TWENTY FIFTH IRISH MATHEMATICAL OLYMPIAD

Saturday, 12 May 2012

First Paper

Time allowed: **Three hours.**

1. Let

$$C = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$$

and let

$$S = \{4, 5, 9, 14, 23, 37\}.$$

Find two sets A and B with the properties

(a) $A \cap B = \emptyset$.

(b) $A \cup B = C$.

(c) The sum of two distinct elements of A is not in S.

(d) The sum of two distinct elements of B is not in S.

2. A, B, C and D are four points in that order on the circumference of a circle K. AB is perpendicular to BC and BC is perpendicular to CD. X is a point on the circumference of the circle between A and D. AX extended meets CD extended at E and DX extended meets BA extended at F.

Prove that the circumcircle of triangle AXF is tangent to the circumcircle of triangle DXE and that the common tangent line passes through the centre of the circle K.

3. Find, with proof, all polynomials f such that f has nonnegative integer coefficients, $f(1) = 8$ and $f(2) = 2012$.

4. There exists an infinite set of triangles with the following properties:

(a) the lengths of the sides are integers with no common factors, and

(b) one and only one angle is $60°$.

One such triangle has side lengths 5, 7 and 8. Find two more.

5. (a) Show that if x and y are positive real numbers, then

$$(x + y)^5 \geq 12xy(x^3 + y^3).$$

(b) Prove that the constant 12 is the best possible. In other words, prove that for any $K > 12$ there exist positive real numbers x and y such that

$$(x + y)^5 < Kxy(x^3 + y^3).$$