Divergence Test

The divergence test is based on the observation that if a series $\sum_{n=0}^{\infty} a_n$ converges, then

$$\lim_{n\to\infty} a_n = 0.$$

Hence we have a necessary condition for the convergence of a series, that is, a series can only converge if the underlying sequence converges towards zero. From this follows the **Divergence Test**, which states:

$$\text{If } \lim_{n\to\infty} a_n \neq 0, \text{ then } \sum_{n=0}^{\infty} a_n \text{ does not converge.}$$

Example 1.1: Consider the series

$$\sum_{n=1}^{\infty} \frac{n}{n+1}.$$

Then the underlying sequence is given by $a_n = \frac{n}{n+1}$, for $n \geq 1$. The divergence test requires us to study the limit of a_n as n tends to infinity. We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1 \neq 0.$$

Hence $\lim_{n\to\infty} a_n \neq 0$, and thus, according to the divergence test, the given series does not converge. Note that since our series only consists of positive terms (that is, $\frac{n}{n+1} > 0$, for all $n \geq 0$) we can conclude that our series diverges to plus infinity, that is,

$$\sum_{n=1}^{\infty} \frac{n}{n+1} = \infty.$$

Example 1.2: Consider the series

$$\sum_{n=0}^{\infty} (-1)^n.$$

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Then the underlying sequence is given by $a_n = (-1)^n$, for $n \geq 0$. Observe that $a_n = 1$, whenever n is an even number, and $a_n = -1$, whenever n is odd. Thus the sequence a_n oscillates between plus and minus one. As a consequence this sequence does not have a limit at all, which in particular means that

$$\lim_{n\to\infty} a_n \neq 0.$$

Hence the divergence test implies that our series does not converge. This makes perfect sense once we realize that the given series is a geometric series with $x = -1$. From the theory about geometric series we know that in this case the limit of the series does not exist. (For more details on this see the handout Geometric Series II.)

Limitations of the Divergence Test

Note that the divergence test only gives a condition under which a given series does not converge, but it does not provide a condition under which a given series does converge. In particular in a situation where $\lim_{n\to\infty} a_n = 0$, the test does not allow us to draw any conclusions about the corresponding series. In fact, in general it is incorrect to say that a series converges because the limit of the underlying sequence happens to equal zero.

Example 1.3: Consider the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

Then the underlying sequence is given by $a_n = \frac{1}{n}$, for all $n \geq 1$. If we want to apply the divergence test we have to study the limit of a_n, as n tends to infinity. We get

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Hence the divergence test can not be applied to this series. But that does not mean that the series converges. In fact the harmonic series diverges to plus infinity (see Example 2.1 on the handout p-Series II).

In summary we can say, that if $\lim\limits_{n\to\infty} a_n \neq 0$, then $\sum a_n$ does not converge. However if $\lim\limits_{n\to\infty} a_n = 0$, then $\sum a_n$ might or might not converge. In this case we need to gather more information about the series to make a statement.