INTEGRAL TEST I (OF II)

A sequence based on a function

Let $f : [1, \infty) \to \mathbb{R}$ be a function. Then f gives rise to the sequence

$$a_n := f(n), \quad \text{for all } n \geq 1.$$

That means the n-th element of the sequence is the value $f(n)$, that is, the value the function f takes on for the integer n.

Example 1.1: Consider the function $f(x) = \frac{1}{x}$, for all $x \geq 1$. Then we can define a sequence based on f by setting

$$a_n := f(n) = \frac{1}{n}, \quad \text{for all } n \geq 1.$$

That means the elements of the sequence $(a_n)_{n \geq 1}$ are

$$a_1 = 1, \quad a_2 = \frac{1}{2}, \quad a_3 = \frac{1}{3} \quad a_4 = \frac{1}{4}, \quad \ldots$$

A series based on a function

Let $f : [1, \infty) \to \mathbb{R}$ be a function. Then we can study the series corresponding to the sequence $(f(n))_{n \geq 1}$, that is, the series

$$\sum_{n=1}^{\infty} f(n).$$

Example 1.2: As in Example 1.1, let $f(x) = \frac{1}{x}$, for all $x \geq 1$. This function gives rise to the series

$$\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} \frac{1}{n},$$

which we know as the harmonic series.

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Integral Test

This idea of connecting sequences to functions and vice versa can be exploited to study the convergence of certain series.

Integral Test: Let $f : [1, \infty) \to \mathbb{R}$ be a continuous, decreasing and positive function. Then the series $\displaystyle\sum_{n=1}^{\infty} f(n)$ converges if and only if the integral $\displaystyle\int_1^{\infty} f(x)\ dx$ converges. Otherwise $\displaystyle\sum_{n=1}^{\infty} f(n)$ diverges to plus infinity.

Example 1.3: We use the integral test to prove that the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to plus infinity. First we observe that the harmonic series can be written as $\displaystyle\sum_{n=1}^{\infty} f(n)$, if we define the function f as

$$f(x) := \frac{1}{x}, \quad \text{for all } x \geq 1.$$

Before we can apply the integral test we need to verify that the function f is continuous, decreasing and positive. All three properties are straightforward to check in this case. It is well-known that f is continuous on the interval $[1, \infty)$. Furthermore f is decreasing on the interval $[1, \infty)$, because as x gets bigger $\frac{1}{x}$ gets smaller. Finally f is positive on the interval $[1, \infty)$, as $x \geq 1$ is positive.

Hence f satisfies the conditions necessary for the integral test. This test now requires us to study the integral

$$\int_1^{\infty} \frac{1}{x}\ dx.$$

We calculate

$$\begin{aligned}\int_1^{\infty} \frac{1}{x}\ dx &= \lim_{t\to\infty} \int_1^{t} \frac{1}{x}\ dx = \lim_{t\to\infty} [\ln(x)]_1^t \\ &= \lim_{t\to\infty} [\ln(t) - \underbrace{\ln(1)}_{=0}] = \lim_{t\to\infty} \ln(t) = \infty\end{aligned}$$

Hence $\displaystyle\int_1^{\infty} \frac{1}{x}\ dx$ does not converge, and thus, by the integral test, the harmonic series does not converge. In particular the harmonic series diverges to plus infinity. (For more information about the harmonic series see Example 1.1. on the handout p-Series I and Example 2.1 on the handout p-Series II.)