INTEGRAL TEST II (OF II)

Example 2.1: Consider the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. We know this series converges. (See the handout p-Series II). We use the integral test to verify this fact. The underlying sequence $\left(\frac{1}{n^2}\right)_{n\geq 1}$ can be linked to the function $f(x) = \frac{1}{x^2}$, for $x \geq 1$, that is,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} f(n).$$

We need to check that f is continuous, positive and decreasing on the interval $[1, \infty)$. It is not hard to show these properties for the given function $f(x) = \frac{1}{x^2}$ and we omit the details here. However in Example 2.3 we show how one can verify such those properties generally.

Next we calculate

$$\int_1^{\infty} \frac{1}{x^2}\, dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\, dx = \lim_{t\to\infty} \left[\frac{-1}{x}\right]_1^t = \lim_{t\to\infty} \frac{-1}{t} + 1 = 1$$

Hence $\int_1^{\infty} \frac{1}{x^2}$ converges. Therefore, by the integral test, the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges as well.

Remark 2.2: Observe that Example 2.1 can be repeated for any p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$. If $p > 1$ we obtain convergence and if $p \leq 1$ we obtain divergence to infinity (see also Example 1.3 on the handout Integral Test I, where we dealt with the case $p = 1$). In particular this confirms the rules about the limit of p-series as given on handout p-Series II.

Example 2.3: Consider the series $\sum_{n=1}^{\infty} ne^{-n^2}$. Observe that we can link the underlying sequence $(ne^{-n^2})_{n\geq 1}$ of this series to the function $f(x) := xe^{-x^2}$, for $x \geq 1$. Then $f(n) = ne^{-n^2}$, for all $n \geq 1$, and

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consequently $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2} = \sum_{n=1}^{\infty} f(n)$. Furthermore we can tell that f is both continuous and positive on the interval $[1, \infty)$, since it is the product of the two continuous and positive functions x and e^{-x^2}, (note that $x \geq 1$). Next we show that f is decreasing. This can be done by studying the derivative of f. Using the product rule we obtain

$$f'(x) = 1 \cdot e^{-x^2} + x \cdot (-2xe^{-x^2}) = e^{-x^2} \cdot (1 - 2x^2).$$

But since $e^{-x^2} > 0$, for all $x \in \mathbb{R}$ and $1 - 2x^2 < 0$, for all $x \geq 1$ we conclude that $f'(x) < 0$, for all $x \geq 1$. Hence f has a negative slope, and thus is decreasing on the interval $[1, \infty)$.

Therefore f satisfies the conditions that are necessary to apply the integral test. The test requires us to calculate the integral $\displaystyle\int_1^{\infty} xe^{-x^2}\ dx$. First we observe that

$$\int_1^{\infty} xe^{-x^2}\ dx = \lim_{t\to\infty} \int_1^{t} xe^{-x^2}\ dx.$$

In order to solve the integral on the right-hand side we use the substitution $u = -x^2$. Then $du = -2x\ dx$ and the integration limits $(1, t)$ for x change to the integration limits $(-1, -t^2)$ for u. Therefore

$$\int_1^{t} xe^{-x^2}\ dx = \int_{-1}^{-t^2} \frac{e^u}{-2}\ du = \left[\frac{e^u}{-2}\right]_{-1}^{-t^2} = \frac{e^{-t^2}}{-2} - \frac{e^{-1}}{-2}.$$

Observe that as t tends to infinity, e^{-t^2} tends towards zero. Thus

$$\int_1^{\infty} xe^{-x^2}\ dx = \lim_{t\to\infty} \int_1^{t} xe^{-x^2}\ dx = -\frac{e^{-1}}{-2} = \frac{1}{2e}.$$

Hence we have shown that $\displaystyle\int_1^{\infty} xe^{-x^2}\ dx$ converges, and thus by the integral test, we conclude that the series $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$ converges.

Note that the integral test only allows us to conclude that the series $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$ converges, but it does not say anything about its limit.

In particular it is **incorrect** to claim that $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$ converges towards $\dfrac{1}{2e}$ just because $\displaystyle\int_1^{\infty} xe^{-x^2}\ dx = \frac{1}{2e}$.