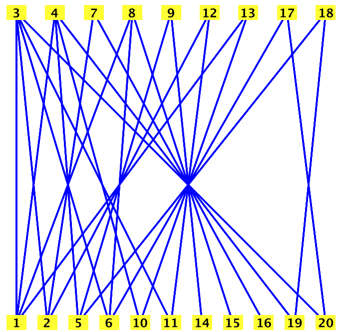
TWENTY FIFTH IRISH MATHEMATICAL OLYMPIAD

Saturday, 12 May 2012

First Paper

Solutions



1. The graph with vertex set C and edges consisting of pairs (a, b) with $a + b \in S$ is shown above and is seen to be bipartite and connected. It has a unique bipartition as shown. Thus there are two possible solutions. $A = \{1, 2, 5, 6, 10, 11, 14, 15, 16, 19, 20\}$ and $B = \{3, 4, 7, 8, 9, 12, 13, 17, 18\}$ or $A = \{3, 4, 7, 8, 9, 12, 13, 17, 18\}$ and $B = \{1, 2, 5, 6, 10, 11, 14, 15, 16, 19, 20\}$

2. Let O be the centre of the circle K. Join A to C and X to O. AC will pass through O since $\angle ABC = 90°$. Then $\angle AFX = \angle FDE$ since FB is parallel to EC and $\angle FDE = \angle XAC$ since $AXDC$ is cyclic. Therefore AO is tangent to the circumcircle of AXF. Therefore XO is tangent to the circumcircle of AXF since $XO = AO$ and X lies on the circumcircle. Similarly XO is tangent to the circumcircle of DXE. That is, the two circumcircles touch and the common tangent XO passes through O.

3. There is only one such polynomial, namely $f(x) = x^{10}+x^9+x^8+x^7+x^6+x^4+x^3+x^2$. To prove this, suppose that g is a polynomial with nonnegative integer coefficients and that $g(2) = 2012$. Suppose further that some coefficient of g is at least 2. Say that the coefficient of x^k is at least 2. Let

$$h(x) = g(x) - 2x^k + x^{k+1}.$$

Then h has nonnegative integer coefficients, $h(2) = g(2) = 2012$ and $h(1) < g(1)$. By applying this observation repeatedly we can find some polynomial q with coefficients 0 or 1 such that $h(1) < g(1)$ Now, there is a unique polynomial with coefficients 0 or 1 whose value at 2 is 2012, namely $f(x) = x^{10} + x^9 + x^8 + x^7 + x^6 + x^4 + x^3 + x^2$ (this follows from the uniqueness of the binary representation of 2012). Thus, if p is a polynomial with nonegative integer coefficients such that $p(2) = 2012$ then $p(1) \geq f(1) = 8$ with equality if and only if $p = f$.

4. One triangle is easily found by finding the point D on BC in the given 5, 7, 8 triangle such that $AB = AD = 7$. The angle at C is the $60°$ angle. Then $\cos\angle ACB = \frac{x^2+64-49}{14x} = \frac{1}{2}$. Therefore

$$x^2 - 7x + 15 = 0,$$

so $x = 5$ or $x = 3$. The solution $x = 5$ is the given triangle, and the solution $x = 3$ gives a new triangle with sides 3, 7 and 8 that has the required properties.

In general, let ABC be any triangle with integer sides and $ACB = 60°$. Let $BC = t$ and $AC = t + a$ and $AB = t + b$ where t is a positive rational, a and b are integers and $a > b$. Then

$$\begin{array}{rrcl} & \cos\angle ACB & = & \cos 60° \\ \Rightarrow & \frac{(t+a)^2+t^2-(t+b)^2}{2t(a-2b)} & = & \frac{1}{2} \\ \Rightarrow & t(a-2b) & = & b^2 - a^2 \\ \Rightarrow & t & = & \frac{a^2-b^2}{2b-a}. \end{array}$$

From this, $2b$ must be greater than a. It can be shown that, in order to get primitive triangles, a and b must be relatively prime but it is not necessary to show that in this case. So the smallest possible choices for a and b are $a = 3$ and $b = 2$. These give $t = 5$ which gives the 5, 7, 8 triangle. If we choose $a = 4$ and $b = 3$, we get $t = \frac{7}{2}$, which gives a triangle with sides $\frac{7}{2}$, $\frac{13}{2}$ and $\frac{15}{2}$. Multiplying across by 2 gives the triangle with sides 7, 13 and 15. Taking $a = 5$ and $b = 3$ gives the triangle with sides 3, 7 and 8 obtained above. All the triangles satisfying the two conditions in the question can be generated in this way. A final example is the triangle with sides 168, 223 and 253 given by taking $a = 17$ and $b = 11$.

5. First Solution. Since

$$x^3 + y^3 = (x+y)(x^2 - xy + y^2),$$

and

$$(x+y)^4 = x^3 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4,$$

the stated inequality is equivalent to the statement that, for all $x, y > 0$,

$$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = (x+y)^4 \geq 12xy(x^2 - xy + y^2).$$

Hence, after transposing and rearranging the terms of this, the stated inequality is equivalent to the next one: $\forall x, y > 0$

$$x^4 - 8x^3y + 18x^2y^2 - 8xy^3 + y^4 \geq 0.$$

But a little thoughtful experimentation shows that

$$x^4 - 8x^3y + 18x^2y^2 - 8xy^3 + y^4 = (x^2 - 4xy + y^2)^2,$$

whence part (a) follows.

Clearly, there is equality iff $x, y > 0$ and $x^2 - 4xy + y^2 = 0$, i.e., when

$$(x - 2y)^2 = 3y^2, \;\; x = (2 \pm \sqrt{3})y.$$

Thus, the constant 12 is best possible because the inequality becomes an equality when

$$x = (2 \pm \sqrt{3})\lambda, \;\; y = \lambda,$$

where $\lambda > 0$, and only then. Hence, part (b) follows.

Second Solution. Since the polynomial expression

$$P(x, y) = (x + y)^5 - 12xy(x^3 + y^3)$$

is homogeneous of degree 5, i.e., $P(tx, ty) = t^5 P(x, y)$, for all x, y, t, it's enough to prove the inequality when $x, y > 0$ and $x + y = 1$. Assuming this, and making the substitution $y = 1 - x$, we must show that $P(x, 1 - x) \geq 0$, i.e.,

$$12x(1 - x)(x^3 + (1 - x)^3) \leq 1,$$

whenever $0 \leq x \leq 1$. But

$$12x(1-x)(x^3+(1-x)^3) = 12x(1-x)(1-3x(1-x)) \equiv 4z(1-z), \;\; z = 3x(1-x),$$

and

$$4z(1 - z) = 1 - (2z - 1)^2 \leq 1, \;\; \forall z.$$

Hence, $P(x, y) \geq 0$ if $x, y > 0$ and $x + y = 1$, and so the stated inequality follows by homogeneity. This proves (a).

As for (b), there is equality in the last inequality iff $z = 1/2$, i.e., $6x(1-x) = 1$, i.e.,

$$x = \frac{3 \pm \sqrt{3}}{3}.$$

And so, for such x, $P(x, 1 - x) = P(1 - x, x) = 0$. In other words, there is equality in the given inequality if (x, y) is a positive multiple of

$$(\frac{3 - \sqrt{3}}{3}, \frac{\sqrt{3}}{3}), \text{ or } (\frac{\sqrt{3}}{3}, \frac{3 - \sqrt{3}}{3}).$$

On the other hand, if $P(x, y) = 0$, for some positive x, y, then $P(u, 1 - u) =$ for some $u \in [0, 1]$, which means that $u = (3 - \sqrt{3})/3$, etc., and so either $x = tu, y = t(1 - u)$ or $x = s(1 - u), y = su$ for some positive s, t.

Third Solution. By homogeneity of $P(x, y,$ it's sufficient to prove that $P(t, 1) \geq 0$ for all $t > 0$. Equivalently, that

$$t^4 - 8t^3 + 18t^2 - 8t + 1 \geq 0, \ \forall t > 0.$$

Observe the symmetry of the coefficients of this quartic polynomial. To exploit this, divide across by $t^2 > 0$ to get

$$t^2 + \frac{1}{t^2} - 8(t + \frac{1}{t}) + 18 \geq 0$$

Let

$$s = t + \frac{1}{t}, \text{ so that } s \geq 2, \text{ and } t^2 + \frac{1}{t^2} = s^2 - 2,$$

whence we have to show that $s^2 - 8s + 16 \geq 0$ for all $s \geq 2$. But this is obvious. Hence (a) holds.

Moreover, there is equality in the last statement iff $s = 4$, i.e., $t = 2 \pm \sqrt{3}$, etc.. Hence no constant smaller than 12 will do.