka and Ian Short) [LOS]

- The isometry groups of all the classical constant-curvature geometries (Short).

- The group of formally-invertible power series in one variable.

- The group of diffeomorphisms of the real line, and its subgroup of orientation-preserving maps. (with Ian Short).

- The group of diffeomorphisms of the circle, and its subgroup of orientation-preserving maps. (with Ian Short and Nick Gill).

- Groups of piecewise-linear transformations on the line and circle (Short and Gill).

- The group of biholomorphic germs in one complex variable (with Patrick Ahern).

We have also learned much about conjugacy, and factorisation, and have developed a systematic methodology. This has resulted in accelerated progress as we tackle each new group of maps.

The plan-of-campaign is as follows:

- Start with the simplest things we don't know, and work up.

- The property of being reversible and the property of being the product of k involutions are conjugacy invariants, so a general strategy is to approach the problem by first clarifying the conjugacy classification as far as possible.

- Another strategy is to look for homomorphic images of the target group, and work on the (automatically simpler) problem in the image. If the image is proper in a larger group, then the problem using maps from the big group to factor, is easier again. For instance, the subgroup of diffeomorphisms of R^d fixing a given point has a homomorphism into the general linear group, and into the group of formally-invertible d-tuples of formal power series.

- The full homeomorphism group of the plane will be studied. We (+ Short, Roginskaya and Le Roux) have begun work on aspects of conjugacy in this group, concentrating on the fixed-point-free maps, which are already complicated.

- We (+ Roginskaya) have classified conjugacy in the diffeomorphism group of the line. One idea used (a certain infinite product) there has the potential to help with the diffeomorphism group of the plane.

- The group of quasiconformal maps of the plane will be studied. This group is intermediate between the Möbius group and the full homeomorphism group, and much is known. It should be relatively manageable.

- The group of formally-invertible pairs of formal power series in two (commuting) symbols. This will play an auxiliary role.

- groups of PL maps on the plane.

- Eventually, corresponding problems in higher dimensions and on more interesting manifolds may be studied. It would also be desirable to understand what happens with discrete subgroups of the group of plane homeomorphisms. This has fascinating connections to number theory, to surface topology, and to the chaotic Lorentz flow.

- At some point, certainly by the case of diffeomorphisms of 3-space, the group will contain the free group on two generators, and one has a healthy respect for this gorilla. This has to mean that the description of the reversible maps will be less than totally explicit. The real objective is to get some description that is informative and useful.

- Where possible, provided it comes in sight, the plan would include pressing on to a classification of the product of k involutions and of k reversibles in each case studied.

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