The Root Test

Let $\sum_{n=1}^{\infty} a_n$ be a series such that

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \rho,$$

(that is, $|a_n|^{\frac{1}{n}}$ tends to the value ρ, as n tends to infinity. Here ρ is allowed to be any finite number as well as ∞.) Then

(a) If $\rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $\rho > 1$, then $\sum_{n=1}^{\infty} a_n$ does not converge.

(c) If $\rho = 1$, then the root test in inconclusive.

Remark 1.1: That means we need to determine $\lim_{n\to\infty} |a_n|^{\frac{1}{n}}$ and if this limit exists and is different from one, then we are able to make a statement about the convergence of the given series. The root test is usually used when there are only powers involved in the expression of a_n.

Examples 1.2: Consider the series $\sum_{n=1}^{\infty} \frac{1}{(\ln n)^n}$. Then $a_n = \frac{1}{(\ln n)^n}$ for all $n \geq 1$, and we need to understand $\lim_{n\to\infty} |a_n|^{\frac{1}{n}}$. Since $a_n > 0$, we have

$$|a_n|^{\frac{1}{n}} = \left(\frac{1}{(\ln n)^n}\right)^{\frac{1}{n}} = \frac{1}{((\ln n)^n)^{\frac{1}{n}}} = \frac{1}{\ln n}$$

So $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\ln n} = 0$. Since this limit is less than one, we conclude from the root test that $\sum_{n=1}^{\infty} \frac{1}{(\ln n)^n}$ converges.

Material developed by the Department of Mathematics & Statistics, NUIM and supported by www.ndlr.com.

Examples 1.3: Consider the series $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$. Then the underlying sequence is $a_n = \frac{2^n}{n^3} \geq 0$, for all $n \geq 1$. So we have

$$|a_n|^{\frac{1}{n}} = \left(\frac{2^n}{n^3}\right)^{\frac{1}{n}} = \frac{(2^n)^{\frac{1}{n}}}{(n^3)^{\frac{1}{n}}} = 2 \cdot \left(\frac{1}{n^{\frac{1}{n}}}\right)^3$$

So $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} 2 \cdot \left(\frac{1}{n^{\frac{1}{n}}}\right)^3 = 2 > 1$. (Note that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$, which is not obvious, but it is a well-known result and we use it here without verifying it.) Hence the root test tells us that $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ does not converge. Moreover since all a_n are positive we know that the series diverges to infinity.

Examples 1.4: Consider the series $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^n$. Then

$$a_n = \left(1 - \frac{1}{n}\right)^n \geq 0, \quad \text{for all } n \geq 1,$$

and all $a_n \geq 0$. Furthermore we have

$$|a_n|^{\frac{1}{n}} = \left(\left(1 - \frac{1}{n}\right)^n\right)^{\frac{1}{n}} = 1 - \frac{1}{n}$$

So $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} 1 - \frac{1}{n} = 1$, and thus the root test is inconclusive.

That means we have to use a different method to understand the given series. In this case we may use the divergence test. Observe that for the underlying sequence a_n we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = \frac{1}{e} \neq 0.$$

(Again it is not obvious that $\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = \frac{1}{e}$, however it is a famous result and as such worth remembering.) That means a_n does not converge to zero, and thus by the divergence test the corresponding series does not converge. (For more details refer to the handout on the divergence test.) Since all sequence elements are positive we can conclude that $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^n$ diverges to infinity.